Homework 1

Problems in 1.6.

3. Notice that

$$I_{2k} \in \mathcal{A}_{2k} \subset \bigcup_{n=1}^{\infty} \mathcal{A}_n \qquad k = 1, 2 \cdots$$

On the other hand,

$$\bigcup_{k=1}^{\infty} I_{2k} = \bigcup_{k=1}^{\infty} [2k-1, 2k) \notin \mathcal{A}_n \qquad n = 1, 2, \cdots$$

Therefore,

$$\bigcup_{k=1}^{\infty} I_{2k} = \bigcup_{k=1}^{\infty} [2k-1, 2k) \notin \bigcup_{n=1}^{\infty} \mathcal{A}_n$$

4. We need to establish the relations

$$\sigma\big(\{A_1, \cdots, A_n\}\big) \subset \sigma\big(\{B_1, \cdots, B_n\}\big) \ \text{ and } \ \sigma\big(\{A_1, \cdots, A_n\}\big) \supset \sigma\big(\{B_1, \cdots, B_n\}\big)$$

Set $A_0 = \phi$. By the fact the $A_1, \cdots, A_n$ are disjoint, $A_k = B_k \setminus B_{k-1} \in \sigma\big(\{B_1, \cdots, B_n\}\big)$ $(k = 1, \cdots n)$. Or, $\{A_1, \cdots, A_n\} \subset \sigma\big(\{B_1, \cdots, B_n\}\big)$. Since $\sigma\big(\{A_1, \cdots, A_n\}\big)$ is the smallest σ-algebra that takes $\{A_1, \cdots, A_n\}$ as its sub-class. So we must have

$$\sigma\big(\{A_1, \cdots, A_n\}\big) \subset \sigma\big(\{B_1, \cdots, B_n\}\big)$$

Another direction can be established in a similar way.

5. (a). Indeed, we can prove that for any $\epsilon > 0$ there is a subsequence $\{n_k\}$ such that

$$P\Big(\bigcap_{k=1}^{\infty} A_{n_k}\Big) \geq 1 - \epsilon$$

By assumption, for each $k \geq 1$, there is n_k such that

$$P(A_{n_k}) \geq 1 - \frac{1}{2^k}\epsilon \ \text{ or } \ P(A_{n_k}^c) \leq \frac{1}{2^k}\epsilon$$

Clearly, we can make $n_k < n_{k+1}$ for any $k \geq 1$. By Morgan's law,

$$P\Big(\Big\{\bigcap_{k=1}^{\infty} A_{n_k}\Big\}^c\Big) = P\Big(\bigcup_{k=1}^{\infty} A_{n_k}^c\Big) \leq \sum_{k=1}^{\infty} P(A_{n_k}^c) \leq \sum_{k=1}^{\infty} \frac{1}{2^k}\epsilon = \epsilon$$

(b). Assume that $\{A_n\} \subset \mathcal{F}$ be an independent sequence with $P(A_n) = \alpha > 0$ (you may think A_n is the event of "success in game n" in a Bernoulli trial). Then for any n_k,

$$P\Big(\bigcap_{k=1}^{\infty} A_{n_k}\Big) = \prod_{k=1}^{\infty} P(A_{n_k}) = 0$$

(c). By continuity theorem (Theorem 3.1, p.11)

$$P\Big(\bigcap_{n=1}^{\infty} A_n\Big) = \lim_{n\to\infty} P(A_n) \geq \alpha$$

(d). By Morgan's law

$$P\Big(\bigcup_{n=1}^{\infty} A_n\Big) = 1 - P\Big(\bigcap_{n=1}^{\infty} A_n^c\Big)$$

Notice A_n^c is non-increasing with

$$P(A_n^c) = 1 - P(A_n) \geq 1 - \alpha$$

Applying the conclusion of Part (c) to the sequence $\{A_n^c\}$,

$$P\Big(\bigcap_{n=1}^{\infty} A_n^c\Big) \geq 1 - \alpha$$

So we have

$$P\Big(\bigcup_{n=1}^{\infty} A_n\Big) \leq \alpha$$

11. By Morgan's law and independence,

$$P\Big(\bigcup_{k=1}^{n} A_k\Big) = 1 - P\Big(\Big\{\bigcup_{k=1}^{n} A_k\Big\}^c\Big) = 1 - P\Big(\bigcap_{k=1}^{n} A_k^c\Big) = 1 - \prod_{k=1}^{n} P(A_k^c) = 1 - \prod_{k=1}^{n} \big(1 - P(A_k)\big)$$

12. By the inequality $e^{-x} \geq 1 - x$ $(x > 0)$ and the identity from Problem 11,

$$P\Big(\bigcup_{k=1}^{n} A_k\Big) = 1 - \prod_{k=1}^{n} \big(1 - P(A_k)\big) \geq 1 - \prod_{k=1}^{n} \exp\big\{ - P(A_k)\big\} = 1 - \exp\Big\{ - \sum_{k=1}^{n} P(A_k)\Big\}$$

Assume that

$$\sum_{n=1}^{\infty} P(A_n) = \infty$$

By continuity theorem

$$P\Big(\bigcup_{n=1}^{\infty} A_n) = P\Big(\bigcup_{n=1}^{\infty}\bigcup_{k=1}^{n} A_k\Big) = \lim_{n\to\infty} P\Big(\bigcup_{k=1}^{n} A_k\Big) \geq 1 - \lim_{n\to\infty} \exp\Big\{-\sum_{k=1}^{n} P(A_k)\Big\} = 1$$

13. The problem is to prove

$$\bigcap_{n=1}^{\infty}\bigcup_{m=n}^{\infty} A_m \neq \phi$$

All we need is to show

$$\lambda\Big(\bigcap_{n=1}^{\infty}\bigcup_{m=n}^{\infty} A_m\Big) \geq \eta > 0$$

Indeed, by continuity theorem

$$\lambda\Big(\bigcap_{n=1}^{\infty}\bigcup_{m=n}^{\infty} A_m\Big) = \lim_{n\to\infty} \lambda\Big(\bigcup_{m=n}^{\infty} A_m\Big)$$

So the conlusion follows from the relation

$$\lambda\Big(\bigcup_{m=n}^{\infty} A_m\Big) \geq \lambda(A_n) \geq \eta$$

14. Let A_k be the event that the father $\#k$ picks his own child ($k = 1, \cdots, n$). By inclusion-exclusion (Problem #10),

$$P\Big(\bigcup_{k=1}^{n} A_k\Big) = \sum_{k=1}^{n}(-1)^{k-1}\sum_{1\leq j_1<\cdots<j_k\leq n} P\Big(A_{j_1}\cap\cdots\cap A_{j_k}\Big)$$

The probability that the fathers $\#(j_1), \cdots \#(j_k)$ pick their own children is

$$P\Big(A_{j_1}\cap\cdots\cap A_{j_k}\Big) = \frac{(n-k)!}{n!}$$

$$\sum_{k=1}^{n}(-1)^{k-1}\sum_{1\leq j_1<\cdots<j_k\leq n} P\Big(A_{j_1}\cap\cdots\cap A_{j_k}\Big) = \frac{(n-k)!}{n!}\sum_{1\leq j_1<\cdots<j_k\leq n} 1$$
$$= \frac{(n-k)!}{n!}\binom{n}{k} = \frac{1}{k!}$$

Thus,

$$P\Big(\bigcup_{k=1}^{n} A_k\Big) = \sum_{k=1}^{n}(-1)^{k-1}\frac{1}{k!}$$