

Homework 2 (Chapter 2)

Exercises in Chapter 2

1.3. Need to show $\sigma(X) = \sigma(X^{-1}((-\infty, x])); \quad x \in \mathbf{R}$) The direction of

$$\sigma(X) \supset \sigma(X^{-1}((-\infty, x])); \quad x \in \mathbf{R})$$

is obvious. Need to show the opposite

$$\sigma(X) \subset \sigma(X^{-1}((-\infty, x])); \quad x \in \mathbf{R})$$

Define

$$\mathcal{E} = \left\{ B \subset \mathbf{R}; \quad X^{-1}(B) \in \sigma(X^{-1}((-\infty, x])); \quad x \in \mathbf{R}) \right\}$$

All we need is

$$\mathcal{R} \subset \mathcal{E}$$

Indeed,

$$\{(-\infty, x]; \quad x \in \mathbf{R}\} \subset \mathcal{E}$$

Next, one can prove that $\mathcal{E}$ is a σ -algebra on $\mathbf{R}$. Hence

$$\mathcal{R} = \sigma\{(-\infty, x]; \quad x \in \mathbf{R}\} \subset \mathcal{E}$$

5.2. All we need is

$$\lim_{n \rightarrow \infty} \sum_{k=1}^{n2^n} \frac{k-1}{2^n} P\left\{ \frac{k-1}{2^n} < X \leq \frac{k}{2^n} \right\} = \sup_{0 \leq Y \leq X} \left\{ EY; \quad Y \text{ is simple} \right\}$$

Set

$$X_n = \sum_{k=1}^{n2^n} \frac{k-1}{2^n} 1\left\{ \frac{k-1}{2^n} < X \leq \frac{k}{2^n} \right\}$$

By the fact that X_n is simple and $X_n \leq X$,

$$\sup_{0 \leq Y \leq X} \left\{ EY; \quad Y \text{ is simple} \right\} \geq EX_n = \sum_{k=1}^{n2^n} \frac{k-1}{2^n} P\left\{ \frac{k-1}{2^n} < X \leq \frac{k}{2^n} \right\}$$

for any $n \geq 1$. Hence

$$\sup_{0 \leq Y \leq X} \left\{ EY; \quad Y \text{ is simple} \right\} \geq \lim_{n \rightarrow \infty} \sum_{k=1}^{n2^n} \frac{k-1}{2^n} P\left\{ \frac{k-1}{2^n} < X \leq \frac{k}{2^n} \right\}$$

On the other hand, let Y be a simple random variable with $0 \leq Y \leq X$. All we need to prove is that (why is this enough?)

$$\lim_{n \rightarrow \infty} EX_n \geq EY \quad (1)$$

We now prove (1) in three alternative ways (in the following argument, EX is defined by Definition 4.2)

(1) By Fatou lemma

$$\lim_{n \rightarrow \infty} EX_n \geq EX \geq EY$$

where the second inequality follows from the relation $Y \leq X$.

(2) Notice that $X_n \uparrow X$. By monotonic convergence

$$\lim_{n \rightarrow \infty} EX_n = EX \geq EY$$

(3). If the left hand of (1) is infinite, then nothing need to be proved. Assume it is finite. Then $EX < \infty$. Notice that $0 \leq X_n \leq X$ or $|X_n| \leq X$. By dominant convergence

$$\lim_{n \rightarrow \infty} EX_n = EX \geq EY$$

Problems in 2.20

1. All we need is to show that $\{\omega \in \Omega; Z(\omega) \leq x\} \in \mathcal{F}$. Notice that

$$Z(\omega) = \begin{cases} X(\omega); & \omega \in A \\ Y(\omega) & \omega \notin A \end{cases}$$

Thus

$$\{\omega \in \Omega; Z(\omega) \leq x\} = \{X \leq x\} \cap A \cup \{Y \leq x\} \cap A^c \in \mathcal{F}$$

2. We first prove that

$$\lim_{n \rightarrow \infty} P\{|X| \geq n\} = 0$$

Indeed,

$$\bigcap_{n=1}^{\infty} \{|X| \geq n\} = \{|X| = \infty\}$$

Hence

$$P\left(\bigcap_{n=1}^{\infty} \{|X| \geq n\}\right) = P\{|X| = \infty\} = 0$$

By continuity theorem

$$\lim_{n \rightarrow \infty} P\{|X| \geq n\} = P\left(\bigcap_{n=1}^{\infty} \{|X| \geq n\}\right) = 0$$

Therefore, there is a integer $N \geq 1$ such that

$$P\{|X| \geq N\} < \epsilon$$

Set

$$X_\epsilon = X 1_{\{|X| < N\}}$$

We have $|X_\epsilon| \leq N$ and

$$P\{X \neq X_\epsilon\} \leq P\{|X| \geq N\} < \epsilon$$

4. (a) Since $F(x) - F(x-0) = P\{X = x\}$ for any $x \in \mathbf{R}$. $P\{X = x\} = 0$ whenever $x \notin J_f$. In addition

$$P\{x - h < X \leq x + h\} = F(x + h) - F(x - h).$$

Therefore, all we need is to show that for any decreasing sequence h_n with $h_n \rightarrow 0$ ($n \rightarrow \infty$)

$$\lim_{n \rightarrow \infty} P\{x - h_n < X \leq x + h_n\} = P\{X = x\} \quad \forall x \in \mathbf{R}$$

Indeed,

$$\bigcap_{n=1}^{\infty} \{x - h_n < X \leq x + h_n\} = \{X = x\}$$

and $\{x - h_n < X \leq x + h_n\}$ is a non-increasing sequence. By continuity theorem,

$$\lim_{n \rightarrow \infty} P\{x - h_n < X \leq x + h_n\} = P\left(\bigcap_{n=1}^{\infty} \{x - h_n < X \leq x + h_n\}\right) = P\{X = x\}$$

5. Write

$$\sum_{n=-\infty}^{\infty} P(n < X \leq n + m) = \sum_{n=-\infty}^0 P(n < X \leq n + m) + \sum_{n=1}^{\infty} P(n < X \leq n + m)$$

and

$$P(n < X \leq n + m) = E 1_{\{n < X \leq n + m\}}$$

By linearity of expectation

$$\sum_{n=1}^N P(n < X \leq n + m) = E\left(\sum_{n=1}^N 1_{\{n < X \leq n + m\}}\right)$$

Apply monotonic convergence to the non-decreasing sequence

$$S_N = \sum_{n=1}^N 1'_{\{n < X \leq n+m\}}$$

$$\sum_{n=1}^{\infty} P(n < X \leq n+m) = \lim_{N \rightarrow \infty} E\left(\sum_{n=1}^N 1_{\{n < X \leq n+m\}}\right) = E\left(\sum_{n=1}^{\infty} 1_{\{n < X \leq n+m\}}\right)$$

Similarly,

$$\sum_{n=-\infty}^0 P(n < X \leq n+m) = \lim_{N \rightarrow \infty} E\left(\sum_{n=-N}^0 1_{\{n < X \leq n+m\}}\right) = E\left(\sum_{n=-\infty}^0 1_{\{n < X \leq n+m\}}\right)$$

By linearity

$$\begin{aligned} \sum_{n=-\infty}^{\infty} P(n < X \leq n+m) &= E \sum_{n=-\infty}^0 1_{\{n < X \leq n+m\}} + E \sum_{n=1}^{\infty} 1_{\{n < X \leq n+m\}} \\ &= E \sum_{n=-\infty}^{\infty} 1_{\{n < X \leq n+m\}} \end{aligned}$$

Notice that

$$\{n < X \leq n+m\} = \{X-m \leq n < X\}$$

Therefore

$$\sum_{n=-\infty}^{\infty} 1_{\{n < X \leq n+m\}} = \sum_{n=X-m}^{X-1} 1 = m$$

Thus,

$$\sum_{n=-\infty}^{\infty} P(n < X \leq n+m) = m$$

11. Let X be the number of the needed trials and let X_k be the number of the trials between the $(k-1)$ -th new face and the k -th new face.

$$X = X_1 + \cdots + X_6$$

Therefore,

$$EX = EX_1 + \cdots + EX_6$$

Notice that $X_1 = 1$ and for each $2 \leq k \leq 6$

$$\begin{aligned} &\{X_k = j\} \\ &= \bigcap_{i=1}^{j-1} \{\text{the } i\text{-th roll after appearance of the } (k-1)\text{-th new face is not a new face}\} \\ &\quad \times \cap \{\text{the } k\text{-th roll after appearance of the } (k-1)\text{-th new face is a new face}\} \end{aligned}$$

Thus, by independence

$$\begin{aligned}
& P\{X_k = j\} \\
&= \prod_{i=1}^{j-1} P\{\text{the } i\text{-th roll after appearance of the } (k-1)\text{-th new face is not a new face}\} \\
&\times P\{\text{the } k\text{-th roll after appearance of the } (k-1)\text{-th new face is a new face}\} \\
&= \left(\frac{k-1}{6}\right)^{j-1} \left(\frac{6-(k-1)}{6}\right) \quad j = 1, 2, \dots
\end{aligned}$$

$$EX_k = \sum_{j=1}^{\infty} j P\{X_k = j\} = \left(\frac{6-(k-1)}{6}\right) \sum_{j=1}^{\infty} j \left(\frac{k-1}{6}\right)^{j-1} = \frac{6}{6-(k-1)}$$

Here the identity

$$(1-x) \sum_{j=1}^{\infty} j x^{j-1} = \frac{1}{1-x} \quad 0 \leq x < 1$$

is used. Finally,

$$EX = EX_1 + \dots + EX_6 = \sum_{k=1}^6 \frac{6}{6-(k-1)} = 6 \left(1 + \frac{1}{2} + \dots + \frac{1}{6}\right)$$

19. First,

$$\lim_{n \rightarrow \infty} n \frac{1}{X(\omega)} 1_{\{X(\omega) > n\}} = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{1}{n} \frac{1}{X(\omega)} 1_{\{X(\omega) > \frac{1}{n}\}} = 0$$

for every $\omega \in \Omega$. Indeed, the first claim follow from $1_{\{X > n\}} = 0$ as $n > X(\omega)$. As for the second claim, given ω , if $X(\omega) = 0$ then $1_{\{X(\omega) > \frac{1}{n}\}} = 0$ and the sequence is constantly zero. When $X(\omega) > 0$,

$$0 \leq \frac{1}{n} \frac{1}{X(\omega)} 1_{\{X(\omega) > \frac{1}{n}\}} \leq \frac{1}{n} \frac{1}{X(\omega)} \longrightarrow 0 \quad (n \rightarrow \infty)$$

Second, notice that

$$0 \leq n \frac{1}{X(\omega)} 1_{\{X(\omega) > n\}} \leq 1 \quad \text{and} \quad 0 \leq \frac{1}{n} \frac{1}{X(\omega)} 1_{\{X(\omega) > \frac{1}{n}\}} \leq 1$$

So the conclusion follows from the dominated convergence with dominating random variable $Y \equiv 1$.