Homework # 2

Chapter 5

5. We use the hint: Notice that

$$E\exp\left\{\xi B_\tau-\frac{1}{2}\xi^2\tau\right\}=1$$

for any stopping time τ with

$$E\exp\left\{\frac{1}{2}\xi^2\tau\right\}<\infty$$

By Taylor expansion

$$\exp\left\{\xi B_\tau-\frac{1}{2}\xi^2\tau\right\}=\exp\{\xi B_\tau\}\exp\left\{-\frac{1}{2}\xi^2\tau\right\}$$

$$=\left(\sum_{n=0}^\infty\frac{1}{n!}B_\tau^n\xi^n\right)\left(\sum_{n=0}^\infty\frac{(-1)^n}{2^n n!}\tau^n\xi^{2n}\right)=\left(\sum_{n=0}^\infty a_n\xi^n\right)\left(\sum_{n=0}^\infty b_n\xi^{2n}\right)$$

Write the right hand side as the single power series

$$\sum_{n=0}^\infty c_n\xi^n$$

and find out

$$c_4=a_0b_2+a_2b_1+a_4b_0=\frac{1}{8}\tau^2-\frac{1}{4}\tau B_\tau^2+\frac{1}{24}B_\tau^4$$

By the unqueness of Taylor expansion and expponential Wald's identity

$$E\left\{\frac{1}{8}\tau^2-\frac{1}{4}\tau B_\tau^2+\frac{1}{24}B_\tau^4\right\}=0$$

This suggests that

$$M_t=\frac{1}{8}t^2-\frac{1}{4}tB_t^2+\frac{1}{24}B_t^4$$

is a martingale. There are two ways to verify it: One is to directly check the martingale identity $E[M_t|\mathcal{F}_s]=M_s$ for any $s<t$. Another way is to consider the function

$$\pi(t,x)=\frac{1}{8}t^2-\frac{1}{4}tx^2+\frac{1}{24}x^4$$

and to verify that

$$\frac{\partial\pi}{\partial t}(t,x)+\frac{1}{2}\frac{\partial^2\pi}{\partial t^2}(t,x)=0$$

10. (a). No, unless $c=0$ (a trivial case). Indeed,

$$EU_t=Ee^{cB_t}=\exp\left\{\frac{1}{2}c^2t\right\}\neq 1=EU_0$$

(b). Yes. Take $f(t,x) = tx$ in Theorem 5.6, p.51.

(c) No, by direct computation for $s < t$

$$\begin{aligned}
&E[W_t|\mathcal{F}_s] = E\big[\big(B_s + (B_t - B_s)\big)^3 - tB_t|\mathcal{F}_s\big] \\
&= E\Big\{B_s^3 + 3B_s(B_t - B_s)^2 + 3B_s^2(B_t - B_s) + (B_t - B_s)^3 - tB_t|\mathcal{F}_s\Big\} \\
&= B_s^3 + 3B_s(t-s) - tB_s \neq W_s
\end{aligned}$$

(d), Yes. Take $f(x) = x^3$ in Theorem 5.6, p.51.

(e). Yes, by a direct computation

$$\begin{aligned}
&E[Y_t|\mathcal{F}_s] = E\big[\frac{1}{3}\big(B_s + (B_t - B_s)\big)^3 - tB_t|\mathcal{F}_s\big] \\
&= E\Big\{\frac{1}{3}B_s^3 + B_s(B_t - B_s)^2 + B_s^2(B_t - B_s) + \frac{1}{3}(B_t - B_s)^3 - tB_t|\mathcal{F}_s\Big\} \\
&= \frac{1}{3}B_s^3 + B_s(t-s) - tB_s = \frac{1}{3}B_s^3 - sB_s = Y_s
\end{aligned}$$

(f). Yes when $c = 1/2$. No when $c \neq 1/2$.

11. Set $\tilde{f}(t,x) = f(t)x$. By Theorem 5.6, p.51.

$$M_t = \tilde{f}(t, B_t) - \int_0^t L\tilde{f}(s, B_s)ds$$

is a martingale. The conclusion follows from

$$L\tilde{f}(t,x) = \frac{\partial \tilde{f}}{\partial t} + \frac{\partial^2 \tilde{f}}{\partial x^2} = f'(t)x$$

20. (a). Write $R = \max\{a, b\}$ and define $\tau_R = \{t \geq 0;\ \ |B_t| \geq R\}$. We have that $\tau \leq \tau_R$. We claim that

$$E\exp\{\theta\tau_R\} < \infty \qquad \forall\theta < \frac{1}{2}\Big(\frac{\pi}{2R}\Big)^2$$

By the relation $e^x \geq x^n/n!$ for any $x \geq 0$, this leads to $E\tau^n \leq E\tau_R^n < \infty$ for any $n \geq 1$.

Let $\xi \in \mathbf{R}$ be fixed and consider the complex valued martingale

$$M_t = \exp\Big\{i\xi B_t + \frac{|\xi|^2}{2}t\Big\}$$

is a martingale. By Doob's stopping rule, $M_{\tau_R \wedge t} = EM_0 = 1$. Or

$$E\exp\Big\{i\xi B_{\tau_R \wedge t} + \frac{|\xi|^2}{2}(\tau \wedge t)\Big\} = 1$$

Replacing ξ by $-\xi$,

$$E\exp\Big\{-i\xi B_{\tau_R\wedge t}+\frac{|\xi|^2}{2}(\tau\wedge t)\Big\}=1$$

Adding two equations together and by Euler's formula,

$$E\cos(\xi B_{\tau_R\wedge t})\exp\Big\{\frac{|\xi|^2}{2}(\tau\wedge t)\Big\}=1$$

We now let $|\xi|<\pi/(2R)$. Notice that

$$0<\cos(|\xi|R)\le\cos(\xi B_{\tau_R\wedge t})$$

So we have that

$$E\exp\Big\{\frac{|\xi|^2}{2}(\tau_R\wedge t)\Big\}\le\cos^{-1}(|\xi|R)<\infty$$

Let $t\to\infty$. By Fatou lemma, or by monotonic convergence,

$$E\exp\Big\{\frac{|\xi|^2}{2}\tau_R\Big\}\le\cos^{-1}(|\xi|R)<\infty$$

as far as $|\xi|<\pi/(2R)$. Letting $\theta=\frac{|\xi|^2}{2}$. leads to the claimed exponential integrability.

(b). Set $f(x)=\frac{1}{3}x^3$. Then $\frac{1}{2}f"(x)=x$. By Theorem 5.6, p.51,

$$M_t=f(B_t)-\frac{1}{2}\int_0^t f"(B_s)ds=\frac{1}{3}B_t^3-\int_0^t B_s ds$$

is a martingale. By Doob's stopping rule, $EM_{\tau\wedge t}=EM_0=0$. Or,

$$E\int_0^{\tau\wedge t}B_s ds=\frac{1}{3}EB_{\tau\wedge t}^3$$

Notice that

$$|B_{\tau\wedge t}|^3\le(a\vee b)^3$$

By dominated control,

$$\lim_{t\to\infty}EB_{\tau\wedge t}^3=EB_\tau^3=(-a)^3P\{B_\tau=-a\}+b^3P\{B_\tau=b\}=\frac{-a^3b\cdot}{a+b}+\frac{b^3a\cdot}{a+b}=ab(b-a)$$

Here we have used Corollary 5.11, p.56.

To justify the step "$t\to\infty$" on the left hand side, notice that

$$\Big|\int_0^{\tau\wedge t}B_s ds\Big|\le\int_0^{\tau\wedge t}|B_s|ds\le\int_0^{\tau\wedge t}(a\vee b)ds\le(a\vee b)\tau$$

By the fact (corollary 5.11, p.56) that $E\tau = ab < \infty$, and by dominated control again,

$$\lim_{t\infty} E\int_0^{\tau\wedge t} B_s ds = E\int_0^{\tau} B_s ds$$

In summary,

$$E\int_0^{\tau} B_s ds = \frac{1}{3}ab(b-a)$$

21. One way to do it is to use Doob's stopping rule to the bounded stopping time $\tau_R \wedge t$ and to the martingale $M_t = |B_t|^2 - dt$. So we have $EM_{\tau\wedge t} = EM_0 = 0$. Or

$$E|B_{\tau_R\wedge t}|^2 = dE(\tau_R\wedge t)$$

By monotonic convergence

$$\lim_{t\to\infty} E(\tau_R\wedge t) = E\tau_R$$

By the fact that $|B_{\tau_R\wedge t}|^2 \le R^2$, the right hand side is finite and, by dominated convergence

$$\lim_{t\to\infty} E|B_{\tau_R\wedge t}|^2 = E|B_{\tau_R}|^2 = R^2$$

In summary,

$$E\tau_R = d^{-1}E|B_{\tau_R}|^2 = d^{-1}R^2$$

22. (a). For any $t \ge 0$,

$$\{\tau\wedge\sigma\le t\} = \{\tau\le t\}\cup\{\sigma\le t\}\in\mathcal{F}_t$$

(b). For any $t\ge 0$,

$$\{\sigma\le\tau\}\cap\{\tau\wedge\sigma\le t\} = \{\sigma\le\tau\}\cap\{\sigma\le t\} = \{\sigma\le\tau\wedge t\}\in\mathcal{F}_{\tau\wedge t}\subset\mathcal{F}_t$$

(c). We first assume that τ and σ are bounded.

$$E(B_\sigma B_\tau) = E(B_\sigma B_\tau)1_{\{\sigma\le\tau\}} + E(B_\sigma B_\tau)1_{\{\tau<\sigma\}}$$

For the first term,

$$\begin{aligned} E(B_\sigma B_\tau)1_{\{\sigma\le\tau\}} &= E\Big(B_{\sigma\wedge\tau}1_{\{\sigma\le\tau\}}B_\tau)\Big) = E(B_\sigma B_\tau)1_{\{\sigma\le\tau\}} \\ &= E\Big(B_{\sigma\wedge\tau}1_{\{\sigma\le\tau\}}E\big[B_\tau)|\mathcal{F}_{\sigma\wedge\tau}\big]\Big) = E\Big(B^2_{\sigma\wedge\tau}1_{\{\sigma\le\tau\}}\Big) \end{aligned}$$

Rotating the roles of σ and τ,

$$E(B_\sigma B_\tau)1_{\{\tau<\sigma\}} = E\Big(B^2_{\sigma\wedge\tau}1_{\{\sigma>\tau\}}\Big)$$

Thus,

$$E(B_\sigma B_\tau) = E\Big(B^2_{\sigma\wedge\tau}\Big) = E(\sigma\wedge\tau)$$

where the last step follows from Wald's second identity.

We now drop the boundedness assumption on τ, σ. Applying what have been proved to $\tau\wedge t$ and $\sigma\wedge t$

$$E(B_{\sigma\wedge t}B_{\tau\wedge t}) = E(\sigma\wedge\tau\wedge t)$$

We now let $t\to\infty$ on the both side. By monotonic convergence,

$$\lim_{t\to\infty} E(\sigma\wedge\tau\wedge t) = E(\sigma\wedge\tau)$$

It remains to show

$$\lim_{t\to\infty} E(B_{\sigma\wedge t}B_{\tau\wedge t}) = E(B_\sigma B_\tau)$$

Indeed, by dominated convergence theorem, all we need is that

$$E\sup_{t\ge 1}|B_{\sigma\wedge t}||B_{\tau\wedge t}| < \infty$$

By Cauchy, maximum inequalities and Wald's identity

$$\begin{aligned}&E\sup_{t\ge 1}|B_{\sigma\wedge t}||B_{\tau\wedge t}| \le \Big\{E\sup_{t\ge 1}|B_{\sigma\wedge t}|^2\Big\}^{1/2}\Big\{E\sup_{t\ge 1}|B_{\tau\wedge t}|^2\Big\}^{1/2}\\ &\le \Big\{4E|B_\sigma|^2\Big\}^{1/2}\Big\{4E|B_\tau|^2\Big\}^{1/2} = 4\Big\{E\sigma\Big\}^{1/2}\Big\{E\tau\Big\}^{1/2} < \infty\end{aligned}$$

(d).

$$\begin{aligned}&E|B_\tau - B_\sigma|^2 = EB^2_\tau + EB^2_\sigma - 2E(B_\sigma B_\tau) = E\tau + E\sigma - 2E(\sigma\wedge\tau)\\ &= E\big(\tau+\sigma-2\sigma\wedge\tau\big) = E|\tau-\sigma|\end{aligned}$$