Test #3 (1:50-2:40pm) **Fall 2024** **Name:**

Do all problems and give the **process** of your solution.

1. (20 points). Evaluate $\iint_D x dA$, where D is bounded by the curves $y = x^2$ and $y = 2 - x^2$.

Solution.

$$\iint_D x dA = \int_{-1}^{1}\int_{x^2}^{2-x^2} x dy dx = \int_{-1}^{1} x(2-2x^2)dx = 0$$

2. (20 points). Evaluate $\iint_D e^{x^2+y^2} dA$, where $D = \{(x,y)|\ \ x^2+y^2 \le 1,\ y \ge 0\}$.

Solution. By the polar substitution

$$\iint_D e^{x^2+y^2} dA = \int_0^{\pi}\int_0^1 e^{r^2} r dr d\theta = \pi \int_0^1 e^{r^2} r dr \overset{u=r^2}{=} \frac{\pi}{2}\int_0^1 e^u du = \frac{\pi}{2}(e-1)$$

3 (20 points). Evaluate $\iiint_E z dV$, where E is bounded by the paraboloid $z = 1 - x^2 - y^2$ and the plane $z = 0$.

Solution. By Fubini's theorem and then polar substitution

$$\iiint_E z dV = \iint_D \left[\int_0^{1-x^2-y^2} z dz\right] dA = \frac{1}{2}\iint_D (1-x^2-y^2)^2 dA$$
$$= \frac{1}{2}\int_0^{2\pi}\int_0^1 (1-r^2)^2 r dr d\theta = \pi\int_0^1 (1-r^2)^2 r dr \overset{u=1-r^2}{=} \frac{\pi}{2}\int_0^1 u^2 du = \frac{\pi}{6}$$

4 (20 points). Evaluate $\iiint_E \sqrt{x^2+y^2+z^2} dV$, where $E = \{(x,y,z)|\ 1 \le x^2+y^2+z^2 \le 4\}$.

Solution. By spherical substitution,

$$\iiint_E \sqrt{x^2+y^2+z^2} dV = \int_0^{2\pi}\int_0^{\pi}\int_1^2 \rho\cdot\rho^2 \sin\varphi d\rho d\varphi d\theta$$
$$= 2\pi\left(\int_0^{\pi}\sin\varphi d\varphi\right)\left(\int_1^2 \rho^3 d\rho\right) = 2\pi\cdot 2\cdot\frac{2^4-1}{4} = 15\pi$$

5. (20 points). Calculate the work done by the force field $\mathbf{F}(x,y,z) = \langle -y, x, z\rangle$ from $(1,0,0)$ to $(1,0,2\pi)$ along the curve $\mathbf{r}(t) = \langle \cos t, \sin t, t\rangle$.

Solution.

$$W = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C -y dx + x dy + z dz = \int_0^{2\pi}\Big((-\sin t\cdot(-\sin t) + \cos t\cdot\cos t + t\Big)dt$$
$$= \int_0^{2\pi}(1+t)dt = 2\pi + 2\pi^2$$