Test #2(12:40-1:30pm) **(Fall 2024)** **Name:**

Do all problems and give the **process** of your solution.

1. (15 points). Evaluate the double integral $\iint_D \sqrt{x+y}dA$, where $D=[0,1]\times[0,2]$

Solution. By Fubini's theorem,

$$\iint_D \sqrt{x+y}dA = \int_0^1 \left[\int_0^2 \sqrt{x+y}dy\right]dx = \iint_D \sqrt{x+y}dA = \int_0^1 \left[\frac{2}{3}(x+y)^{3/2}\right]_{y=0}^{y=2} dx$$
$$= \frac{2}{3}\int_0^1 \left[(x+2)^{3/2} - x^{3/2}\right]dx = \frac{2}{3}\times\frac{2}{5}\left[(x+2)^{5/2}\Big|_{x=0}^{x=1} - x^{5/2}\Big|_{x=0}^{x=1}\right]$$
$$= \frac{4}{15}(3^{5/2} - 2^{5/2} - 1)$$

2 (15 points). Let $z = \ln(x^2+xy+y^4)$. $x=\sin t$ and $y=\cos t$. Find $\left.\frac{dz}{dt}\right|_{t=0}$.

Solution.

$$\frac{\partial z}{\partial x} = \frac{2x+y}{x^2+xy+y^4}, \qquad \frac{\partial z}{\partial y} = \frac{x+4y^3}{x^2+xy+y^4}$$

By chain rule,

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = \frac{2x+y}{x^2+xy+y^4}\cos t - \frac{x+4y^3}{x^2+xy+y^4}\sin t$$

When $t=0$, $x=0$ and $y=1$.

$$\left.\frac{dz}{dt}\right|_{t=0} = 1$$

3. (15. points). Find equation for the tangent plane to the surface $z = \left(\frac{1}{2}x + \frac{1}{4}y\right)^{100}$ at the point $(1,2,1)$.

Solution.

$$\frac{\partial z}{\partial x} = 50\left(\frac{1}{2}x + \frac{1}{4}y\right)^{99}, \qquad \frac{\partial z}{\partial y} = 25\left(\frac{1}{2}x + \frac{1}{4}y\right)^{99}, \qquad \frac{\partial z}{\partial x}(1,2) = 50, \qquad \frac{\partial z}{\partial y}(1,2) = 25$$

Thus, the tangent plane is

$$z-1 = 50(x-1) + 25(y-2) = 50x + 25y - 100$$

4. (15 points). Find equation for the tangent plane to the surface $x^2+2y^2+z^2+yz=3$ at the point $(1,1,-1)$.

Solution. Let $F(x, y, z) = x^2 + 2y^2 + z^2 + yz$. $\nabla F(x, y, z) =< 2x, 4y + z, 2z + y >$. Thus, the normal vector is

$$\mathbf{n} = \nabla F(1, 1, -1) =< 2, 3, -1 >$$

The equation of the tangent plane: $2(x - 1) + 3(y - 1) - (z + 1) = 0$. Or

$$2x + 3y - z = 6$$

5. (20 points). Let $f(x, y, z) = \dfrac{1}{x^2 + y^2 + z^2}$.

(a). Find the maximum rate of change of f at the point $(1, 1, 2)$ and the direction in which it occurs.

Solution.

$$\nabla f = -\frac{1}{(x^2 + y^2 + z^2)^2} \langle 2x, 2y, 2z \rangle$$

The direction of the maximum rate

$$\nabla f(1, 1, 2) = -\frac{1}{36} \langle 2, 2, 4 \rangle$$

and the maximal rate is

$$\|\nabla f(1, 1, 2)\| = \frac{1}{36} \sqrt{2^2 + 2^2 + 4^2} = \frac{\sqrt{24}}{36}$$

(b). Find the direction derivative of f at the same point $(1, 1, 2)$ in the direction of the vector $\mathbf{v} = \langle 1, 2, 2 \rangle$

Solution. the unit vector $\mathbf{u}$ in the direction of $\mathbf{v}$:

$$\mathbf{u} = \frac{1}{\|\mathbf{v}\|} \mathbf{v} = \langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \rangle$$

Thus,

$$D_{\mathbf{u}} f(1, 1, 2) = \nabla f(1, 1, 2) \cdot \mathbf{u} = -\frac{1}{36} \langle 2, 2, 4 \rangle \cdot \langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \rangle = -\frac{7}{54}$$

6. (20 points). Use Lagrange multiplier to find the maximum and minimum values of the function $f(x, y, z) = 2x + 4y + 4z$ subject to the constraint $x^2 + y^2 + z^2 = 1$.

Solution. $\nabla f(x, y, z) =< 2, 4, 4 >$, $\nabla g(x, y, z) =< 2x, 2y, 2z >$. We solve

$$\begin{cases} \nabla f(x, y, z) = \lambda \nabla g(x, y, z) \\ x^2 + y^2 + z^2 = 1 \end{cases} \quad \text{or} \quad \begin{cases} 2 = \lambda 2x \\ 4 = \lambda 2y \\ 4 = \lambda 2z \\ x^2 + y^2 + z^2 = 1 \end{cases}$$

$$\begin{cases} x = \frac{1}{\lambda} \\ y = \frac{2}{\lambda} \\ z = \frac{2}{\lambda} \\ \left(\frac{1}{\lambda}\right)^2 + \left(\frac{2}{\lambda}\right)^2 + \left(\frac{2}{\lambda}\right)^2 = 1 \end{cases} \quad \text{or} \quad \begin{cases} x = \frac{1}{\lambda} \\ y = \frac{2}{\lambda} \\ z = \frac{2}{\lambda} \\ \lambda^2 = 9 \end{cases}$$

As $\lambda = 3$: $x = \frac{1}{3}$, $y = \frac{2}{3}$ and $z = \frac{2}{3}$

As $\lambda = -3$: $x = -\frac{1}{3}$, $y = -\frac{2}{3}$ and $z = -\frac{2}{3}$

$$f\left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right) == 2 \times 3 + 4 \times \frac{3}{2} + 4 \times \frac{2}{3} = 18$$

$$f\left(-\frac{1}{3}, -\frac{2}{3}, -\frac{2}{3}\right) == -2 \times 3 - 4 \times \frac{3}{2} - 4 \times \frac{2}{3} = -18$$

So the maximal value is 18, the minimal value is -18.