Test #3 (12:40-1:30pm) **Fall 2024** **Name:**

Do all problems and give the **process** of your solution.

1 (20 points). Evaluate $\iint_D x dA$, where D is bounded by the curves $y = x^2$ and $y = \sqrt{x}$.

Solution.

$$\iint_D x dA = \int_0^1 \int_{x^2}^{\sqrt{x}} x dy dx = \int_0^1 x(\sqrt{x} - x^2) dx = \frac{2}{5} - \frac{1}{4} = \frac{3}{20}$$

2 (20 points). Evaluate $\iint_D \sqrt{1 - x^2 - y^2} dA$, where $D = \{(x, y) | x^2 + y^2 \leq 1\}$

Solution. By the polar substitution

$$\iint_D \sqrt{1 - x^2 - y^2} dA = \int_0^{2\pi} \int_0^1 \sqrt{1 - r^2} r dr \theta$$
$$= 2\pi \int_0^1 \sqrt{1 - r^2} r dr \overset{u=1-x^2}{=} \pi \int_0^1 \sqrt{u} du = \frac{2}{3}\pi$$

3. Evaluate $\iiint_E x dV$, where E is bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$.

Solution. Let $D = \{(x, y) | x + y \leq 1,\ x, y \geq 0\}$. By Fubini theorem

$$\iiint_E x dV = \iint_D \Big[\int_0^{1-x-y} x dz\Big] dA = \iint_D x\Big[1 - x - y\Big] dA$$
$$= \int_0^1 x\Big[\int_0^{1-x} [1 - x - y] dy\Big] dx = \int_0^1 x\Big[(1 - x)^2 - \frac{1}{2}y^2\Big|_{y=0}^{y=1-x}\Big] dx$$
$$= \frac{1}{2}\int_0^1 x(1 - x)^2 dx = \frac{1}{2}\int_0^1 \Big[x - 2x^2 + x^3\Big] dx = \frac{1}{2}\Big[\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\Big] = \frac{1}{24}$$

4. (20 points). Evaluate $\iiint_E (x^2 + y^2 + z^2)^{3/2} dV$, where $E = \{(x, y, z) |\ x^2 + y^2 + z^2 \leq 1,\ x, y, z \geq 0\}$.

Solution.

$$\iiint_E (x^2 + y^2 + z^2)^{3/2} dV = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^1 \rho^3 \cdot \rho^2 \sin\phi d\rho d\phi d\theta$$
$$= \frac{\pi}{2}\Big(\int_0^{\frac{\pi}{2}} \sin\phi d\phi\Big)\Big(\int_0^1 \rho^5 d\rho\Big) = \frac{\pi}{2} \cdot 1 \cdot \frac{1}{6} = \frac{\pi}{12}$$

5 (20 points). Find the work done by the force field $\mathbf{F}(x, y, z) = y\mathbf{i} + z\mathbf{j} + x\mathbf{k}$ in moving a particle along the trajectory $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$ $(0 \le t \le 1)$.

Solution.

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C y dx + z dy + x dz = \int_0^1 \left(t^2 + t^3 \cdot 2t + t \cdot 3t^2\right) dt = \frac{1}{3} + \frac{2}{5} + \frac{3}{4} = \frac{89}{60}$$