Homework # 1 (Chapter 2)

1. The density of U is

$$f_U(u) = \int_{-\infty}^{\infty} f(u,v)dv = g(u)\int_{-\infty}^{\infty} g(v)(1-\sin u \sin v)dv$$
$$= g(u)\Big\{\int_{-\infty}^{\infty} g(v)dv - \sin u \int_{-\infty}^{\infty} g(v)\sin v dv\Big\} = g(u)$$

Similarly, $f_V(v) = g(v)$. Take

$$g(u) = \frac{1}{\sqrt{2\pi}}\exp\Big\{-\frac{u^2}{2}\Big\}$$

Then $U, V \sim N(0,1)$ but (U,V) is not normal as $f(u,v)$ is not two-dimentional normal density.

8. One can check that all of them are mean-zero Gaussian processes. $X_t = 2B_{t/4}$ is BM by 2.16 (scaling), p.15 with $c = 1/4$.

The rest are not Brownian motion by checking the covariance function.

16. (a). As W_t is mean-zero Gaussian with continuous sample path and the covariance

$$E(W_sW_t) = \frac{1}{2}E(b_t+\beta_t)(b_s+\beta_s) = \frac{1}{2}\Big\{Eb_tb_s + Eb_t\beta_s + E\beta_tb_s + E\beta_t\beta_s\Big\}$$
$$= \frac{1}{2}\Big\{s\wedge t + 0 + 0 + s\wedge t\Big\} = s\wedge t$$

(b). In both cases, the components are 1-dimensional Brownian motions. So the answer depends on whether the components are independent.

Notice that $EW_t\beta_1 = \frac{1}{\sqrt{2}}E(b_t+\beta_t)\beta_t = \frac{1}{\sqrt{2}}t \neq 0$. Consequently, the processes W_t and β_1 are not independent.

On the other hand, for any $s, t \geq 0$,

$$E(b_s+\beta_s)(b_t-\beta_t) = E(b_sb_t) - E(\beta_s\beta_t) = 0$$

So $\frac{b_t+\beta_t}{\sqrt{2}}$ and $\frac{b_t-\beta_t}{\sqrt{2}}$ are two independent 1-dim Brownian motions.

17. The condition is $\lambda^2 + \mu^2 = 1$.

18. No, as $\gamma_t = \beta_{s-t} - \beta_t$ $(0 \leq t \leq s)$ is not 1-dim BM. Indeed,

$$E\gamma_t^2 = E\big[\beta_{s-t} - \beta_t\big]^2 = s - 2t \neq t$$

as $t < s/2$

22. Since $-B_s$ is an 1-dimensional Brownian motion

$$\tau_{-a} = \inf\{s \geq 0; \quad B_s = -a\} = \inf\{s \geq 0; \quad -B_s = a\} \stackrel{d}{=} \inf\{s \geq 0; \quad B_s = a\} = \tau_a$$

Similarly, by the fact that

$$\tau_a = \inf\{s \geq 0; \quad B_s = a\} = \inf\{s \geq 0; \quad a^{-1}B_s = 1\} = \inf\{a^2 s \geq 0; \quad a^{-1}B_{a^2 s} = 1\}$$
$$= a^2 \inf\{s \geq 0; \quad a^{-1}B_{a^2 s} = 1\} \stackrel{d}{=} a^2 \inf\{s \geq 0; \quad B_s = 1\} = a^2 \tau_1$$

24. First, W_t is continuous in t: The continuity at $t \neq 1$ is obvious. The continuity at $t = 1$ follows from

$$\lim_{t \to 1} W_t = B_1 = W_1 \qquad \text{(follows from continuity of } B_t \text{ and } \beta_t)$$

Second, W_t is a mean-zero Gaussian: Indeed, it is obvious that $EW_t = 0$ (need to check for $t \leq 1$ and $t > 1$ separately). In addition, for any $t_1, \cdots, t_n > 0$, any linear combination $C_1 W_{t_1} + \cdots C_n W_{t_n}$ can be written as a linear combination of $B_{t_1}, \cdots, B_{t_n}; (\beta_{1/t_1} - \beta_1), \cdots, (\beta_{1/t_n} - \beta_1)$. Or, in the form

$$\Big(C_1' B_{t_1} + \cdots + C_n' B_{t_n}\Big) + \Big(C"_1(\beta_{1/t_1} - \beta_1) + \cdots + C"_n(\beta_{1/t_n} - \beta_1)\Big)$$

which is the sum of two independent 1-dimensional normals, and is therefore normal.

Finally, it remain to show that $EW_s W_t = s \wedge t$. This is straightforward by showing $EW_s W_t = s \wedge t$ for $s, t \leq$ and for $s, t > 1$, and $EW_s W_t = s$ for $s \leq 1 < t$.

26. Clearly, the process $W_t = B_{a-t} - B_a$ $(0 \leq t \leq a)$ is mean-zero Gaussian and path-wise continuous. It remains to verify its covariance function. For any $0 \leq s, t \leq a$,

$$E(W_s W_t) = E\big(B_{a-s} - B_a\big)\big(B_{a-t} - B_a\big) = E\Big\{B_{a-s}B_{a-t} - B_a B_{a-s} - B_a B_{a-t} + B_a^2\Big\}$$
$$= (a-s) \wedge (a-t) - (a-s) - (a-t) + a = s \wedge t$$