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Brownian motion and parabolic Anderson model in a renormalized Poisson potential

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Abstract. A method known as renormalization is proposed for constructing some more physically realistic random potentials in a Poisson cloud. The Brownian motion in the renormalized random potential and related parabolic Anderson models are modeled. With the renormalization, for example, the models consistent to Newton's law of universal attraction can be rigorously constructed.

Résumé. Nous présentons une méthode de renormalisation pour construire certains modèles de potentiel aléatoire dans un nuage Poissonnien qui sont physiquement plus réalistes. Nous obtenons le mouvement brownien dans un potentiel aléatoire renormalisé et les modèles Anderson parabolique associés. Par exemple, avec la renormalisation, nous pouvons construire rigoureusement les modèles qui sont consistant avec la loi de la gravitation de Newton.

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1. Introduction

The model of Brownian motion in Poisson potential is used to describe the trajectory of a Brownian particle that survived being trapped by the obstacles randomly located in the space. More precisely, let $\omega(\mathrm{d}x)$ be a Poisson field in $\mathbb{R}^d$ with the d-dimensional Lebesgue measure as its intensity measure. In our model, the obstacles are a family of particles distributed in the space according to the Poisson law $\omega(\mathrm{d}x)$. An additional particle executes random movement in the space $\mathbb{R}^d$ whose trajectory B_s $(s \geq 0)$ is a d-dimensional Brownian motion. Throughout, $\omega(\mathrm{d}x)$ and B_s are independent and the notations "$\mathbb{E}$" and "$\mathbb{E}_0$" are used for the expectations with respect to the Poisson field and the Brownian motion,[3] respectively. The correspondent laws $\mathbb{P}_0$ and $\mathbb{P}$ are introduced analogously. Let $K(x) \geq 0$ be a properly chosen function (known as shape function) on $\mathbb{R}^d$ and let the random function (known as potential)

$$V(x) = \int_{\mathbb{R}^d} K(y-x)\omega(\mathrm{d}y) \tag{1.1}$$

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[3]The notation "$\mathbb{E}_0$" reflects the usual convention that the Brownian motion starts at 0. In the case $B_0 = x$, we use "$\mathbb{E}_x$" instead.

represents the total trapping energy at $x \in \mathbb{R}^d$ generated by the Poisson obstacles. The model of Brownian motion in Poisson potential is considered in two different settings. In the quenched setting, where the set-up is conditioned on the random environment created by the Poisson obstacles, it is formulated in terms of the Gibbs measure

$$\frac{d\mu_{t,\omega}}{d\mathbb{P}_0} = \frac{1}{Z_{t,\omega}} \exp\left\{-\theta \int_0^t V(B_s)\, ds\right\} \tag{1.2}$$

defined on the space $C\{[0,t];\mathbb{R}^d\}$ of the continuous functions $f : [0,t] \longrightarrow \mathbb{R}^d$.

In the annealed setting, where the model averages on both the Brownian motion and the environment, the Gibbs measure is given as

$$\frac{d\mu_t}{d(\mathbb{P}_0 \otimes \mathbb{P})} = \frac{1}{Z_t} \exp\left\{-\theta \int_0^t V(B_s)\, ds\right\}. \tag{1.3}$$

Clearly, to make $\mu_{t,\omega}$ and $\mu_{t,\omega}$ probability measures, it has to be that

$$Z_{t,\omega} = \mathbb{E}_0 \exp\left\{-\int_0^t V(B_s)\, ds\right\} \quad \text{and} \quad Z_t = \mathbb{E}_0 \otimes \mathbb{E} \exp\left\{-\int_0^t V(B_s)\, ds\right\}. \tag{1.4}$$

In this model, the integral

$$\int_0^t V(B_s)\, ds$$

represents the total action from the Poisson obstacles and over the Brownian particle up to the time t. Under the law $\mu_{t,\omega}$ or μ_t, therefore, the Brownian paths heavily impacted by the Poisson obstacles are penalized and become less likely. The interested reader is referred to Chapter 7, [22] for the physical background on the study of the kinetics of trapping processes.

Another mathematically related problem is known as the parabolic Anderson model formulated in terms of the initial value problem

$$\begin{cases} \partial_t u(t,x) = \kappa \Delta u(t,x) \pm V(x) u(t,x), \\ u(0,x) = 1, \quad x \in \mathbb{R}^d. \end{cases} \tag{1.5}$$

Among other things, this model describes the evolution of mass density $u(t,x)$ distributed in the space $\mathbb{R}^d$. The system is driven by two activities: The mass diffuses (with the coefficient $\kappa > 0$) from region of high concentration to region of low concentration according to Fick's law which claims that the flux is in the direction $-\nabla u(t,x)$. At every site $x \in \mathbb{R}^d$, an amount of mass proportional to $u(t,x)$ is created or absorbed with the coefficient $\pm V(x)$. The initial configuration is an uniform distribution (i.e., $u(0,x) = 1$).

To derive (1.5), let $D \subset \mathbb{R}^d$ be a bounded domain with sufficiently smooth boundary ∂D. The change rate of the total population in D is given as

$$\frac{d}{dt} \int_D u(t,x)\, dx = \kappa \int_{\partial D} \big(\nabla u(t,x) \cdot \mathbf{n}\big)\, dS \pm \int_D V(x) u(t,x)\, dx,$$

where $\mathbf{n} = \mathbf{n}(x)$ is the unit vector normal to the surface ∂D at $x \in \partial D$ in the out-bond direction, the surface integral on the right-hand side represents the migration flux crossing the boundary ∂D at the time t, while the another integral symbols the mass loss (gain) in the region D due to absorption (creation). By divergence theorem,

$$\int_{\partial D} \big(\nabla u(t,x) \cdot \mathbf{n}\big)\, dS = \int_D \operatorname{div}\big(\nabla u(t,x)\big)\, dx = \int_D \Delta u(t,x)\, dx.$$

Consequently,

$$\int_D \partial_t u(t,x)\, dx = \kappa \int_D \Delta u(x,t)\, dx + \int_D V(x) u(x,t)\, dx.$$

Since D is arbitrary, $u(t,x)$ satisfies (1.5) under some smooth assumptions on $V(x)$.

The mathematical relevance to the topic in this paper comes from Feynman–Kac representation

$$\begin{aligned} u(t,x) &= \mathbb{E}_x \exp\left\{\pm \int_0^t V(B_{2\kappa s})\,\mathrm{d}s\right\} \\ &= \mathbb{E}_x \exp\left\{\pm (2\kappa)^{-1} \int_0^{2\kappa t} V(B_s)\,\mathrm{d}s\right\}, \quad x \in \mathbb{R}^d, \end{aligned} \tag{1.6}$$

which holds when the potential $V(x)$ is Hölder continuous and bounded.

Brownian motions in Poisson potentials have been extensively investigated in recent two decades and the main achievements (up to the year 1998) are collected in Sznitman's book [31]. See also [23] for a survey and [2,21,27,32] for specific topics or for recent development on this subject. The existing results on the parabolic Anderson models are scattered in a long list of the papers. Here we cite [3,5,6,11,12,15,17–20,30] as a partial list of the publications on this subject.

Sznitman [31] considers two kinds of shape functions. In one case

$$K(x) = \begin{cases} \infty, & x \in C, \\ 0, & \text{otherwise}, \end{cases}$$

for a non-polar set $C \subset \mathbb{R}^d$ while in another case, the shape function $K(x)$ is assumed to be bounded and compactly supported. The corresponding potential functions are called hard and soft obstacles, respectively. In the case of hard obstacles, the Brownian particle is completely free from the influence of the obstacles until hitting the C-neighborhood of the Poisson cloud which serves as the death trap. In the setting of the soft obstacles, only the obstacles in a local neighborhood of the Brownian particle act on the particle and the collision does not create extreme impact. In the parabolic Anderson model, a localized shape is analogous to the usual set-up in discrete Anderson model, where the potential $\{V(x); x \in \mathbb{Z}^d\}$ is an i.i.d. sequence.

The major goal of this paper is to make the models more realistic by considering unbounded and non-local shape functions. According to Newton's law of universal attraction, for example, the integrals

$$\int_{\mathbb{R}^d} \frac{1}{|y-x|^{d-1}}\omega(\mathrm{d}y) \quad \text{and} \quad \int_{\mathbb{R}^d} \frac{1}{|y-x|^{d-2}}\omega(\mathrm{d}y), \quad x \in \mathbb{R}^d,$$

represent (up to constant multiples), respectively, the total gravitational force and the total gravitational potential at the location x in the gravitational field generated by the Poisson obstacles in the case when $d \geq 3$.

Thus, it makes physical sense to consider the shape function of the form

$$K(x) = \theta|x|^{-p}, \quad x \in \mathbb{R}^d. \tag{1.7}$$

For the shape function (1.7), truncated near the origin and henceforth bounded, the large t behavior of the annealed exponential moment Z_t (given in (1.4)) has been studied by Donsker and Varadhan in the case $p > d+2$ (Theorem 2, [14]), and by Pastur in the case $d < p < d+2$ [25]. After the first draft of our paper was completed, we learned the recent investigation by Fukushima [16] on the second order asymptotics of annealed and quenched exponential moments Z_t and $Z_{t,\omega}$ in the setting of $d < p < d+2$.

According to Proposition 2.1 below, however, one concludes that the integral

$$\int_{\mathbb{R}^d} \frac{1}{|y-x|^p}\omega(\mathrm{d}y)$$

blows up at every $x \in \mathbb{R}^d$ as $p \leq d$. To resolve such discrepancy between mathematics and physics, we propose a procedure known as renormalization. To explain our point, we go back to the case where $K(x)$ is bounded and compactly supported. The key observation is that

$$\mathbb{E}V(x) = \mathbb{E}\int_{\mathbb{R}^d} K(y-x)\omega(\mathrm{d}y) = \int_{\mathbb{R}^d} K(y-x)\,\mathrm{d}y = \int_{\mathbb{R}^d} K(y)\,\mathrm{d}y = \text{constant}.$$

Consequently, replacing $V(x)$ by $\overline{V}(x) = V(x) - \mathbb{E}V(x)$ does not change the Gibbs measures defined in (1.2) and (1.3) if the normalizers $Z_{t,\omega}$ and Z_t are respectively replaced by[4]

$$\overline{Z}_{t,\omega} = \mathbb{E}_0 \exp\left\{-\int_0^t \overline{V}(B_s)\,\mathrm{d}s\right\} \quad \text{and} \quad \overline{Z}_t = \mathbb{E}_0 \otimes \mathbb{E} \exp\left\{-\int_0^t \overline{V}(B_s)\,\mathrm{d}s\right\}. \tag{1.8}$$

It appears to be possible for a unbounded and non-local shape function $K(x)$, such as the one defined in (1.7), that the potential $V(x)$ blows up but the renormalized potential $\overline{V}(x)$ formally written (for the time being) as

$$\overline{V}(x) = \int_{\mathbb{R}^d} K(y-x)\big[\omega(\mathrm{d}y) - \mathrm{d}y\big], \quad x \in \mathbb{R}^d, \tag{1.9}$$

is properly defined with the integrability which makes $\overline{Z}_{t,\omega}$ and $\overline{Z}_t$ finite. In this case, we define the Gibbs measures

$$\frac{\mathrm{d}\overline{\mu}_{t,\omega}}{\mathrm{d}\mathbb{P}_0} = \frac{1}{\overline{Z}_{t,\omega}} \exp\left\{-\int_0^t \overline{V}(B_s)\,\mathrm{d}s\right\} \tag{1.10}$$

and

$$\frac{\mathrm{d}\overline{\mu}_t}{\mathrm{d}(\mathbb{P}_0 \otimes \mathbb{P})} = \frac{1}{\overline{Z}_t} \exp\left\{-\int_0^t \overline{V}(B_s)\,\mathrm{d}s\right\} \tag{1.11}$$

for the replacements of $\mu_{t,\omega}$ and μ_t given in (1.2) and (1.3), respectively. The random trajectories under the laws $\overline{\mu}_{t,\omega}$ and $\overline{\mu}_t$ are called the Brownian motions of the renormalized Poisson potentials $\overline{V}$ (in quenched and annealed settings, respectively).

Renormalization also makes sense for Anderson model. First, there are real needs to consider the case where the environment (represented by $V(x)$ in (1.16)) has a long range dependency. In this regard, the non-local shape $K(x)$ of the form given in (1.7) is natural and physically sound. Second, the renormalization combines the birth and death factors in a single model. In particular, the equation

$$\begin{cases} \partial_t u(t,x) = \kappa \Delta u(t,x) - \overline{V}(x) u(t,x), \\ u(0,x) = 1 \end{cases} \tag{1.12}$$

models the problem where the death (absorption) depends on, while the birth is independent of the environment. Alternatively, the equation

$$\begin{cases} \partial_t u(t,x) = \kappa \Delta u(t,x) + \overline{V}(x) u(t,x), \\ u(0,x) = 1 \end{cases} \tag{1.13}$$

reflects a reality that the birth (creation) depends on, while the death is independent of the environment.

The following are the main results of this paper.

Theorem 1.1. *Let the shape function $K(x) \geq 0$ satisfy*

$$\int_{\mathbb{R}^d} \big(\mathrm{e}^{-K(x)} - 1 + K(x)\big)\,\mathrm{d}x < \infty. \tag{1.14}$$

Then the random field $\overline{V}(x)$ in (1.9) *can be properly defined* (*Proposition* 2.8). *Further, for any $t > 0$,*

$$\mathbb{E}_0 \otimes \mathbb{E} \exp\left\{-\int_0^t \overline{V}(B_s)\,\mathrm{d}s\right\} < \infty. \tag{1.15}$$

Consequently, the Gibbs measures $\overline{\mu}_{t,\omega}$ in (1.10) *and $\overline{\mu}_t$ in* (1.11) *are well defined.*

[4]This is where the word "renormalization" comes from.

The parabolic Anderson models are not always solvable in the classic sense (see [20] for the discussion on the existence and uniqueness of the solutions to the parabolic Anderson models). Given the fact (Proposition 2.9) that with positive probability $\overline{V}(x)$ is unbounded and therefore discontinuous in any neighborhood if $K(x)$ is unbounded at somewhere, it is unlikely in general for the Eqs (1.12) and (1.13) to have path-wise solutions. For this reason we look for solutions in some weaker sense.

Recall (e.g. [26], p. 184, Definition 1.1) that a function $v(t,x)$ ($(t,x)\in\mathbb{R}^+\times\mathbb{R}^d$) is a *mild solution* to the initial value problem

$$\begin{cases}\partial_t v(t,x)=\kappa\Delta v(t,x)+\xi(x)v(t,x),\\ v(0,x)=v_0(x), \quad x\in\mathbb{R}^d,\end{cases} \tag{1.16}$$

if

$$\int_0^t\int_{\mathbb{R}^d} p_{2\kappa(t-s)}(x-y)\big|\xi(y)v(s,y)\big|\,\mathrm{d}y\,\mathrm{d}s<+\infty, \quad x\in\mathbb{R}^d, t>0,$$

and

$$v(t,x)=v_0(x)+\int_0^t\int_{\mathbb{R}^d} p_{2\kappa(t-s)}(x-y)\xi(y)v(s,y)\,\mathrm{d}y\,\mathrm{d}s, \quad x\in\mathbb{R}^d, t>0,$$

where $p_t(x)=(2\pi t)^{-d/2}\exp(-|x|^2/2)$ is the transition probability density of B_t.

In connection to the Anderson problems (1.12) and (1.13), we write

$$\bar{u}_\pm(t,x)=\mathbb{E}_x\exp\left\{\pm(2\kappa)^{-1}\int_0^{2\kappa t}\overline{V}(B_s)\,\mathrm{d}s\right\}, \quad x\in\mathbb{R}^d. \tag{1.17}$$

Theorem 1.1 leads to the conclusion that $\mathbb{E}\bar{u}_-(t,0)<\infty$. Further, by the homogeneity of the random field $\overline{V}(x)$ (see (2.23)), for any $x\in\mathbb{R}^d$ one has

$$\big\{\bar{u}_-(t,x);t\ge 0\big\}\stackrel{d}{=}\big\{\bar{u}_-(t,0);t\ge 0\big\}. \tag{1.18}$$

In particular, $\mathbb{E}\bar{u}_-(t,x)=\mathbb{E}\bar{u}_-(t,0)<\infty$. Consequently, $\bar{u}_-(t,x)<\infty$ a.s. for each $x\in\mathbb{R}^d$ under (1.14).

Proposition 1.2. *Under the assumption* (1.14), *the random field* $\bar{u}_-$ *in* (1.17) *is a mild solution to the Anderson problem* (1.12).

The integrability requested by the Anderson model formulated in (1.13) appears to be much more delicate and we investigate it in the special case where $K(x)$ is given in (1.7). An immediate observation is that, for $K(x)=\theta|x|^{-p}$ with $\theta>0$, condition (1.14) is equivalent to $d/2<p<d$.

Corollary 1.3. *Let* $K(x)$ *be given in* (1.7). *Under* $d/2<p<d$, *all conclusions stated in Theorem* 1.1 *and Proposition* 1.2 *hold.*

As for the integrability associated to the Anderson model formulated in (1.13), we first bring an unfortunate news.

Theorem 1.4. *Let* $K(x)$ *be given in* (1.7). *Under* $d/2<p<d$,

$$\mathbb{E}_0\otimes\mathbb{E}\exp\left\{\int_0^t\overline{V}(B_s)\,\mathrm{d}s\right\}=\infty \tag{1.19}$$

for every $\theta>0$ (*hiding in* (1.7)) *and* $t>0$.

Theorem 1.4 reflects the fact that it is the unboundedness of the shape function $K(x) = \theta|x|^{-p}$ at $x = 0$ that destroys the integrability as far as the positive tail is concerned. On the other hand, such unboundedness does not affect the integrability stated in (1.15).

Bad as it is, (1.19) does not say anything conclusive about the correspondent "quenched" integrability, which is the one needed for solving the Anderson model formulated in (1.13). The next theorem shows a phase transition associated with the value of p.

Theorem 1.5. *Let $K(x)$ be given in* (1.7) *with $d/2 < p < d$. For every $\theta > 0$ (hiding in* (1.7)*) and $t > 0$*

$$\mathbb{E}_0 \exp\left\{\int_0^t \overline{V}(B_s)\,\mathrm{d}s\right\} \begin{cases} < \infty & \text{a.s. when } p < 2, \\ = \infty & \text{a.s. when } p > 2. \end{cases} \tag{1.20}$$

Under $p = 2$ (necessarily, the constraint "$d/2 < 2 < d$" leads to $d = 3$), there is a $\theta_0 > 0$ such that when $\theta > \theta_0$

$$\mathbb{E}_0 \exp\left\{\int_0^t \overline{V}(B_s)\,\mathrm{d}s\right\} = \infty \quad a.s. \tag{1.21}$$

for any $t > 0$.

The story in the critical case $p = 2$ and $d = 3$ was recently finalized in [10], where it is shown that for any $t > 0$

$$\mathbb{E}_0 \exp\left\{\int_0^t \overline{V}(B_s)\,\mathrm{d}s\right\} \begin{cases} < \infty & \text{a.s. when } \theta < \frac{1}{16}, \\ = \infty & \text{a.s. when } \theta > \frac{1}{16}. \end{cases} \tag{1.22}$$

Let $\bar{u}_+(t,x)$ be defined in (1.17) with specified shape function $K(\cdot)$ given in (1.7). By Theorem 1.5 and by the fact that $\bar{u}_+(t,x) \stackrel{d}{=} \bar{u}_+(t,0)$, we have $\bar{u}_+(t,x) < \infty$ a.s. for each $x \in \mathbb{R}^d$ when $d/2 < p < \min\{d, 2\}$.

Proposition 1.6. *Let $K(x)$ be given in* (1.7) *and $d/2 < p < \min\{2, d\}$. Then the random field $\bar{u}_+$ is a mild solution to the Anderson problem* (1.13).

We now outline the rest of the paper. Section 2 is devoted to the construction of Poisson integrals and the random field $\overline{V}(x)$. The Poisson integration was defined and the necessary-sufficient conditions for integrability were obtained in the paper by Rajput and Rosinski [28]. Considering the fact that the theory of Poisson integration is relatively less well known compared to Gaussian integration, and limited by our knowledge on the literature of Poisson integration, we list some properties and identities which are possibly known to the people in the area.[5] Further, we make use a result by Rosinski [29] in Proposition 2.9 showing that with unboundedness of $K(x)$, the sample path of random potential $\overline{V}(x)$ is unbounded (and therefore discontinuous) in any neighborhood with positive probability. These facts are crucially needed for the establishment of our main results. The difficulty we often encounter is the loss of monotonicity of the Poisson integrals due to renormalization.

In Section 3, we prove Theorem 1.1 and establish the Fubini-type identity

$$\int_0^t \overline{V}(B_s)\,\mathrm{d}s = \int_{\mathbb{R}^d} \left[\int_0^t K(B_s - x)\,\mathrm{d}s\right]\left[\omega(\mathrm{d}x) - \mathrm{d}x\right] \quad \text{a.s.} \tag{1.23}$$

The proof of Theorem 1.1 depends on a simple application of Jensen inequality. On the other hand, it is not obvious that the time integral on the left-hand side of (1.23) is well defined given the discontinuity of $\overline{V}(x)$ stated in Proposition 2.9. In addition, some justification of the identity (1.23) is needed.

In Section 4, we consider the special case when $K(\cdot) = \theta|\cdot|^{-p}$ and prove Theorem 1.4 and Theorem 1.5. Given the annealed un-integrability stated in Theorem 1.4, the proof of Theorem 1.5 is highly non-trivial and the strategy developed here has potential for future study in this area. To show the quenched un-integrability in $p \geq 2$, the crucial

[5] A reader with certain understanding of Poisson integration may skip Section 2.

task is to choose the Brownian trajectories which generates most of "energy." To this end, we use small ball estimate for Brownian motions combined with some fine geometrical construction to localize the Poisson potential to a neighborhood of its pole. The establishment of the quenched integrability appears as combination of the large deviation [1] for

$$\sup_{x\in\mathbb{R}^d}\int_0^t \frac{\mathrm{d}s}{|x-B_s|^p}$$

and some techniques on the estimation of the principal eigenvalues developed in [18] and [19]. The proof of Theorem 1.5 is informative in the sense that it shows how much unboundedness from $K(x)$ that our models can tolerate.

In Section 5, we prove Proposition 1.2 and Proposition 1.6. Our approach relies on a formula derived from fundamental calculus theorem and the Markov property of the Brownian motion.

Finally, in Section 6 we introduce the construction of a partially normalized Poisson potential, which contains the Poisson potentials $V(x)$, $\overline{V}(x)$ as particular cases and unifies the definition of the respective Gibbs measures.

The set-up of the models (1.10), (1.11), (1.12) and (1.13) raises further problems on the Brownian motions in the renormalized Poisson potentials and the parabolic Anderson models in the setting of unbounded and non-local (heavy tail) shape functions. The case when $K(x)$ is given in (1.7) deserves some special attention due to its association with Newton's law of attraction. An important subject is the long term behavior of the system. To this topic we mention the follow-up works [8,9] and [10] for some very recent progress on the large-t asymptotics of the quenched and annealed exponential moments

$$\mathbb{E}_0\exp\left\{\pm\int_0^t\overline{V}(B_s)\,\mathrm{d}s\right\}\quad\text{and}\quad\mathbb{E}_0\otimes\mathbb{E}\exp\left\{\pm\int_0^t\overline{V}(B_s)\,\mathrm{d}s\right\}.$$

Among others, here are some interesting observations made in these papers:

1. The main theorems in [8,9] and [10] indicates some drastic changes in dynamics caused by the non-locality and unboundedness of the shape function. In contrary to the shape insensitivity in the setting of local and bounded shapes, for example, the new results present a strong shape-dependence.
2. The systems of the potential $-\overline{V}(x)$ and of $\overline{V}(x)$ correspond to the shape $K(\cdot)=\theta|\cdot|^{-p}$ in different ways. It is the non-locality of the shape that plays the major role in the system with the potential $-\overline{V}(x)$; while the singularity of $K(\cdot)$ at 0 is the one responsible for the dynamical change in the system of the potential $\overline{V}(x)$ (see also the comment next to Theorem 1.4).
3. In connection to Theorem 1.5 and (1.22), the usual deterministic tendency of the log-exp moment

   $$\log\mathbb{E}_0\exp\left\{\int_0^t\overline{V}(B_s)\,\mathrm{d}s\right\}\quad(t\to\infty)$$

   gives way to a highly uncharacteristic behavior in the critical case $d=3$ and $p=2$ [10]. The driving force behind this phase transition is the celebrated Hardy's inequality

   $$\int_{\mathbb{R}^3}\frac{f^2(x)}{|x|^2}\,\mathrm{d}x\le 4\|\nabla f\|_2^2,\quad f\in W^{1,2}(\mathbb{R}^3).$$

   These observed differences, together with the physical importance, make it worthwhile to exam all major aspects of the models of Brownian motion in renormalized Poisson potentials with the shape function given in (1.7). We leave this topic to future investigation.

2. Poisson integrals

In the remaining of the paper, we denote

$$\varphi(\theta)=1-\mathrm{e}^{-\theta},\qquad\Phi(\theta)=\mathrm{e}^{\theta}-1,\qquad\psi(\theta)=\mathrm{e}^{-\theta}-1+\theta,\qquad\Psi(\theta)=\mathrm{e}^{\theta}-1-\theta,\quad\theta\in[0,\infty).$$

It is easy to see that all functions defined above are non-negative, increasing, and vanishing at 0. In addition, Φ and Ψ are Young functions on $[0, \infty)$ for their convexity.

We start this section by a brief review of some existing results on the random integration; for a detailed exposition see Rajput and Rosinski [28], Sections 1 and 2. Let Λ be an (independently scattered) random measure defined on a δ-ring $\mathcal{R}$ (i.e. a ring closed w.r.t. countable intersections) which generates the Borel σ-algebra in $\mathbb{R}^d$. The random measure Λ is said to be infinitely divisible (**ID**) if for any $A \in \mathcal{R}$ the random variable $\Lambda(A)$ is infinitely divisible.

The characteristic function for the values of an **ID** random measure $\Lambda(A)$ has the representation (Rajput and Rosinski [28], Proposition 2.4)

$$\mathbb{E}e^{it\Lambda(A)} = \exp\left\{\int_A K(t,x)\lambda(dx)\right\}, \quad t \in \mathbb{R},\ A \in \mathcal{R}, \tag{2.1}$$

with

$$K(t,x) = ita(x) - \frac{1}{2}t^2\sigma^2(x) + \int_{\mathbb{R}} \left[e^{ity} - 1 - it\tau(y)\right]\rho(x, dy), \quad \tau(y) := \min\{1, y^2\}.$$

The measure $\lambda(dx)$, functions $a(x), \sigma^2(x)$, and the kernel $\rho(x, dy)$ are called *the deterministic characteristics* of the **ID** random measure Λ.

Given an infinite divisible random measure $\Lambda(dx)$ on $\mathbb{R}^d$ and a Borel set $A \subset \mathbb{R}^d$, the random integral

$$\int_A f(x)\Lambda(dx)$$

is constructed as following. For a simple function

$$f(x) = \sum_{k=1}^n c_k 1_{E_k},$$

where $E_1, \dots, E_n \subset \mathbb{R}^d$ are disjoint and $A \cap E_k \in \mathcal{R}$ for $1 \le k \le n$, the integral is defined as

$$\int_A f(x)\Lambda(dx) = \sum_{k=1}^n c_k \Lambda(A \cap E_k).$$

In general, a Borel measurable function $f(x)$ on A is said to be Λ-integrable if there is a sequence $f_n(x)$ of simple functions on A given as above such that

(1) $f_n \longrightarrow f$ λ-a.e. on A;
(2) for any Borel set $B \subset A$, the sequence

$$\int_B f_n(x)\Lambda(dx), \quad n = 1, 2, \dots,$$

converges in probability.

For a Λ-integrable function f on A, the Λ-integral is defined as

$$\int_A f(x)\Lambda(dx) = \mathbb{P} - \lim_{n\to\infty} \int_A f_n(x)\Lambda(dx).$$

The definition of the above random integral does not depend on the choice of the simple sequence f_n. Clearly, the integrability on a set A implies the integrability on any Borel subset of A, with the integral of f over $B \subset A$ being equal to the integral of $f 1_B$. By standard argument, it can be shown that the usual properties in the general integration theory, such as linearity and additivity, remain true in our setting.

The necessary and sufficient condition for the integrability of f over Λ in the terms of the deterministic characteristics of Λ has been obtained in Rajput and Rosinski [28]. We apply their result in the following two different settings.

1. $\Lambda(\mathrm{d}x) = \omega(\mathrm{d}x)$, which is an **ID** random measure with

$$\lambda(\mathrm{d}x) = \mathrm{d}x, \qquad a(x) = 1, \qquad \sigma(x) = 0, \qquad \rho(x, \mathrm{d}y) = \delta_1(\mathrm{d}y).$$

2. $\Lambda(\mathrm{d}x) = \omega(\mathrm{d}x) - \mathrm{d}x$, which is an **ID** random measure with

$$\lambda(\mathrm{d}x) = \mathrm{d}x, \qquad a(x) = 0, \qquad \sigma(x) = 0, \qquad \rho(x, \mathrm{d}y) = \delta_1(\mathrm{d}y).$$

By Theorem 2.7 in Rajput and Rosinski [28], we have

Proposition 2.1. *A Borel-measurable function $f(x)$ is integrable on $\mathbb{R}^d$ with respect to $\omega(\mathrm{d}x)$ if and only if*

$$\int_{\mathbb{R}^d} \varphi\bigl(\bigl|f(x)\bigr|\bigr)\,\mathrm{d}x < \infty. \tag{2.2}$$

A Borel-measurable function $f(x)$ is integrable on $\mathbb{R}^d$ with respect to $\omega(\mathrm{d}x) - \mathrm{d}x$ if and only if

$$\int_{\mathbb{R}^d} \psi\bigl(\bigl|f(x)\bigr|\bigr)\,\mathrm{d}x < \infty. \tag{2.3}$$

Proof. Clearly (2.2) and (2.3) are equivalent to

$$\int_{\mathbb{R}^d} \min\bigl\{1, \bigl|f(x)\bigr|\bigr\}\,\mathrm{d}x < \infty \tag{2.4}$$

and

$$\int_{\mathbb{R}^d} \bigl(\bigl|f(x)\bigr| - 1\bigr)_+\,\mathrm{d}x < +\infty, \qquad \int_{\mathbb{R}^d} \min\bigl\{1, f^2(x)\bigr\}\,\mathrm{d}x < +\infty, \tag{2.5}$$

respectively. Theorem 2.7 in Rajput and Rosinski [28] characterizes the integrability by a group of conditions listed as (i), (ii) and (iii). The condition (ii) is automatic in our setting as $\sigma(x) = 0$. In the case of $\Lambda(\mathrm{d}x) = \omega(\mathrm{d}x)$, the condition (i) becomes (2.4) and (iii) becomes the second part in (2.5) (which is obviously weaker than (2.4) and therefore omitted). In the case of $\Lambda(\mathrm{d}x) = \omega(\mathrm{d}x) - \mathrm{d}x$, the conditions (i) and (iii) correspond the first and second parts of (2.5), respectively. □

The following formulas for the characteristic functions of the stochastic integrals over $\omega(\mathrm{d}x)$ and $\omega(\mathrm{d}x) - \mathrm{d}x$ come from the part (iv) of Theorem 2.7 in Rajput and Rosinski [28] straightforwardly.

Proposition 2.2. *Under the condition* (2.2),

$$\mathbb{E}\exp\left\{\mathrm{i}t\int_{\mathbb{R}^d} f(x)\omega(\mathrm{d}x)\right\} = \exp\left\{\int_{\mathbb{R}^d} \Phi\bigl(\mathrm{i}tf(x)\bigr)\,\mathrm{d}x\right\}, \quad t \in \mathbb{R}. \tag{2.6}$$

Under the condition (2.3),

$$\mathbb{E}\exp\left\{\mathrm{i}t\int_{\mathbb{R}^d} f(x)\bigl[\omega(\mathrm{d}x) - \mathrm{d}x\bigr]\right\} = \exp\left\{\int_{\mathbb{R}^d} \Psi\bigl(\mathrm{i}tf(x)\bigr)\,\mathrm{d}x\right\}, \quad t \in \mathbb{R}. \tag{2.7}$$

Like in the general integration theory, the Poisson integration can be continuous over the integrand f under proper topology. Below we give an analogue of (the sufficiency part of) the Vitali convergence theorem. Recall that a sequence $\{f_n\}_{n\ge 1}$ is said to be uniformly integrable w.r.t. a σ-finite measure λ on $\mathbb{R}^d$ if

$$\lim_{c\to+\infty} \sup_n \int_{|f_n|>c} |f_n|\,\mathrm{d}\lambda = 0, \qquad \inf_{A:\lambda(A)<\infty} \sup_n \int_{\mathbb{R}^d\setminus A} |f_n|\,\mathrm{d}\lambda = 0.$$

Proposition 2.3. *Assume* $\{f_n\}_{n\ge 1}$ *converge to* f *in measure* $\lambda(\mathrm{d}x)=\mathrm{d}x$. *If the sequence* $\{\varphi(|f_n|)\}_{n\ge 1}$ *is uniformly integrable w.r.t.* λ *then*

$$\int_{\mathbb{R}^d} f_n(x)\omega(\mathrm{d}x) \xrightarrow{\mathbb{P}} \int_{\mathbb{R}^d} f(x)\omega(\mathrm{d}x). \tag{2.8}$$

If the sequence $\{\psi(|f_n|)\}_{n\ge 1}$ *is uniformly integrable w.r.t.* λ *then*

$$\int_{\mathbb{R}^d} f_n(x)\bigl[\omega(\mathrm{d}x)-\mathrm{d}x\bigr] \xrightarrow{\mathbb{P}} \int_{\mathbb{R}^d} f(x)\bigl[\omega(\mathrm{d}x)-\mathrm{d}x\bigr]. \tag{2.9}$$

Proof. We only prove (2.8) as the proof of (2.9) is analogous. We will use (2.6). It can be shown straightforwardly that for every $t\in\mathbb{R}$ there exists $C_t>0$ such that

$$\bigl|\Phi(\mathrm{i}tz)\bigr| = \bigl|\mathrm{e}^{\mathrm{i}tz}-1\bigr| \le C_t\varphi\bigl(|z|/2\bigr), \quad z\in\mathbb{R}.$$

By monotonicity,

$$\varphi\bigl(\bigl|f_n(x)-f(x)\bigr|/2\bigr) \le \varphi\bigl(\max\bigl\{\bigl|f_n(x)\bigr|, f(x)\bigr\}\bigr) = \max\bigl\{\varphi\bigl(\bigl|f_n(x)\bigr|\bigr), \varphi\bigl(\bigl|f(x)\bigr|\bigr)\bigr\},$$

and consequently the sequence $\{\varphi(|f_n(x)-f(x)|/2)\}_{n\ge 1}$ is uniformly integrable. Then by the Vitali convergence theorem

$$\left|\int_{\mathbb{R}^d} \Phi\bigl(\mathrm{i}t\bigl(f_n(x)-f(x)\bigr)\bigr)\,\mathrm{d}x\right| \le C_t \int_{\mathbb{R}^d} \varphi\bigl(\bigl|f_n(x)-f(x)\bigr|/2\bigr)\,\mathrm{d}x \to 0, \quad n\to\infty.$$

It shows that the characteristic functions of Poisson integrals of $f-f_n$ are asymptotically degenerate, which implies (2.8).

The following analogues of the dominated convergence and monotone convergence theorems are available, as well. □

Proposition 2.4.

1. *Assume* $\{f_n\}_{n\ge 1}$ *converge to* f *in measure* $\lambda(\mathrm{d}x)=\mathrm{d}x$ *and* $|f_n|\le g, n\ge 1$ *with* $\int_{\mathbb{R}^d}\varphi(g(x))\,\mathrm{d}x<\infty$ (*resp.* $\int_{\mathbb{R}^d}\psi(g(x))\,\mathrm{d}x<\infty$). *Then* (2.8) (*resp.* (2.9)) *holds.*
2. *Let* $0\le f_n\le f_{n+1}, n\ge 1$, *and* $f=\lim_n f_n$ λ-*a.e. Assume every* $f_n, n\ge 1$, *be integrable w.r.t.* $\omega(\mathrm{d}x)$ (*resp.* $[\omega(\mathrm{d}x)-\mathrm{d}x]$). *Then, for the sequence of respective integrals to converge in probability it is necessary and sufficient that* f *satisfies* (2.2) (*resp.* (2.3)). *In that case*, (2.8) (*resp.* (2.9)) *holds true.*

Proof. Statement 1 and the sufficiency part of statement 2 follow immediately from Proposition 2.3 because a sequence dominated by an integrable function is uniformly integrable. For similarity, we only verify the necessity part for the integrals w.r.t. $\omega(\mathrm{d}x)$. Since every f_n is integrable w.r.t. $\omega(\mathrm{d}x)$, there exist non-negative simple functions $\tilde{f}_n\le f_n, n\ge 1$, such that

$$\lambda\bigl(\bigl\{x\colon f_n(x)-\tilde{f}_n(x)\ge n^{-1}\bigr\}\bigr)\le n^{-1}, \qquad \mathbb{P}\left(\int_{\mathbb{R}^d} f_n(x)\omega(\mathrm{d}x) - \int_{\mathbb{R}^d}\tilde{f}_n(x)\omega(\mathrm{d}x)\ge n^{-1}\right)\le n^{-1}.$$

Then $\tilde{f}_n\to f$ in measure λ and respective integrals converge in probability to the same limit with the integrals of $f_n, n\ge 1$. Therefore f is integrable w.r.t. $\omega(\mathrm{d}x)$ by the definition, and the integral coincides with the limit of $\int_{\mathbb{R}^d} f_n(x)\omega(\mathrm{d}x), n\ge 1$. □

Continuity property exposed in Propositions 2.3 and 2.4 is an efficient tool for the further analysis of the Poisson integrals.

Proposition 2.5.

1. *Any* $f \in \mathcal{L}(\mathbb{R}^d)$ *is integrable w.r.t. both* $\omega(\mathrm{d}x)$ *and* $\omega(\mathrm{d}x) - \mathrm{d}x$. *In addition,*

$$\int_{R^d} f(x)\big[\omega(\mathrm{d}x) - \mathrm{d}x\big] = \int_{\mathbb{R}^d} f(x)\omega(\mathrm{d}x) - \int_{R^d} f(x)\,\mathrm{d}x, \tag{2.10}$$

$$\mathbb{E}\biggl(\int_{\mathbb{R}^d} f(x)\omega(\mathrm{d}x)\biggr) = \int_{\mathbb{R}^d} f(x)\,\mathrm{d}x. \tag{2.11}$$

2. *Any* $f \in \mathcal{L}^2(\mathbb{R}^d)$ *is integrable w.r.t.* $\omega(\mathrm{d}x)$ *and* $\omega(\mathrm{d}x) - \mathrm{d}x$, *and*

$$\mathbb{E}\biggl(\int_{\mathbb{R}^d} f(x)\big[\omega(\mathrm{d}x) - \mathrm{d}x\big]\biggr) = 0, \qquad \mathbb{E}\biggl(\int_{\mathbb{R}^d} f(x)\big[\omega(\mathrm{d}x) - \mathrm{d}x\big]\biggr)^2 = \int_{\mathbb{R}^d} f^2(x)\,\mathrm{d}x. \tag{2.12}$$

Proof. Integrability issues follow from Proposition 2.1 and the inequalities

$$\varphi(\theta) \le \theta, \qquad \psi(\theta) \le \theta^2, \quad \theta \in [0,\infty).$$

For every simple f, relations (2.10)–(2.12) can be verified straightforwardly. Let f be general; we may assume that $f \ge 0$, for otherwise we consider f^+ and f^- instead. Consider a non-decreasing sequence of simple functions such that $0 \le f_n \le f$ and $f_n \to f$ a.e. In the case 1, Proposition 2.3 provides that the integrals of f_n both over $\omega(\mathrm{d}x)$ and $\omega(\mathrm{d}x) - \mathrm{d}x$ converge to respective integrals of f in probability. In addition, the sequences $\{f_n\}_{n\ge1}$ and $\{\int_{\mathbb{R}^d} f_n(x)\omega(\mathrm{d}x)\}_{n\ge1}$ are monotonous. Therefore (2.10), (2.11) are provided by the same relations for simple functions and the monotone convergence theorem.

In the case 2, the second identity in (2.12) shows that $\{\int_{\mathbb{R}^d} f_n(x)[\omega(\mathrm{d}x) - \mathrm{d}x]\}_{n\ge1}$ is a Cauchy sequence in mean square. On the other hand, by Proposition 2.3 the integrals of f_n with respect to $\omega(\mathrm{d}x) - \mathrm{d}x$ converge to respective integral of f in probability. Therefore, the convergence holds in the mean square, and (2.12) is passed from the simple functions f_n to the general f. □

We now comment on the range of application of the definitions given above. It is natural to use standard isometry argument to define either the integrals over $\omega(\mathrm{d}x)$ and $\omega(\mathrm{d}x) - \mathrm{d}x$ for $f \in \mathcal{L}(\mathbb{R}^d)$, or the integral over $\omega(\mathrm{d}x) - \mathrm{d}x$ for $f \in \mathcal{L}^2(\mathbb{R}^d)$. It should be noticed, however, that the class defined by (2.3) is genuinely larger than $\mathcal{L}(\mathbb{R}^d) \cup \mathcal{L}^2(\mathbb{R}^d)$. This fact is crucial for our considerations because the shape function $K(x)$ defined in (1.7) does not belong to $\mathcal{L}(\mathbb{R}^d) \cup \mathcal{L}^2(\mathbb{R}^d)$ for any p. On the other hand, $K(x)$ defined in (1.7) satisfies (2.3) for $d/2 < p < d$.

When $f \in \mathcal{L}(\mathbb{R}^d) \cup \mathcal{L}^2(\mathbb{R}^d)$, the integral with respect to $\omega(\mathrm{d}x) - \mathrm{d}x$ has expectation 0. The next proposition demonstrates that this property still holds under (most general) integrability condition (2.3).

Proposition 2.6. *Under the condition* (2.3),

$$\mathbb{E}\biggl|\int_{\mathbb{R}^d} f(x)\big[\omega(\mathrm{d}x) - \mathrm{d}x\big]\biggr| < \infty \quad \textit{and} \quad \mathbb{E}\biggl(\int_{\mathbb{R}^d} f(x)\big[\omega(\mathrm{d}x) - \mathrm{d}x\big]\biggr) = 0. \tag{2.13}$$

Proof. As in the previous proof, we may assume that $f \ge 0$. Take $h > 0$ and consider the decomposition

$$f(x) = \big(f(x) - h\big)^+ + \min\{h, f(x)\} = f^h(x) + f_h(x) \quad \text{(say)}.$$

Notice that the condition (2.3) implies that $f^h \in \mathcal{L}(\mathbb{R}^d)$ and $f_h \in \mathcal{L}^2(\mathbb{R}^d)$. By linearity,

$$\int_{\mathbb{R}^d} f(x)\big[\omega(\mathrm{d}x) - \mathrm{d}x\big] = \int_{\mathbb{R}^d} f^h(x)\big[\omega(\mathrm{d}x) - \mathrm{d}x\big] + \int_{\mathbb{R}^d} f_h(x)\big[\omega(\mathrm{d}x) - \mathrm{d}x\big].$$

Applying (2.10), (2.11) with $f = f^h$ and (2.12) with $f = f_h$ completes the proof. □

The next proposition describes the exponential moments of the Poisson integrals.

Proposition 2.7. *Under* (2.2) *and* (2.3) *respectively,*

$$\mathbb{E}\exp\left\{\int_{\mathbb{R}^d} f(x)\omega(\mathrm{d}x)\right\} = \exp\left\{\int_{\mathbb{R}^d} \Phi\big(f(x)\big)\,\mathrm{d}x\right\}, \tag{2.14}$$

$$\mathbb{E}\exp\left\{\int_{\mathbb{R}^d} f(x)\big[\omega(\mathrm{d}x) - \mathrm{d}x\big]\right\} = \exp\left\{\int_{\mathbb{R}^d} \Psi\big(f(x)\big)\,\mathrm{d}x\right\}. \tag{2.15}$$

These identities hold in the sense that one side is finite if another side is finite. In particular, for every $f \ge 0$ satisfying (2.2) *and* (2.3) *respectively,*

$$\mathbb{E}\exp\left\{-\int_{\mathbb{R}^d} f(x)\omega(\mathrm{d}x)\right\} = \exp\left\{\int_{\mathbb{R}^d} \varphi\big(f(x)\big)\,\mathrm{d}x\right\} < \infty, \tag{2.16}$$

$$\mathbb{E}\exp\left\{-\int_{\mathbb{R}^d} f(x)\big[\omega(\mathrm{d}x) - \mathrm{d}x\big]\right\} = \exp\left\{\int_{\mathbb{R}^d} \psi\big(f(x)\big)\,\mathrm{d}x\right\} < \infty. \tag{2.17}$$

Proof. Write $f = f^+ - f^-$ and decompose an integral of f into the sum of respective integrals of f^+, f^-. Since f^+, f^- are supported by disjoint sets, respective integrals are independent. Therefore, it is sufficient to consider separately the cases $f \ge 0$, $f \le 0$ or, in other words, to prove (2.14)–(2.17) under assumption $f \ge 0$.

It is easy to see that (2.14)–(2.17) holds when $f \ge 0$ is simple. In the non-simple case, let f_n be a non-decreasing sequence of non-negative simple functions such that $f_n \to f$ pointwise. We intend to take limits as $n \to \infty$ on the both sides of the Eqs (2.14)–(2.17) with f_n instead of f. Clearly, for the right-hand sides such limit pass is legible by the monotonic convergence theorem. We now consider separately the left-hand sides of (2.14)–(2.17).

Expressions under the expectation in the left-hand side of (2.14) are non-decreasing, hence the monotonic convergence theorem can be applied. Expressions under the expectation in the left-hand side of (2.16) are dominated by 1, hence the dominated convergence theorem can be applied. Furthermore, there exists a constant C such that

$$\psi(2\theta) \le C\psi(\theta), \quad \theta \in [0, \infty),$$

and consequently (2.17) with f_n instead of f provides

$$\begin{aligned}\sup_n \mathbb{E}\exp\left\{-2\int_{\mathbb{R}^d} f_n(x)\big[\omega(\mathrm{d}x) - \mathrm{d}x\big]\right\} &= \sup_n \exp\left\{\int_{\mathbb{R}^d} \psi\big(2f_n(x)\big)\,\mathrm{d}x\right\} \\ &\le \exp\left\{C\int_{\mathbb{R}^d} \psi\big(f(x)\big)\,\mathrm{d}x\right\} < \infty.\end{aligned}$$

Therefore, by the de la Vallée–Poussin theorem, expressions under the expectation in the left-hand side of (2.17) form a uniformly integrable sequence, and the Vitali convergence theorem can be applied.

Finally, the Poisson integral in the left-hand side of (2.15) can be expressed as

$$\int_{\mathbb{R}^d} (f_n)^h(x)\omega(\mathrm{d}x) - \int_{\mathbb{R}^d} (f_n)^h(x)\,\mathrm{d}x + \int_{\mathbb{R}^d} (f_n)_h(x)\big[\omega(\mathrm{d}x) - \mathrm{d}x\big], \tag{2.18}$$

see the proof of Proposition 2.6 for the notation f^h, f_h. The second term in (2.18) is deterministic, while two others are independent because $(f_n)^h$ and $(f_n)_h$ have disjoint supports. Hence it is enough to consider the limit behavior of these three terms separately. By the monotone convergence theorem,

$$\int_{\mathbb{R}^d} (f_n)^h(x)\,\mathrm{d}x \to \int_{\mathbb{R}^d} f^h(x)\,\mathrm{d}x,$$

$$\mathbb{E}\exp\left\{\int_{\mathbb{R}^d} (f_n)^h(x)\omega(\mathrm{d}x)\right\} \to \mathbb{E}\exp\left\{\int_{\mathbb{R}^d} f^h(x)\omega(\mathrm{d}x)\right\}.$$

On the other hand, there exists a constant C_h such that

$$\Psi(2\theta) \le C_h\psi(\theta), \quad \theta \in [0, h].$$

Then the argument based on uniform integrability analogous to the one exposed above provides

$$\mathbb{E}\exp\left\{\int_{\mathbb{R}^d}(f_n)_h(x)\big[\omega(\mathrm{d}x)-\mathrm{d}x\big]\right\}\to\mathbb{E}\exp\left\{\int_{\mathbb{R}^d}f_h(x)\big[\omega(\mathrm{d}x)-\mathrm{d}x\big]\right\}$$

because $(f_n)_h\le h$. □

Let $K(x)\ge 0$ be an non-negative function on $\mathbb{R}^d$. We intend to view the random integrals

$$\int_{\mathbb{R}^d}K(y-x)\omega(\mathrm{d}y)\quad\text{and}\quad\int_{\mathbb{R}^d}K(y-x)\big[\omega(\mathrm{d}y)-\mathrm{d}y\big]\tag{2.19}$$

as the (random) functions in x. By the shifting invariance of Lebesgue measure the above integrals are well defined for every $x\in\mathbb{R}^d$ under the assumptions

$$\int_{\mathbb{R}^d}\varphi\big(K(x)\big)\,\mathrm{d}x<\infty\quad\text{and}\quad\int_{\mathbb{R}^d}\psi\big(K(x)\big)\,\mathrm{d}x<\infty,$$

respectively. On the other hand, for each $x\in\mathbb{R}^d$ the random integrals in (2.19) are defined as equivalence classes of random variables. A usual treatment is to work with some modifications of them which have satisfy some measurability assumptions. Recall that given a family $\{U(x);x\in\Theta\}$ of equivalence classes of random variables indexed by a metric space Θ, the process $\{\widetilde{U}(x);x\in\Theta\}$ is called a modification of $\{U(x);x\in\Theta\}$ if for each $x\in\Theta$, $\widetilde{U}(x)\in U(x)$. Further, $\{\widetilde{U}(x);x\in\Theta\}$ is said to be measurable if it is measurable as a function $\widetilde{U}:\Theta\times\Omega\to\mathbb{R}$. In that case, in particular, for each $\omega\in\Omega$ the realization $\{\widetilde{U}(x,\omega);x\in\Theta\}$ is a measurable function on Θ. It is well known (Chapter 6, [4]) that the family $\{U(x);x\in\Theta\}$ has a measurable modification if $\{U(x);x\in\Theta\}$ is continuous in probability: For given $\varepsilon>0$ and $x_0\in\Theta$,

$$\lim_{x\to x_0}\mathbb{P}\big\{\big|U(x)-U(x_0)\big|\ge\varepsilon\big\}=0.$$

Proposition 2.8. *Under the assumption*

$$\int_{\mathbb{R}^d}\varphi\big(K(x)\big)\,\mathrm{d}x<\infty\quad\textit{and}\quad\int_{\mathbb{R}^d}\psi\big(K(x)\big)\,\mathrm{d}x<\infty,\tag{2.20}$$

respectively, the random integrals

$$\int_{\mathbb{R}^d}K(y-x)\omega(\mathrm{d}y)\quad\textit{and}\quad\int_{\mathbb{R}^d}K(y-x)\big[\omega(\mathrm{d}y)-\mathrm{d}y\big],\quad x\in\mathbb{R}^d,$$

are continuous in probability as functions of x. Consequently, they yield measurable modifications.

Proof. We only consider the second integral as the proof for the first part is analogous. Consider the family of the operators in $\mathcal{L}(\mathbb{R}^d)$:

$$S_xg(y)=g(x-y).$$

Clearly, the norm of any S_x is 1, and $S_{x_n}g\to S_xg$, $x_n\to x$ in $\mathcal{L}(\mathbb{R}^d)$ for any continuous g with compact support. Therefore, by standard argument $S_{x_n}g\to S_xg$, $x_n\to x$ in $\mathcal{L}(\mathbb{R}^d)$ for any $g\in\mathcal{L}(\mathbb{R}^d)$.

Take $g=\psi(K)$, then $\psi(K(x_n-\cdot))\to\psi(K(x-\cdot))$, $x_n\to x$ in $\mathcal{L}(\mathbb{R}^d)$. This convergence provides both convergence $\psi(K(x_n-\cdot))\to\psi(K(x-\cdot))$, $x_n\to x$ in measure $\mathrm{d}x$ and, by the necessity part of the Vitali convergence theorem, uniform integrability of the sequence $\{\psi(K(x_n-\cdot))\}_{n\ge1}$. Hence, the requested continuity in probability is provided by Proposition 2.3. □

In the remaining of the paper, we use the notations $V(x)$ and $\overline{V}(x)$ for the measurable modifications of the Poisson integrals given in (2.19) and still write

$$V(x)=\int_{\mathbb{R}^d} K(y-x)\omega(\mathrm{d}y) \quad \text{and} \quad \overline{V}(x)=\int_{\mathbb{R}^d} K(y-x)\bigl[\omega(\mathrm{d}y)-\mathrm{d}y\bigr]. \tag{2.21}$$

By a standard procedure, we may assume that $V(x)$ and $\overline{V}(x)$ are separable, so that the measurability is guaranteed for the supremum and infimum of the random field $V(x)$ and $\overline{V}(x)$.

By the shifting invariance of the Poisson field $\omega(\mathrm{d}x)$, we have that for any $z\in\mathbb{R}^d$,

$$\bigl\{V(z+x), x\in\mathbb{R}^d\bigr\} \stackrel{d}{=} \bigl\{V(x), x\in\mathbb{R}^d\bigr\}, \tag{2.22}$$

$$\bigl\{\overline{V}(z+x), x\in\mathbb{R}^d\bigr\} \stackrel{d}{=} \bigl\{\overline{V}(x), x\in\mathbb{R}^d\bigr\}. \tag{2.23}$$

One might further intend to install path-wise continuity for the random fields $V(x)$ and $\overline{V}(x)$. Here we refer the reader to [24] for some recent discussion on this topic. In the next proposition we show that this is not the case for unbounded shape $K(x)$. In a slightly different setting, Rosinski (Theorem 4, [29]) shows that the random field given as the random integral of a parametric integrand is in general no nicer than the integrand. Applying this result we have

Proposition 2.9. *Assume* (2.2) *and* (2.3), *respectively, for the statement concerning* $V(x)$ *and for the statement concerning* $\overline{V}(x)$. *Suppose that*

$$\lim_{x\to x_0} K(x)=\infty$$

for some $x_0\in\mathbb{R}^d$. *We have that*

$$\mathbb{P}\Bigl\{\sup_{x\in\mathcal{N}} V(x)=\infty\Bigr\}>0 \quad \textit{and} \quad \mathbb{P}\Bigl\{\sup_{x\in\mathcal{N}} \overline{V}(x)=\infty\Bigr\}>0 \tag{2.24}$$

for any non-empty open set $\mathcal{N}\subset\mathbb{R}^d$.

Proof. We only prove the second part in (2.24) as the proof of the first part is analogous. We first consider the case $\mathcal{N}=[-\delta,\delta]^d$. Let $\tilde{\omega}(\mathrm{d}y)$ be an independent copy of $\omega(\mathrm{d}y)$. Applying Theorem 4, [29] to the special case $T=\mathcal{N}$, $k=1$, and $\Lambda(\mathrm{d}y)=\omega(\mathrm{d}y)-\tilde{\omega}(\mathrm{d}y)$ gives that

$$\mathbb{P}\Bigl\{\sup_{x\in\mathcal{N}}\int_{\mathbb{R}^d} K(y-x)\Lambda(\mathrm{d}y)=\infty\Bigr\}>0.$$

By the triangular inequality, we obtain (2.24) for $\mathcal{N}=[-\delta,\delta]^d$. The proof is completed by the homogeneity identity (2.23). □

3. Brownian motion in renormalized Poisson potentials

Let $K(x)\ge 0$ be a shape function. Throughout this section, we assume that

$$\int_{\mathbb{R}^d}\varphi\bigl(K(x)\bigr)\,\mathrm{d}x<\infty$$

when our discussion involves the random potential

$$V(x)=\int_{\mathbb{R}^d} K(y-x)\omega(\mathrm{d}y)$$

and assume that

$$\int_{\mathbb{R}^d} \psi\big(K(x)\big)\,\mathrm{d}x < \infty$$

when the discussion involves the renormalized random potential

$$\overline{V}(x) = \int_{\mathbb{R}^d} K(y-x)\big[\omega(\mathrm{d}y) - \mathrm{d}y\big].$$

Our aim in this section is to define and study the time integrals

$$\int_0^t V(B_s)\,\mathrm{d}s, \qquad \int_0^t \overline{V}(B_s)\,\mathrm{d}s. \tag{3.1}$$

Given the local unboundedness of the random fields $V(x)$ and $\overline{V}(x)$ observed in Proposition 2.24, it is not obvious that the integrals in (3.1) are well defined. To justify that definition and to provide a tool for further study of the integrals in (3.1), it is convenient to consider the random field

$$\xi(t,x) = \int_0^t K(x - B_s)\,\mathrm{d}s, \quad x \in \mathbb{R}^d, t \ge 0, \tag{3.2}$$

and represent the integrals in (3.1) as the Poisson integrals

$$\int_{\mathbb{R}^d} \xi(t,x)\omega(\mathrm{d}x), \qquad \int_{\mathbb{R}^d} \xi(t,x)\big[\omega(\mathrm{d}x) - \mathrm{d}x\big]. \tag{3.3}$$

Proposition 3.1. *For every $t \ge 0$,*

$$\int_0^t \big|\overline{V}(B_s)\big|\,\mathrm{d}s < \infty \quad a.s., \tag{3.4}$$

$$\int_0^t \overline{V}(B_s)\,\mathrm{d}s = \int_{\mathbb{R}^d} \xi(t,x)\big[\omega(\mathrm{d}x) - \mathrm{d}x\big] \quad a.s. \tag{3.5}$$

In addition, if the set $\{x; K(x) \ge h\}$ is bounded in $\mathbb{R}^d$ for some $h > 0$, then

$$\int_0^t V(B_s)\,\mathrm{d}s < \infty \quad a.s., \tag{3.6}$$

$$\int_0^t V(B_s)\,\mathrm{d}s = \int_{\mathbb{R}^d} \xi(t,x)\omega(\mathrm{d}x) \quad a.s. \tag{3.7}$$

Proof. By the shift invariance of the field $\overline{V}$ and the Fubini's theorem, we have

$$\mathbb{E}\otimes\mathbb{E}_0\bigg(\int_0^t \big|\overline{V}(B_t)\big|\,\mathrm{d}t\bigg) = \mathbb{E}_0\bigg(\int_0^t \mathbb{E}\big|\overline{V}(B_t)\big|\,\mathrm{d}t\bigg) = \mathbb{E}_0\bigg(\int_0^t \mathbb{E}\big|\overline{V}(0)\big|\,\mathrm{d}t\bigg) = t\mathbb{E}\big|\overline{V}(0)\big| < \infty,$$

where the last step follows from Proposition 2.6. Hence, (3.4) holds.

We begin to establish (3.5) by showing that the right-hand side is well defined. Indeed, the function ψ is convex, hence Jensen's inequality provides

$$\int_{\mathbb{R}^d} \psi\big(\xi(t,x)\big)\,\mathrm{d}x \le \frac{1}{t}\int_0^t \int_{\mathbb{R}^d} \psi\big(tK(x-B_s)\big)\,\mathrm{d}x\,\mathrm{d}s = \int_{\mathbb{R}^d} \psi\big(tK(x)\big)\,\mathrm{d}x. \tag{3.8}$$

Notice that the function ψ is sub-homogeneous in the sense that for every $t > 0$ there exists a constant C_t such that

$$\psi(t\theta) \le C_t\psi(\theta), \quad \theta \in [0,\infty). \tag{3.9}$$

Consequently, we conclude that

$$\int_{\mathbb{R}^d} \psi\big(\xi(t,x)\big)\,\mathrm{d}x < \infty \quad \text{a.s.}$$

By Proposition 2.1, therefore, $\xi(t,x)$ is a.s. integrable w.r.t. $\omega(\mathrm{d}x) - \mathrm{d}x$.

Beyond technicalities, (3.7) comes from order change of the integrals:

$$\begin{aligned}\int_0^t \overline{V}(B_s)\,\mathrm{d}s &= \int_0^t \bigg[\int_{\mathbb{R}^d} K(y - B_s)\big[\omega(\mathrm{d}y) - \mathrm{d}y\big]\bigg]\mathrm{d}s \\ &= \int_{\mathbb{R}^d}\bigg[\int_0^t K(y-B_s)\,\mathrm{d}s\bigg]\big[\omega(\mathrm{d}y) - \mathrm{d}y\big] = \int_{\mathbb{R}^d} \xi(t,y)\big[\omega(\mathrm{d}y) - \mathrm{d}y\big].\end{aligned}$$

The above calculation is mathematically rigorous when $K(x)$ is simple.

In the general case, consider an increasing sequence of simple functions $K_n \geq 0, n \geq 1$, such that $0 \leq K_n(x) \uparrow K(x)$ a.e. and write

$$\xi_n(t,x) = \int_0^t K_n(x - B_s)\,\mathrm{d}s \quad \text{and} \quad \overline{V}_n(x) = \int_{\mathbb{R}^d} K_n(y-x)\big[\omega(\mathrm{d}y) - \mathrm{d}y\big].$$

Notice that by monotonic convergence theorem $\xi_n(t,x) \uparrow \xi(t,x)$ a.s. for a.e. $x \in \mathbb{R}^d$ and therefore (Proposition 2.4),

$$\int_{\mathbb{R}^d} \xi_n(t,x)\big[\omega(\mathrm{d}x) - \mathrm{d}x\big] \xrightarrow{P} \int_{\mathbb{R}^d} \xi(t,x)\big[\omega(\mathrm{d}x) - \mathrm{d}x\big]$$

as $n \to \infty$. On the other hand,

$$\mathbb{E} \otimes \mathbb{E}_0 \bigg|\int_0^t \big(\overline{V}(B_s) - \overline{V}_n(B_s)\big)\,\mathrm{d}s\bigg| \leq t\mathbb{E}\big|\overline{V}(0) - \overline{V}_n(0)\big|.$$

By slightly modifying the argument in the proof of Proposition 2.6, one can show that the right-hand side tends to 0 as $n \to \infty$. Summarizing our argument, we have proved (3.5).

It remains to prove (3.6) and (3.7). The extra difficulties are: $V(0)$ is not necessarily in $\mathcal{L}(\Omega, \mathcal{A}, \mathbb{P})$ (take the case (1.7) for an example) under our assumption so the argument for (3.4) does not apply here; the function φ is not convex so Jensen's inequality can not be used for the integrability of $\xi(t,x)$ w.r.t. $\omega(\mathrm{d}x)$. This is the reason for us to introduce the additional assumption for the set $\{x\colon\ K(x) \geq h\}$ to be bounded.

For a given $h > 0$, there exist positive constants c_h, C_h such that

$$c_h \min(h,y) \leq \varphi(y) \leq C_h \min(h,y), \qquad y \geq 0. \tag{3.10}$$

Denote τ_x the hitting time of the closure of the set $\{y\colon\ K(y) \geq h\}$ by the process $B_\cdot - x$. On the set $\{\tau_x \geq t\}$,

$$\xi(t,x) = \int_0^t K(B_s - x)\,\mathrm{d}s = \int_0^t \min\big(h, K(B_s - x)\big)\,\mathrm{d}s.$$

Therefore

$$\mathbb{E}_0 \min\big(h, \xi(t,x)\big) \leq \mathbb{E}_0 \int_0^t \min\big(h, K(B_s - x)\big)\,\mathrm{d}s + h\mathbb{P}_0(\tau_x < t),$$

and consequently

$$\mathbb{E}_0 \int_{\mathbb{R}^d} \min\big(h, \xi(t,x)\big)\,\mathrm{d}x \leq t\int_{\mathbb{R}^d} \min\big(h, K(x)\big)\,\mathrm{d}x + h\int_{\mathbb{R}^d} \mathbb{P}_0(\tau_x < t)\,\mathrm{d}x.$$

The second integral in the r.h.s. is finite because of the standard estimate for a Brownian motion to hit a given ball:

$$\mathbb{P}_0\big(\exists s \le t:\ B_s - x \in B(0,R)\big) \le C_{1,R,t}\mathrm{e}^{-C_{2,R,t}|x|}.$$

By (3.10), therefore,

$$\int_{\mathbb{R}^d} \varphi\big(\xi(t,x)\big)\,\mathrm{d}x < \infty \quad \text{a.s.},$$

which leads to the a.s. integrability of $\xi(t,x)$ w.r.t. $\omega(\mathrm{d}x)$.

Let $K_n(x)$ and $\xi_n(t,x)$ be defined as above. Then (3.7) holds for the simple shape function $K_n(x)$:

$$\int_0^t V_n(B_s)\,\mathrm{d}s = \int_{\mathbb{R}^d} \xi_n(t,x)\omega(\mathrm{d}x) \quad \text{a.s.}$$

In particular,

$$\int_0^t V_n(B_s)\,\mathrm{d}s \le \int_{\mathbb{R}^d} \xi(t,x)\omega(\mathrm{d}x) < \infty \quad \text{a.s.}, n = 1, 2, \dots.$$

By the monotonic convergence theorem, (3.6) holds and

$$\int_{\mathbb{R}^d} \xi(t,x)\omega(\mathrm{d}x) = \lim_{n\to\infty}\int_{\mathbb{R}^d} \xi_n(t,x)\omega(\mathrm{d}x) = \lim_{n\to\infty}\int_0^t V_n(B_s)\,\mathrm{d}s = \int_0^t V(B_s)\,\mathrm{d}s \quad \text{a.s.}$$

□

Proof of Theorem 1.1. All left to prove is (1.15). By (3.5), (2.17), (3.8) and (3.9),

$$\begin{aligned}
\mathbb{E}\exp\left\{-\int_0^t \overline{V}(B_s)\,\mathrm{d}s\right\} &= \mathbb{E}\exp\left\{-\int_{\mathbb{R}^d} \xi(t,x)\big[\omega(\mathrm{d}x) - \mathrm{d}x\big]\right\} \\
&= \exp\left\{\int_{\mathbb{R}^d} \psi\big(\xi(t,x)\big)\,\mathrm{d}x\right\} \le \exp\left\{\int_{\mathbb{R}^d} \psi\big(tK(x)\big)\,\mathrm{d}x\right\} \\
&\le \exp\left\{C_t\int_{\mathbb{R}^d} \psi\big(K(x)\big)\,\mathrm{d}x\right\} < \infty.
\end{aligned}$$

So the conclusion follows from Fubini's theorem. □

4. $K(x) = \theta|x|^{-p}$

Throughout this section, $K(x) = \theta|x|^{-p}$ with $\theta, p > 0$. It is easy to see that the condition (1.14) is equivalent to $d/2 < p < d$, which is assumed throughout this section. As a side remark, we point out that the random fields

$$V(x) = \int_{\mathbb{R}^d} \frac{1}{|y-x|^p}\omega(\mathrm{d}y) \quad \text{and} \quad \overline{V}(x) = \int_{\mathbb{R}^d} \frac{1}{|y-x|^p}\big[\omega(\mathrm{d}y) - \mathrm{d}y\big], \quad x \in \mathbb{R}^d,$$

appear as the Riesz transforms of the (signed) measures $\omega(\mathrm{d}y)$ and $\omega(\mathrm{d}y) - \mathrm{d}y$, respectively.

The main objective of this section is to prove Theorem 1.4 and Theorem 1.5.

4.1. *Proof of Theorem* 1.4

By Fubini's theorem, (3.5) and (2.15),

$$\mathbb{E}\otimes\mathbb{E}_0\exp\left\{\int_0^t \overline{V}(B_s)\,\mathrm{d}s\right\} = \mathbb{E}_0\exp\left\{\int_{\mathbb{R}^d} \Psi\big(\xi(t,x)\big)\,\mathrm{d}x\right\}.$$

It remains to show that

$$\mathbb{E}_0 \exp\left\{\int_{\mathbb{R}^d} \Psi\big(\xi(t,x)\big)\,\mathrm{d}x\right\} = \infty. \tag{4.1}$$

To this end, first notice that, for $\max_{0\le s\le t}|B_s|\le\varepsilon$ and $|x|\ge\varepsilon$,

$$\xi(t,x) = \theta\int_0^t \frac{1}{|x-B_s|^p}\,\mathrm{d}s \ge \frac{t\theta}{2^p|x|^p}.$$

Consequently,

$$\mathbb{E}_0 \exp\left\{\int_{\mathbb{R}^d} \Psi\big(\xi(t,x)\big)\,\mathrm{d}x\right\} \ge \exp\left\{\int_{\{|x|\ge\varepsilon\}} \Psi\left(\frac{t\theta}{2^p|x|^p}\right)\mathrm{d}x\right\}\mathbb{P}_0\left\{\max_{0\le s\le t}|B_s|\le\varepsilon\right\}.$$

By a classic result [13] on small deviations for Brownian motions, there is a constant $C_d>0$ such that

$$\mathbb{P}_0\left\{\max_{0\le s\le t}|B_s|\le\varepsilon\right\} \sim \exp\left\{-\big(C_d+\mathrm{o}(1)\big)t\varepsilon^{-2}\right\} \quad \big(\varepsilon\to 0^+\big).$$

Hence, (4.1) follows from the rough estimate that

$$\int_{\{|x|\ge\varepsilon\}} \Psi\left(\frac{t\theta}{2^p|x|^p}\right)\mathrm{d}x \ge \mathrm{e}^{c_t\varepsilon^{-p}}$$

for sufficiently small ε.

Remark. *More than claimed in* (4.1), *we can show that with positive probability*

$$\int_{\mathbb{R}^d} \Psi\big(\xi(t,x)\big)\,\mathrm{d}x = \infty.$$

We show this by contradiction. Assume that

$$\int_{\mathbb{R}^d} \Psi\big(\xi(t,x)\big)\,\mathrm{d}x < \infty \quad \text{a.s.}$$

for some $t>0$. *Write*

$$Z_t = t\int_{\mathbb{R}^d} \Psi\left(\frac{1}{t}\xi(t,x)\right)\mathrm{d}x.$$

Since Ψ *is a Young function, by our assumption* $Z_t<\infty$ *for some* $t>0$. *For any* $s,t>0$, *by Jensen inequality*

$$\Psi\left(\frac{\xi(s+t,x)}{s+t}\right) \le \frac{s}{s+t}\Psi\left(\frac{\xi(s,x)}{s}\right) + \frac{t}{s+t}\Psi\left(\frac{\xi(s+t,x)-\xi(s,x)}{t}\right).$$

Consequently, $Z_{s+t}\le Z_s+Z_t'$ *with*

$$Z_t' = t\int_{\mathbb{R}^d} \Psi\left(\frac{\xi(s+t,x)-\xi(s,x)}{t}\right)\mathrm{d}x.$$

One can see that $Z_t' \overset{d}{=} Z_t$ *and* Z_t' *is independent of* $\{B_u; 0\le u\le s\}$. *The processes like* Z_t *are called sub-additive processes. By sub-additivity and a standard deterministic argument, one can show that* $Z_t<\infty$ *a.s. for all* $t>0$. *By the way that* Z_t *is defined, it is not hard to see that the almost finiteness implies the path-wise continuity. By Theorem* 1.3.5 *in* [7],

$$\mathbb{E}\exp\{CZ_t\}<\infty, \quad C,t>0.$$

With θ *being replaced by* $t\theta$, *the above conclusion contradicts* (4.1).

4.2. *Proof of Theorem* 1.5: *Case* $p \geq 2$

We now prove that under $p > 2$,

$$\mathbb{E}_0 \exp\left\{\int_0^t \overline{V}(B_s)\,\mathrm{d}s\right\} = \infty \quad \text{a.s.} \tag{4.2}$$

for any $t > 0$ and $\theta > 0$; and that under $p = 2$, there is a $\theta_0 > 0$ such that, when $\theta \geq \theta_0$, (4.2) holds for all $t > 0$.

We adopt the following notations: For any $r > 0$, $Q_r = [-r, r]^d$ and $Q = Q_1$. Write

$$K_1(x) = \theta|x|^{-p} 1_{\{x \in Q\}} \quad \text{and} \quad K_2(x) = \theta|x|^{-p} 1_{\{x \notin Q\}},$$
$$\overline{V}_i(x) = \int_{\mathbb{R}^d} K_i(y - x)\big[\omega(\mathrm{d}y) - \mathrm{d}y\big] \quad \text{and} \quad i = 1, 2.$$

For any $C > 0$, by Jensen's inequality

$$\begin{aligned}\mathbb{E} \exp\left\{C \int_0^t \overline{V}_2(B_s)\,\mathrm{d}s\right\} &\leq \frac{1}{t}\int_0^t \mathbb{E}\exp\big\{Ct\overline{V}_2(B_s)\big\}\,\mathrm{d}s \\ &= \mathbb{E}\exp\big\{Ct\overline{V}_2(0)\big\} = \exp\left\{\int_{\{x \notin Q\}} \Psi\left(\frac{Ct\theta}{|x|^p}\right)\mathrm{d}x\right\}.\end{aligned}$$

The right-hand side is deterministic and finite. By Fubini's theorem,

$$\mathbb{E} \otimes \mathbb{E}_0 \exp\left\{C \int_0^t \overline{V}_2(B_s)\,\mathrm{d}s\right\} < \infty.$$

Since θ is arbitrary, and

$$\overline{V}_1(x) = \int_{\mathbb{R}^d} K_1(y - x)\omega(\mathrm{d}x) - \theta \int_{\{|x| \leq 1\}} \frac{1}{|x|^p}\,\mathrm{d}x = V_1(x) - \theta \int_{\{|x| \leq 1\}} \frac{1}{|x|^p}\,\mathrm{d}x,$$

the proof is reduced to show

$$\mathbb{E}_0 \exp\left\{\int_0^t V_1(B_s)\,\mathrm{d}s\right\} = \infty \quad \text{a.s.} \tag{4.3}$$

For each $r > 0$, define the exit time

$$\tau_r = \inf\{s \geq 0; B_s \notin Q_r\}.$$

For each $t > 0$, let $p_t(x)$ be the probability density of B_t. Let $m, n \geq 1$ be large integers with $n \gg m$. By the fact that $V_1(\cdot) \geq 0$ and by the Markov property, for each $z \in 2^{-n}\mathbb{Z}^d \cap Q_{2^m}$

$$\begin{aligned}\mathbb{E}_0 \exp\left\{\int_0^t V_1(B_s)\,\mathrm{d}s\right\} &\geq \mathbb{E}_0\left(1_{\{B_{t/2} \in z + Q_{2^{-n}}\}} \exp\left\{\int_{t/2}^t V_1(B_s)\,\mathrm{d}s\right\}\right) \\ &= \int_{z + Q_{2^{-n}}} p_{t/2}(x)\mathbb{E}_0 \exp\left\{\int_0^{t/2} V_1(x + B_s)\,\mathrm{d}s\right\}\mathrm{d}x.\end{aligned}$$

Notice that on $\{\tau_{2^{-n}} \geq t/2\}$,

$$\begin{aligned}V_1(x + B_s) &= \theta \int_{x + B_s + Q} \frac{1}{|y - x - B_s|^p}\omega(\mathrm{d}y) \\ &\geq \theta \int_{z + Q_{2^{-n}}} \frac{1}{|y - x - B_s|^p}\omega(\mathrm{d}y) \geq c\theta 2^{np}\omega(z + Q_{2^{-n}})\end{aligned}$$

for every $x \in z + Q_{2^{-n}}$ and large n, where $c > 0$ is a fixed constant. Hence,

$$\mathbb{E}_0 \exp\left\{\int_0^{t/2} V_1(x + B_s)\,\mathrm{d}s\right\}\mathrm{d}x \geq \exp\left\{c\theta(t/2)2^{np}\omega(z + Q_{2^{-n}})\right\}\mathbb{P}_0\{\tau_{2^{-n}} \geq 1/2\}.$$

We have obtained that

$$\mathbb{E}_0 \exp\left\{\int_0^t V_1(B_s)\,\mathrm{d}s\right\} \geq \left(\int_{z+Q_{2^{-n}}} p_{t/2}(x)\,\mathrm{d}x\right)\exp\left\{c\theta(t/2)2^{np}\omega(z + Q_{2^{-n}})\right\}\mathbb{P}_0\{\tau_{2^{-n}} \geq t/2\}.$$

Uniformly over $z \in 2^{-n}\mathbb{Z}^d \cap Q_{2^m}$, we have the following bound

$$\int_{z+Q_{2^{-n}}} p_{t/2}(x)\,\mathrm{d}x \geq \mathrm{e}^{-c_1 4^m/t}.$$

Notice the small deviation bound [13]

$$\mathbb{P}_0\{\tau_{2^{-n}} \geq t/2\} \geq \mathrm{e}^{-c_2 t 4^n}.$$

Taking the supremum over $z \in 2^{-n}\mathbb{Z}^d \cap Q_{2^m}$ gives that

$$\mathbb{E}_0 \exp\left\{\int_0^t V_1(B_s)\,\mathrm{d}s\right\} \geq \exp\left\{-c_1 4^m/t - c_2 t 4^n\right\}\exp\left\{\frac{c\theta t}{2}2^{np}\max_{z\in 2^{-n}\mathbb{Z}^d\cap Q_{2^m}}\omega(z + Q_{2^{-n}})\right\}.$$

Notice that the random variable $\max_{z\in 2^{-n}\mathbb{Z}^d\cap Q_{2^m}}\omega(z + Q_{2^{-n}})$ takes only integers. Consequently, when $p > 2$

$$\left\{\mathbb{E}_0 \exp\left\{\int_0^t V_1(B_s)\,\mathrm{d}s\right\} = \infty\right\} \supset \left\{\limsup_{n\to\infty}\max_{z\in 2^{-n}\mathbb{Z}^d\cap Q_{2^m}}\omega(z + Q_{2^{-n}}) \geq 1\right\}. \tag{4.4}$$

When $p = 2$, this relation remains true as $\theta > 2c_2/c$.

Let $m \geq 1$ be fixed at this time and write

$$X_n = \max_{z\in 2^{-n}\mathbb{Z}^d\cap Q_{2^m}}\omega(z + Q_{2^{-n}}), \quad n = 1, 2, \ldots.$$

It is easy to see $\{X_n\}_{n\geq 1}$ is a non-increasing sequence. Thus,

$$\mathbb{P}\left\{\limsup_{n\to\infty} X_n = 0\right\} = \mathbb{P}\left(\bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty}\{X_k = 0\}\right) = \mathbb{P}\left(\bigcup_{n=1}^{\infty}\{X_n = 0\}\right) = \lim_{n\to\infty}\mathbb{P}\{X_n = 0\}.$$

In addition, notice that the family

$$\omega(z + Q_{2^{-n}}), \quad z \in 2^{-n}\mathbb{Z}^d \cap Q_{2^m},$$

forms an i.i.d. sequence. So we have

$$\mathbb{P}\{X_n = 0\} = \left(\mathbb{P}\left\{\omega(Q_{2^{-n}}) = 0\right\}\right)^{2^{(n+m)d}} = \exp\left\{-2^{(m-1)d}\right\}.$$

Summarizing our computation

$$\mathbb{P}\left\{\limsup_{n\to\infty} X_n = 0\right\} = \exp\left\{-2^{(m-1)d}\right\}.$$

In view of (4.4) we have

$$\mathbb{P}\left\{\mathbb{E}_0 \exp\left\{\int_0^t V_1(B_s)\,\mathrm{d}s\right\} = \infty\right\} \ge 1 - \exp\{-2^{(m-1)d}\}.$$

Letting $m \to \infty$ on the right-hand side leads to (4.3).

4.3. *Proof of Theorem* 1.5: *Case* $p < 2$

We now prove under $p < 2$, that for any $t > 0$ and $\theta > 0$ (hiding in (1.7)),

$$\mathbb{E}_0 \exp\left\{\int_0^t \overline{V}(B_s)\,\mathrm{d}s\right\} < \infty \quad \text{a.s.} \tag{4.5}$$

We adopt all notations introduced in Section 4.2. By an argument same as the one carried out in Section 4.2, the problem is reduced to the proof of

$$\mathbb{E}_0 \exp\left\{\int_0^t V_1(B_s)\,\mathrm{d}s\right\} < \infty \quad \text{a.s.} \tag{4.6}$$

Write

$$\eta(t,x) = \int_0^t K_1(x - B_s)\,\mathrm{d}s, \quad x \in \mathbb{R}^d,$$

and recall our notation $Q_r = [-r, r]^d$ and

$$\tau_r = \inf\{s \ge 0; B_s \notin Q_r\}.$$

Consider the decomposition

$$\mathbb{E}_0 \exp\left\{\int_0^t V_1(B_s)\,\mathrm{d}s\right\} = \mathbb{E}_0\left[\exp\left\{\int_0^t V_1(B_s)\,\mathrm{d}s\right\}; \tau \ge t\right] + \sum_{n=0}^{\infty} \mathbb{E}_0\left[\exp\left\{\int_0^t V_1(B_s)\,\mathrm{d}s\right\}; \tau_{2^n} < t \le \tau_{2^{n+1}}\right].$$

By Cauchy–Schwarz inequality,

$$\mathbb{E}_0\left[\exp\left\{\int_0^t V_1(B_s)\,\mathrm{d}s\right\}; \tau_{2^n} < t \le \tau_{2^{n+1}}\right] \le \left(\mathbb{P}_0\{\tau_n \le t\}\right)^{1/2}\left(\mathbb{E}_0\left[\exp\left\{2\int_0^t V_1(B_s)\,\mathrm{d}s\right\}; \tau_{2^{n+1}} \ge t\right]\right)^{1/2}.$$

By the classic result on Gaussian tail of Brownian motions,

$$\mathbb{P}_0\{\tau_{2^n} \le t\} = \mathbb{P}\left\{\max_{0\le s\le t} |B_s|_\infty \ge 2^n\right\} \le \exp\{-c_t 2^{2n}\}$$

for sufficiently large n, where $|\cdot|_\infty$ stands for the maximum norm on $\mathbb{R}^d$ and $c_t > 0$ is a constant independent of n.

Since $\theta > 0$ appearing in (1.7) can be arbitrary, it is sufficient to show that

$$\log \mathbb{E}_0\left[\exp\left\{\int_0^t V_1(B_s)\,\mathrm{d}s\right\}; \tau_{2^n} \ge t\right] = \mathrm{o}(2^{2n}) \quad \text{a.s.} \tag{4.7}$$

as $n \to \infty$.

Write $c = \frac{2d}{2-p}$. By Cauchy–Schwarz inequality

$$\begin{aligned}
&\mathbb{E}_0\left[\exp\left\{\int_0^t V_1(B_s)\,\mathrm{d}s\right\}; \tau_{2^n} \ge t\right] \\
&\quad \le \left\{\mathbb{E}_0\left[\exp\left\{2\int_0^{2^{-cn}} V_1(B_s)\,\mathrm{d}s\right\}; \tau_{2^n} \ge t\right]\right\}^{1/2} \\
&\qquad \times \left\{\mathbb{E}_0\left[\exp\left\{2\int_{2^{-cn}}^t V_1(B_s)\,\mathrm{d}s\right\}; \tau_{2^n} \ge t\right]\right\}^{1/2}.
\end{aligned} \tag{4.8}$$

We now claim that

$$\log \mathbb{E}_0\left[\exp\left\{2\int_0^{2^{-cn}} V_1(B_s)\,\mathrm{d}s\right\}; \tau_{2^n} \ge 1\right] = \mathrm{O}(1) \quad \text{a.s.} \tag{4.9}$$

and that

$$\log \mathbb{E}_0\left[\exp\left\{2\int_{2^{-cn}}^t V_1(B_s)\,\mathrm{d}s\right\}; \tau_{2^n} \ge t\right] = \mathrm{O}\big(n^{2/(2-p)}\big) \quad \text{a.s.}, \tag{4.10}$$

which clearly imply (4.7).

Since $K_1(\cdot)$ is supported on Q,

$$\int_0^{2^{-cn}} V_1(B_s)\,\mathrm{d}s = \int_{\mathbb{R}^d} \eta\big(2^{-cn}, x\big)\omega(\mathrm{d}x) = \int_{Q_{2^{n+1}}} \eta\big(2^{-cn}, x\big)\omega(\mathrm{d}x)$$

on $\{\tau_{2^{n+1}} \ge t\}$. Hence,

$$\begin{aligned}
\mathbb{E}_0\left[\exp\left\{2\int_0^{2^{-cn}} V_1(B_s), \mathrm{d}s\right\}; \tau_{2^n} \ge t\right] &\le \mathbb{E}_0 \exp\left\{2\int_{Q_{2^{n+1}}} \eta\big(2^{-cn}, x\big)\omega(\mathrm{d}x)\right\} \\
&\le \mathbb{E}_0 \exp\left\{2\omega(Q_{2^{n+1}}) \sup_{x\in\mathbb{R}^d} \xi\big(2^{-cn}, x\big)\right\} \\
&= \mathbb{E}_0 \exp\left\{2\cdot 2^{-dn}\omega(Q_{2^{n+1}}) \sup_{x\in\mathbb{R}^d} \xi(1, x)\right\},
\end{aligned}$$

where the last step follows from Brownian scaling.

Taking $p = 1$ (p has a different meaning in [1]), $\beta = 2$ and $\sigma = p$ (p is given in this paper) in Theorem 1.3, [1], we have

$$\lim_{a\to\infty} a^{-2/(2-p)} \log \mathbb{E}_0 \exp\left\{a \sup_{x\in\mathbb{R}^d} \xi(1, x)\right\} < \infty.$$

Consequently, there is a constant $C_0 > 0$ such that

$$\mathbb{E}_0\left[\exp\left\{2\int_0^{2^{-cn}} V_1(B_s)\,\mathrm{d}s\right\}; \tau_{2^n} \ge t\right] \le \exp\big\{C_0\big(2^{-dn}\omega(Q_{2^{n+1}})\big)^{2/(2-p)}\big\}. \tag{4.11}$$

By Cramér large deviation there is $C > 0$ such that

$$\mathbb{P}\big\{\omega(Q_{2^{n+1}}) \ge C2^{dn}\big\} \le \exp\big\{-2^{dn}\big\}$$

for large n. By Borel–Cantelli lemma, therefore,

$$2^{-dn}\omega(Q_{2^{n+1}}) = \mathrm{O}(1) \quad \text{a.s.}$$

Thus, (4.9) follows from (4.11).

We now come to the proof of (4.10). Write $\tau'_{2^n} = \inf\{s \ge 2^{-cn}; B_s \notin Q_{2^n}\}$.

$$\begin{aligned}
&\mathbb{E}_0\left[\exp\left\{2\int_{2^{-cn}}^{t} V_1(B_s)\,\mathrm{d}s\right\}; \tau_{2^n} \ge t\right] \\
&\quad \le \mathbb{E}_0\left[\exp\left\{2\int_{2^{-cn}}^{t} V_1(B_s)\,\mathrm{d}s\right\}; B_{2^{-cn}} \in Q_{2^n}, \tau'_{2^n} \ge t\right] \\
&\quad = \int_{Q_{2^{n+1}}} p_{2^{-cn}}(x)\,\mathrm{d}x\,\mathbb{E}_x\left[\exp\left\{2\int_0^{t-2^{-cn}} V_1(B_s)\,\mathrm{d}s\right\}; \tau_{2^n} \ge t - 2^{-cn}\right] \\
&\quad \le \frac{2^{cdn/2}}{(2\pi)^{d/2}}\int_{Q_{2^{n+1}}} \mathrm{d}x\,\mathbb{E}_x\left[\exp\left\{2\int_0^{t-2^{-cn}} V_1(B_s)\,\mathrm{d}s\right\}; \tau_{2^n} \ge t - 2^{-cn}\right],
\end{aligned}$$

where $p_{2^{-cn}}(x)$ is the density of $B_{2^{-cn}}$ and the equality follows from the Markov property.

Consider a non-negative and smooth function $\gamma(x)$ on $\mathbb{R}^d$ whose support is contained in the 1-neighborhood of the grid $2\mathbb{Z}^d$. Assume $\gamma(x)$ is periodic with period 2:

$$\gamma(x + 2z) = \gamma(x); \quad x \in \mathbb{R}^d, z \in \mathbb{Z}^d.$$

Finally, we assume that

$$K \equiv \int_{Q_2} \gamma(x)\,\mathrm{d}x < \infty.$$

Write $\gamma^y(x) = \gamma(x - y)$. For any $x \in Q_{2^n}$ by periodicity and Jensen's inequality

$$\begin{aligned}
&\mathbb{E}_x\left[\exp\left\{2\int_0^{t-2^{-cn}} V_1(B_s)\,\mathrm{d}s\right\}; \tau_{2^n} \ge t - 2^{-cn}\right] \\
&\quad = \mathrm{e}^K\,\mathbb{E}_x\left[\exp\left\{2\int_0^{t-2^{-cn}} V_1(B_s)\,\mathrm{d}s - \int_{Q_2} \gamma^y(B_s)\,\mathrm{d}y\right\}; \tau_{2^n} \ge t - 2^{-cn}\right] \\
&\quad \le \mathrm{e}^K \int_{Q_2} \mathrm{d}y\,\mathbb{E}_x\left[\exp\left\{\int_0^{t-2^{-cn}} \big(2V_1 - \gamma^y\big)(B_s)\,\mathrm{d}s\right\}; \tau_{2^n} \ge t - 2^{-cn}\right].
\end{aligned}$$

Integrating on the both sides,

$$\begin{aligned}
&\int_{Q_{2^{n+1}}} \mathrm{d}x\,\mathbb{E}_x\left[\exp\left\{2\int_0^{t-2^{-cn}} V_1(B_s)\,\mathrm{d}s\right\}; \tau_{2^n} \ge t - 2^{-cn}\right] \\
&\quad \le \mathrm{e}^K \int_{Q_2} \mathrm{d}y \int_{Q_{2^{n+1}}} \mathrm{d}x\,\mathbb{E}_x\left[\exp\left\{\int_0^{t-2^{-cn}} \big(2V_1 - \gamma^y\big)(B_s)\,\mathrm{d}s\right\}; \tau_{2^n} \ge t - 2^{-cn}\right].
\end{aligned}$$

Summarizing our argument, we have proved that

$$\begin{aligned}
&\mathbb{E}_0\left[\exp\left\{2\int_{2^{-cn}}^{t} V_1(B_s)\,\mathrm{d}s\right\}; \tau_{2^n} \ge t\right] \\
&\quad \le \mathrm{e}^K \frac{2^{cdn/2}}{(2\pi)^{d/2}} \int_{Q_2} \mathrm{d}y \int_{Q_{2^{n+1}}} \mathrm{d}x\,\mathbb{E}_x\left[\exp\left\{\int_0^{t-2^{-cn}} \big(2V_1 - \gamma^y\big)(B_s)\,\mathrm{d}s\right\}; \tau_{2^n} \ge t - 2^{-cn}\right]. \qquad (4.12)
\end{aligned}$$

To continue, we introduce the following notations. Given a open domain $D \subset \mathbb{R}^d$, let $H_0^1(D)$ be the Sobolev space over D, defined to be the closure of the inner product space consists of the infinitely differentiable functions compactly supported in D under the Sobolev norm

$$\|g\|_H = \{\|g\|^2_{\mathcal{L}^2(D)} + \|\nabla g\|^2_{\mathcal{L}^2(D)}\}^{1/2}.$$

Write

$$H^1_{0,1}(D) = \left\{g \in H_0^1(D); \int_D g^2(x)\,\mathrm{d}x = 1\right\}.$$

In particular, $H^1_{0,1} = H^1_{0,1}(\mathbb{R}^d)$. Given a function $\zeta(x)$ on D, define

$$\lambda^\zeta(D) = \sup_{g\in H^1_{0,1}(D)} \left\{\int_D \zeta(x)g^2(x)\,\mathrm{d}x - \frac{1}{2}\int_D \left|\nabla g(x)\right|^2 \mathrm{d}x\right\}.$$

By (2.33) in [19],

$$\begin{aligned}
&\int_{Q_{2^{n+1}}} \mathrm{d}x\,\mathbb{E}_x\left[\exp\left\{\int_0^{t-2^{-cn}} \big(2V_1 - \gamma^y\big)(B_s)\,\mathrm{d}s\right\}; \tau_{2^n} \ge t - 2^{-cn}\right] \\
&\quad \le \exp\big\{\big(t - 2^{-cn}\big)\lambda^{2V_1-\gamma^y}(Q_{2^{n+1}})\big\} \le \exp\big\{t\lambda^{2V_1-\gamma^y}(Q_{2^{n+1}})\big\}.
\end{aligned}$$

According to Proposition 1 (with $r = 2$) in [18], or Lemma 4.6 in [19], there is a deterministic function $\gamma(\cdot)$ satisfying all assumptions posted above, such that

$$\lambda^{2V_1-\gamma^y}(Q_{2^{n+1}}) \le \max_{z\in 4\mathbb{Z}^d \cap Q_{2^{n+1}+2}} \lambda^{2V_1}(z + Q_3).$$

By now, we have come to the conclusion

$$\begin{aligned}
&\mathbb{E}_0\left[\exp\left\{2\int_{2^{-cn}}^t V_1(B_s)\,\mathrm{d}s\right\}; \tau_{2^n} \ge t\right] \\
&\quad \le 4^d \mathrm{e}^K \frac{2^{cdn/2}}{(2\pi)^{d/2}} \exp\left\{t \max_{z\in 4\mathbb{Z}^d \cap Q_{2^{n+1}+2}} \lambda^{2V_1}(z + Q_3)\right\}.
\end{aligned} \tag{4.13}$$

In order to establish (4.10), and therefore to complete our proof, we need to show that

$$\max_{z\in 4\mathbb{Z}^d \cap Q_{2^{n+1}+2}} \lambda^{2V_1}(z + Q_3) = \mathrm{O}\big(n^{2/(2-p)}\big) \quad \text{a.s.} \tag{4.14}$$

By (2.22), the random varibles $\lambda^{2V_1}(z + Q_3)$ ($z \in 4\mathbb{Z}^d \cap Q_{2^{n+1}+2}$) are identically distributed. Thus, there is a constant $C > 0$ such that

$$\mathbb{P}\left\{\max_{z\in 4\mathbb{Z}^d \cap Q_{2^{n+1}+2}} \lambda^{2V_1}(z + Q_3) \ge Cn^{2/(2-p)}\right\} \le C2^n \mathbb{P}\big\{\lambda^{2V_1}(Q_3) \ge Cn^{2/(2-p)}\big\} \tag{4.15}$$

for $n = 1, 2, \ldots$.

Notice that

$$\begin{aligned}
\lambda^{2V_1}(Q_3) &= \sup_{g\in H^1_{0,1}(Q_3)} \left\{2\int_{Q_3} V_1(x)g^2(x) - \frac{1}{2}\int_{Q_3} \left|\nabla g(x)\right|^2 \mathrm{d}x\right\} \\
&= \sup_{g\in H^1_{0,1}(Q_3)} \left\{2\theta \int_{\mathbb{R}^d} \left[\int_{Q_3} \frac{1_{\{y-x\in Q\}}}{|y - x|^p} g^2(x)\,\mathrm{d}x\right]\omega(\mathrm{d}y) - \frac{1}{2}\int_{Q_3} \left|\nabla g(x)\right|^2 \mathrm{d}x\right\}.
\end{aligned}$$

When $y-x\in Q$ and $x\in Q_3$, we must have that $y\in Q_4$. Consequently,

$$\int_{\mathbb{R}^d}\left[\int_{Q_3}\frac{1_{\{y-x\in Q\}}}{|y-x|^p}g^2(x)\,\mathrm{d}x\right]\omega(\mathrm{d}y)\le\int_{Q_4}\left[\int_{Q_3}\frac{g^2(x)}{|y-x|^p}\,\mathrm{d}x\right]\omega(\mathrm{d}y)$$
$$\le\omega(Q_4)\sup_{y\in\mathbb{R}^d}\int_{Q_3}\frac{g^2(x)}{|y-x|^p}\,\mathrm{d}x.$$

Summarizing our estimate, $\lambda^{2V_1}(Q_3)\le M_3(2\theta\omega(Q_4))$, where for any $R>0$ and $\theta>0$,

$$M_R(\theta)=\sup_{g\in H^1_{0,1}(Q_R)}\left\{\theta\sup_{y\in\mathbb{R}^d}\int_{Q_R}\frac{g^2(x)}{|y-x|^p}\,\mathrm{d}x-\frac{1}{2}\int_{Q_R}\left|\nabla g(x)\right|^2\mathrm{d}x\right\}.$$

Clearly, $M_R(\theta)$ is non-decreasing in R. Therefore,

$$M_3(\theta)\le\sup_{g\in H^1_{0,1}}\left\{\theta\sup_{y\in\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{g^2(x)}{|y-x|^p}\,\mathrm{d}x-\frac{1}{2}\int_{\mathbb{R}^d}\left|\nabla g(x)\right|^2\mathrm{d}x\right\}$$
$$=\sup_{y\in\mathbb{R}^d}\sup_{g\in H^1_{0,1}}\left\{\theta\int_{\mathbb{R}^d}\frac{g^2(x)}{|y-x|^p}\,\mathrm{d}x-\frac{1}{2}\int_{\mathbb{R}^d}\left|\nabla g(x)\right|^2\mathrm{d}x\right\}.$$

Notice that for any $g\in H^1_{0,1}$ and any $y\in\mathbb{R}^d$, the function $g^y(x)\equiv g(y-x)$ is also in $H^1_{0,1}$. Consequently, the right-hand side is equal to

$$M(\theta)\equiv\sup_{g\in H^1_{0,1}}\left\{\theta\int_{\mathbb{R}^d}\frac{g^2(x)}{|x|^p}\,\mathrm{d}x-\frac{1}{2}\int_{\mathbb{R}^d}\left|\nabla g(x)\right|^2\mathrm{d}x\right\}.$$

Take $p=1$ (p has a different meaning in [1]), $\beta=2$ and $\sigma=p$ (p here is given in this paper) in (1.19), [1], we have that $M(\theta)<\infty$ for all $\theta>0$. By the substitution

$$g(x)=\theta^{d/(2(2-p))}f\left(\theta^{1/(2-p)}x\right)$$

we have that $M(\theta)=\theta^{2/(2-p)}M(1)$. In particular,

$$\lambda^{2V_1}(Q_3)\le M(1)\left(2\theta\omega(Q_4)\right)^{2/(2-p)}.$$

Thus, there is a constant $C>0$ such that

$$\mathbb{P}\left\{\lambda^{2V_1}(Q_3)\ge Cn^{2/(2-p)}\right\}\le 4^{-n}.$$

In view of (4.15), the requested (4.14) follows from Borel–Cantelli lemma.

Remark. *Slightly more than claimed in Theorem* 1.5, (4.5) *can be strengthened into*

$$\mathbb{E}_0\exp\left\{\int_0^t\left|\overline{V}(B_s)\right|\mathrm{d}s\right\}<\infty\quad\text{a.s.}\tag{4.16}$$

This fact will be used in the next section.

Indeed, by Jensen's inequality and the homogeneity (2.23)

$$\mathbb{E}_0\otimes\mathbb{E}\exp\left\{\int_0^t\left|\overline{V}_2(B_s)\right|\mathrm{d}s\right\}\le\frac{1}{t}\int_0^t\mathbb{E}_0\otimes\mathbb{E}\exp\left\{t\left|\overline{V}_2(B_s)\right|\right\}\mathrm{d}s$$
$$=\mathbb{E}\exp\left\{t\left|\overline{V}_2(0)\right|\right\}\le\mathbb{E}\exp\left\{-t\overline{V}_2(0)\right\}+\mathbb{E}\exp\left\{t\overline{V}_2(0)\right\}.$$

The right-hand side is finite by Proposition 2.7. *Since the parameter θ hiding in* (1.7) *is arbitrary*, *the claimed* (4.16) *follows from* (4.6).

5. Proofs of Proposition 1.2 and Proposition 1.6

The representation (by a simple time-change)

$$\bar{u}_\pm(t,x) = \mathbb{E}_x \exp\left\{\pm \int_0^t \overline{V}(B_{2\kappa s})\,\mathrm{d}s\right\}$$

will be frequently used in this section.

By the fundamental theorem,

$$\exp\left\{\pm \int_0^t \overline{V}(B_{2\kappa s})\,\mathrm{d}s\right\} = 1 \pm \int_0^t \overline{V}(B_{2\kappa s}) \exp\left\{\pm \int_s^t \overline{V}(B_{2\kappa u})\,\mathrm{d}u\right\}\mathrm{d}s, \quad t \ge 0. \tag{5.1}$$

Our intent is to take the expectation $\mathbb{E}_x$ on the both sides of (5.1) and then apply the Fubini's theorem and the Markov property on the right-hand side:

$$\begin{aligned}
\bar{u}_\pm(t,x) &= 1 \pm \int_0^t \mathbb{E}_x\left[\overline{V}(B_{2\kappa s}) \exp\left\{\pm \int_s^t \overline{V}(B_{2\kappa u})\,\mathrm{d}u\right\}\right]\mathrm{d}s \\
&= 1 \pm \int_0^t \mathbb{E}_x\left[\overline{V}(B_{2\kappa s})\bar{u}_\pm(t-s, B_s)\right]\mathrm{d}s \\
&= 1 \pm \int_0^t \int_{\mathbb{R}^d} p_{2\kappa s}(y-x)\overline{V}(y)\bar{u}_\pm(t-s,y)\,\mathrm{d}y\,\mathrm{d}s \\
&= 1 \pm \int_0^t \int_{\mathbb{R}^d} p_{2\kappa(t-s)}(y-x)\overline{V}(y)\bar{u}_\pm(s,y)\,\mathrm{d}y\,\mathrm{d}s.
\end{aligned}$$

In order to make the above calculation rigorous, we need to prove

$$\mathbb{E}_x \int_0^t \left|\overline{V}(B_{2\kappa s})\right| \exp\left\{\pm \int_s^t \overline{V}(B_{2\kappa u})\,\mathrm{d}u\right\}\mathrm{d}s < +\infty \quad \text{a.s.}, x \in \mathbb{R}^d, t \ge 0,$$

for appropriate choice of the sign $\pm$ under the exponent.

By homogeneity (2.23), all we need to prove is

$$\mathbb{E}_0 \int_0^t \left|\overline{V}(B_{2\kappa s})\right| \exp\left\{\pm \int_s^t \overline{V}(B_{2\kappa u})\,\mathrm{d}u\right\}\mathrm{d}s < +\infty \quad \text{a.s.}, t \ge 0. \tag{5.2}$$

In the context of Proposition 1.6, by (4.16) we have

$$\mathbb{E}_0 \exp\left\{2 \int_0^t \left|\overline{V}(B_{2\kappa s})\right|\mathrm{d}s\right\} < \infty \quad \text{a.s.}$$

This fact, combined with elementary inequalities

$$\int_0^t \left|\overline{V}(B_{2\kappa s})\right| \exp\left\{\int_s^t \left|\overline{V}(B_{2\kappa u})\right|\mathrm{d}u\right\}\mathrm{d}s \le \left\{\int_0^t \left|\overline{V}(B_{2\kappa s})\right|\mathrm{d}s\right\} \exp\left\{\int_0^t \left|\overline{V}(B_{2\kappa s})\right|\mathrm{d}s\right\}$$

and $x\mathrm{e}^x \le \mathrm{e}^{2x}$, $x > 0$, provides (5.2).

In the context of Proposition 1.2, we take $h > 0$ and write $K = K_1 + K_2$ with $K_1(x) = (K(x) - h)_+$, and $K_2(x) = \min(K(x), h)$. The random potential $\overline{V}(x)$ is decomposed into a sum $\overline{V} = V_1 - C + \overline{V}_2$ with

$$V_1(x) = \int_{\mathbb{R}^d} K_1(y-x)\omega(\mathrm{d}y), \qquad C = \int_{\mathbb{R}^d} K_1(y)\,\mathrm{d}y, \qquad \overline{V}_2(x) = \int_{\mathbb{R}^d} K_2(y-x)\left[\omega(\mathrm{d}y) - \mathrm{d}y\right].$$

Therefore,

$$\begin{aligned}\int_0^t \overline{V}(B_{2\kappa s})\exp\left\{-\int_s^t \overline{V}(B_{2\kappa u})\,\mathrm{d}u\right\}\mathrm{d}s &= \int_0^t V_1(B_{2\kappa s})\exp\left\{-\int_s^t \overline{V}(B_{2\kappa u})\,\mathrm{d}u\right\}\mathrm{d}s \\ &\quad - C\int_0^t \exp\left\{-\int_s^t \overline{V}(B_{2\kappa u})\,\mathrm{d}u\right\}\mathrm{d}s \\ &\quad + \int_0^t \overline{V}_2(B_{2\kappa s})\exp\left\{-\int_s^t \overline{V}(B_{2\kappa u})\,\mathrm{d}u\right\}\mathrm{d}s. \end{aligned}\tag{5.3}$$

Write

$$\xi_2(t,x) = \int_0^t K_2(B_{2\kappa s} - x)\,\mathrm{d}s.$$

By Fubini's theorem, (3.5), (2.15) and Jensen's inequality

$$\begin{aligned}\mathbb{E}\otimes\mathbb{E}_0\exp\left\{2\int_0^t \overline{V}_2(B_{2\kappa s})\,\mathrm{d}s\right\} &= \mathbb{E}_0\exp\left\{\int_{\mathbb{R}^d}\Psi\big(2\xi_2(t,y)\big)\,\mathrm{d}y\right\} \\ &\le \exp\left\{\int_{\mathbb{R}^d}\Psi\big(2tK_2(x)\big)\,\mathrm{d}x\right\} < \infty,\end{aligned}$$

where the last step follows from the boundedness of $K_2(\cdot)$ which gives the bound $\Psi(2tK_2(x)) \le C_t K_2^2(x)$.

Combined with Theorem 1.1, this yields

$$\mathbb{E}_0\exp\left\{2\int_0^t \left|\overline{V}_2(B_{2\kappa s})\right|\mathrm{d}s\right\} < +\infty \quad \text{a.s.}\tag{5.4}$$

Because $V_1(x)\ge 0$, we have

$$-\overline{V}(x) \le C + \left|\overline{V}_2(x)\right|.$$

Therefore the 2nd and the 3rd terms in the right-hand side of (5.3) can be estimated by

$$C\mathrm{e}^{Ct}\int_0^t \exp\left\{\int_s^t \left|\overline{V}_2(B_{2\kappa u})\right|\mathrm{d}u\right\}\mathrm{d}s \quad \text{and} \quad \mathrm{e}^{Ct}\int_0^t \left|\overline{V}_2(B_{2\kappa s})\right|\exp\left\{\int_s^t \left|\overline{V}_2(B_{2\kappa u})\right|\mathrm{d}u\right\}\mathrm{d}s,$$

respectively, and these expressions are a.s. integrable w.r.t. $\mathbb{E}_0$ because of (5.4). On the other hand, by the fundamental theorem, the left-hand side of (5.3) equals

$$1 - \exp\left\{-\int_0^t \overline{V}(B_{2\kappa s})\,\mathrm{d}s\right\}.$$

Consequently, it is bounded and therefore is integrable w.r.t. $\mathbb{E}_x$. This, together with positivity of V_1, provides integrability for the 1st term in the right-hand side of (5.3):

$$\begin{aligned}&\int_0^t \mathbb{E}_0 V_1(B_{2\kappa s})\exp\left\{-\int_s^t \overline{V}(B_{2\kappa u})\,\mathrm{d}u\right\}\mathrm{d}s \\ &\quad = \mathbb{E}_0\int_0^t V_1(B_{2\kappa s})\exp\left\{-\int_s^t \overline{V}(B_{2\kappa u})\,\mathrm{d}u\right\}\mathrm{d}s < +\infty \quad \text{a.s.}\end{aligned}$$

This, combined with the above estimates for 2nd and 3rd terms in the right-hand side of (5.3), provides (5.2).

6. Partial renormalization

For $K(x) = \theta|x|^{-p}$, corresponding Poisson potentials $\overline{V}(x)$ and $V(x)$ are well defined for $p \in (d/2, d)$ and $p > d$, respectively. Corresponding Gibbs measures are defined, for these two intervals of the values of p, by different relations: (1.10), (1.11) for $p \in (d/2, d)$ and (1.2), (1.3) for $p \in (d, +\infty)$. This is inconvenient, and it is a natural question whether the definition can be designed in a uniform way for the whole range of p. Another inconvenience is that our basic definition does not work for $p = d$, and this exception is non-natural. In order to resolve these inconveniences we introduce the following construction, which contains the Poisson potentials $V(x), \overline{V}(x)$ as particular cases.

For a measurable $K : \mathbb{R}^d \to \mathbb{R}^+$ and $h \in [0, \infty]$ denote

$$K_h(x) = \min\bigl(h, K(x)\bigr), \qquad K^h(x) = \bigl(K(x) - h\bigr)_+, \quad x \in \mathbb{R}^d.$$

Clearly, $K = K_h + K^h$ and the functions K_h, K^h are supported by distinct sets. Define the *Poisson potential renormalized at the level h* by

$$V^h(x) = \int_{\mathbb{R}^d} K_h(x-y)\bigl[\omega(\mathrm{d}y) - \mathrm{d}y\bigr] + \int_{\mathbb{R}^d} K^h(x-y)\omega(\mathrm{d}y)$$

assuming respective integrals to be well defined. By the definition, $V^0 = V, V^\infty = \overline{V}$, and the conditions for the respective integrals to exist are given in Section 2. For $h \in (0, +\infty)$ the potential V^h, which we call a *partially renormalized* one, is well defined if and only if

$$\int_{\mathbb{R}^d} \bigl[\psi\bigl(K_h(x)\bigr) + \varphi\bigl(K^h(x)\bigr)\bigr]\,\mathrm{d}x < +\infty.$$

One can easily verify that this condition is equivalent to

$$\int_{\mathbb{R}^d} \min\bigl(1, K^2(x)\bigr)\,\mathrm{d}x < +\infty, \tag{6.1}$$

which is weaker than the conditions that provide existence of either V or $\overline{V}$. The most important example is $K(x) = |x|^{-p}$. In this case:

- $V = V^0$ is well defined for $p > d$;
- $\overline{V} = V^\infty$ is well defined for $p \in (d/2, d)$;
- V^h is well defined for $p > d/2$ for any $h \in (0, \infty)$.

The following proposition shows that the Poisson potentials renormalized at levels h_1, h_2 coincide up to an explicit constant assuming these potentials exist. Heuristically this means that the partial normalization procedure is consistent in the sense that the choice of the renormalization level does not effect respective "physical" objects, see Proposition 6.2 below. What does this choice effect indeed is the range of the definition of the model.

Proposition 6.1. *Let $h_1, h_2 \in [0, \infty]$, $h_1 < h_2$ be such that V^{h_1}, V^{h_2} are well defined. Then $V^{h_1}(x) = V^{h_2}(x) + C_{K,h_1,h_2}$ with*

$$C_{K,h_1,h_2} = \int_{\mathbb{R}^d} \bigl(\min\bigl(h_2, K(y)\bigr) - h_1\bigr)_+\,\mathrm{d}y \in [0, +\infty).$$

Proof. It is sufficient to consider three cases: (i) $h_1 = 0, h_2 \in (0, \infty)$; (ii) $h_1, h_2 \in (0, \infty)$; (iii) $h_1 \in (0, \infty), h_2 = \infty$. By similarity, we consider case (i), only. Denote $h = h_2$. Since V is well defined, we have

$$\int_{\mathbb{R}^d} \min\bigl(1, K(x)\bigr)\,\mathrm{d}x < +\infty$$

and, in particular, $C_{K,0,h} = \int_{\mathbb{R}^d} \min(h, K(y))\,\mathrm{d}y \in [0, +\infty)$. Consider an increasing sequence of non-negative simple functions $\{K_n\}_{n\geq 1}$ that tends to K. Writing the integrals of simple functions as integral sums, one can easily verify the identity

$$\int_{\mathbb{R}^d} K_n(y-x)\omega(\mathrm{d}y) = C_{K_n,0,h} + \int_{\mathbb{R}^d} (K_n)^h(y-x)\omega(\mathrm{d}y) + \int_{\mathbb{R}^d} (K_n)_h(y-x)\tilde{\omega}(\mathrm{d}y). \tag{6.2}$$

One has

$$C_{K_n,0,h} = \int_{\mathbb{R}^d} \min\bigl(h, K_n(y)\bigr)\,\mathrm{d}y \to \int_{\mathbb{R}^d} \min\bigl(h, K(y)\bigr)\,\mathrm{d}y = C_{K,0,h}.$$

Applying Proposition 2.4 to the last two summations in the r.h.s. of (6.2) completes the proof. □

Let K satisfy (6.1). Consider $h \in (0,\infty)$ and consider separately the integrals in the definition of the potential V^h:

$$U^h(x) = \int_{\mathbb{R}^d} K_h(x-y)\bigl[\omega(\mathrm{d}y) - \mathrm{d}y\bigr], \qquad W^h(x) = \int_{\mathbb{R}^d} K^h(x-y)\omega(\mathrm{d}y).$$

Clearly, the proof of Proposition 2.8 applies to U^h, W^h, hence both these summations and the potential V^h itself can be chosen in a measurable way. Proposition 3.1 applied to K_h and K^h instead of K provides that

$$\int_0^t \bigl|V^h(B_s)\bigr|\,\mathrm{d}s \leq \int_0^t \bigl[\bigl|U^h(B_s)\bigr| + W_h(B_s)\bigr]\,\mathrm{d}s < \infty \quad \text{a.s.},$$

$$\int_0^t V^h(B_s)\,\mathrm{d}s = \int_0^t U^h(B_s)\,\mathrm{d}s + \int_0^t W^h(B_s)\,\mathrm{d}s = \int_{\mathbb{R}^d} \xi_h(t,x)\bigl[\omega(\mathrm{d}x) - \mathrm{d}x\bigr] + \int_{\mathbb{R}^d} \xi^h(t,x)\omega(\mathrm{d}x)$$

with

$$\xi_h(t,x) = \int_0^t K_h(x-B_s)\,\mathrm{d}s, \qquad \xi^h(t,x) = \int_0^t K^h(x-B_s)\,\mathrm{d}s.$$

Analogously to the proof of (1.15) in the proof of Theorem 1.1, one can verify that all the negative exponential moments for $\int_0^t U^h(B_s)\,\mathrm{d}s$ are finite. In addition, $\int_0^t W^h(B_s)\,\mathrm{d}s$ is non-negative. Combining these observations with Proposition 6.1 we get the following extension of Theorem 1.1.

Proposition 6.2. *Let the shape function $K(x) \geq 0$ and $h \in [0,\infty]$ be such that the renormalized Poisson potential $V^h(x)$ is well defined in the point-wise sense* (*see Section* 3 *and* (6.1)).

Then the random field $\{V^h(x), x \in \mathbb{R}^d\}$ has a measurable modification and for any $t > 0$

$$\mathbb{E}_0 \otimes \mathbb{E} \exp\biggl\{-\int_0^t V^h(B_s)\,\mathrm{d}s\biggr\} < \infty.$$

The Gibbs measures

$$\mu_{t,\omega}(A) = \frac{1}{Z^h_{t,\omega}} \exp\biggl\{-\int_0^t V^h(B_s)\,\mathrm{d}s\biggr\}\mathbb{P}_0(A), \qquad Z^h_{t,\omega} = \mathbb{E}_0 \exp\biggl\{-\int_0^t V^h(B_s)\,\mathrm{d}s\biggr\} \tag{6.3}$$

and

$$\mu_t(A) = \frac{1}{Z^h_t} \exp\biggl\{-\int_0^t V^h(B_s)\,\mathrm{d}s\biggr\}\mathbb{P}_0 \otimes \mathbb{P}(A), \qquad Z^h_t = \mathbb{E}_0 \otimes \mathbb{E} \exp\biggl\{-\int_0^t V^h(B_s)\,\mathrm{d}s\biggr\} \tag{6.4}$$

are well defined and do not depend on the choice of h in the sense that, as soon as the renormalized Poisson potentials exist for two values $h = h_1, h_2$, respective Gibbs measures coincide.

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