Test #2 (1:50-2:40pm) **(Fall 2024)** **Name:**

Do all problems and give the **process** of your solution.

1. (15 points). Evaluate the double integral $\iint_D (x+y)^2 dA$, where $D = [0,1] \times [0,2]$

Solution. By Fubini theorem

$$\iint_D (x+y)^2 dA = \int_0^1 \left[\int_0^2 (x+y)^2 dy\right] dx = \int_0^1 \left[\frac{1}{3}(x+y)^3\right]_{y=0}^{y=2} dx$$

$$= \frac{1}{3}\int_0^1 \left[(x+2)^3 - x^3\right] dx = \frac{1}{3} \times \frac{1}{4}\left[(x+2)^4\Big|_{x=0}^{x=1} - x^4\Big|_{x=0}^{x=1}\right]$$

$$= \frac{1}{3} \times \frac{1}{4} \times (3^4 - 2^4 - 1) = \frac{16}{3}$$

2. (15 points). Let $w = \dfrac{1}{x+y^2+z^3}$, $x = \cos t$, $y = \sin t$ and $z = 1+t$. Find $\dfrac{dw}{dt}\Big|_{t=0}$.

Solution.

$$\frac{\partial w}{\partial x} = -\frac{1}{(x+y^2+z^3)^2}, \quad \frac{\partial w}{\partial y} = -\frac{2y}{(x+y^2+z^3)^2}, \quad \frac{\partial w}{\partial z} = -\frac{3z^2}{(x+y^2+z^3)^2}$$

By chain rule

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

$$= -\frac{1}{(x+y^2+z^3)^2}(-\sin t) - \frac{2y}{(x+y^2+z^3)^2}\cos t - \frac{3z^2}{(x+y^2+z^3)^2}$$

When $t = 0$, $x = 1$ and $y = 1$. Therefore,

$$\frac{dw}{dt}\Big|_{t=0} = 0 + 0 - \frac{3}{4} = -\frac{3}{4}$$

3. (15 points). Find equation for the tangent plane to the surface $z = x^2 + xy + 3y^2$ at the point $(1,1,5)$.

Solution.

$$\frac{\partial z}{\partial x} = 2x + y, \qquad \frac{\partial z}{\partial y} = x + 6y, \qquad \frac{\partial z}{\partial x}(1,1) = 3, \qquad \frac{\partial z}{\partial y}(1,1) = 7$$

Thus, the tangent plane is

$$z - 5 = 3(x-1) + 7(y-1) \ \text{ or) } \ 3x + 7y - z = 5$$

4. (15 points). Find equation for the tangent planeto the surface $e^{xyz} + x + y + z = 1$ at the point $(1, -1, 0)$.

Solution.

$$\nabla F = \langle yze^{xyz} + 1, xze^{xyz} + 1, xye^{xyz} + 1\rangle$$

$$\nabla F(1, -1, 0) = \langle 1, 1, 0\rangle$$

The equation of the tangent plane: $(x - 1) + (y + 1) + 0 \cdot z = 0$ or $x + y = 0$

5. (20 points). Let $f(x, y, z) = x^2 + 2y^2 + 3z^2$.

(a). Find the maximum rate of change of f at the point $(1, 1, 1)$ and the direction in which it occurs.

Solution.

$$\nabla f = \langle 2x, 4y, 6z\rangle$$

The direction the maximal rate: $\nabla f(1, 1, 1) = \langle 2, 4, 6\rangle$. The maximal rate is

$$\|\nabla f(1, 1, 1)\| = \sqrt{2^2 + 4^2 + 6^2} = \sqrt{56}$$

(b). Find the direction derivative of f at the same point $(1, 1, 1)$ in the direction of the vector $\mathbf{v} = \mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$

Solution.

$$\mathbf{u} = \frac{1}{\|\mathbf{v}\|}\mathbf{v} = \langle \frac{1}{3}, \frac{2}{3}, -\frac{2}{3}\rangle$$

$$D_{\mathbf{u}}f(1, 1, 1) = \nabla f(1, 1, 1) \cdot \mathbf{u} = \langle 2, 4, 6\rangle \cdot \langle \frac{1}{3}, \frac{2}{3}, -\frac{2}{3}\rangle = -\frac{2}{3}$$

6. (20 points). Use Lagrange multipliers to find the minimal value of the function $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraint $2x + 4y + 4z = 1$.

Solution. $f(x, y, z) = x^2 + y^2 + z^2$ and $g(x, y, z) = 2x + 4y + 4z$.

$$\nabla f(x, y, z) = \langle 2x, 2y, 2z\rangle, \qquad \nabla g(x, y, z) = \langle 2, 4, 4\rangle$$

We solve

$$\begin{cases} \nabla f(x, y, z) = \lambda \nabla g(x, y, z) \\ \\ 2x + 4y + 4z = 1 \end{cases} \quad \text{or} \quad \begin{cases} 2x = 2\lambda \\ 2y = 4\lambda \\ 2z = 4\lambda \\ 2x + 4y + 4z = 1 \end{cases}$$

$$\begin{cases} x = \lambda \\ y = 2\lambda \\ z = 2\lambda \\ 2\lambda + 8\lambda + 8\lambda = 1 \end{cases} \quad \text{or} \quad \begin{cases} x = \lambda \\ y = 2\lambda \\ z = 2\lambda \\ \lambda = \frac{1}{18} \end{cases}$$

Hence, $x = 1/18$, $y = 2/18$, $z = 2/18$

$$f\left(\frac{1}{18}, \frac{2}{18}, \frac{2}{18}\right) = \left(\frac{1}{18}\right)^2 + \left(\frac{2}{18}\right)^2 + \left(\frac{2}{18}\right)^2 = \frac{9}{18^2} = \frac{1}{36}$$

So the minimal value is $1/36$.