Homework # 3 (Chapter 6)

1. (a), (b).

$$E\big(u(X_{s+t})|\mathcal{F}_s\big) = E\big(u(|B_{s+t}|)|\mathcal{F}_s\big) = Eu(|x+B_t|)\Big|_{x=B_s}$$

Replacing B_t by $-B_t$,

$$Eu(|x+B_t|) = Eu(|-x+B_t|) \qquad x \in \mathbf{R}$$

Consequently,

$$E\big(u(|x+B_t|)|\mathcal{F}_s\big) = Eu(|x+B_t|)\Big|_{x=|B_s|}$$

So X_t is a Markov process with transition given by

$$Eu\big(X_{s+t}|X_s\big) = Eu(|x+B_t|)\Big|_{x=X_s} \tag{1}$$

(c). Write $M_t = \max_{u\le t} B_u$. Notice that

$$Y_{s+t} = M_{s+t} - B_{s+t} = M_s \vee \max_{s\le u\le s+t} B_u - B_{s+t} = Y_s \vee \max_{u\le t}(B_{s+u} - B_s) - (B_{s+t} - B_s)$$

Hence, by increment independence,

$$E\big(u(Y_{s+t})|\mathcal{F}_s\big) = Eu\big(y \vee \max_{u\le t} B_u - B_t\big)\Big|_{y=Y_s} \tag{2}$$

This shows that the process Y_t is a Markov process with the transition mechanism given as above.

(d). We claim that for any $y \ge 0$,

$$Eu\big(y \vee \max_{u\le t} B_u - B_t\big) = Eu(|y+B_t|) \tag{3}$$

In comparison (3) to (1) and (2), we find that Y_t and X_t have the same transition function. Further, notice that $X_0 = Y_0 = 0$. Hence X_t and Y_t have identical finite dimensional distributions. So the comparison leads to Lévy's identity in law

$$\{|B_t|;\quad t\ge 0\} \overset{d}{=} \{\max_{s\le t} B_s - B_t;\quad t \ge 0\}$$

To prove (3), all we need is to show that for any fixed $y \ge 0$,

$$y \vee \max_{u\le t} B_u - B_t \overset{d}{=} |y + B_t| \tag{4}$$

Indeed,

$$y \vee \max_{u \le t} B_u - B_t = (y - B_t) \vee \max_{u \le t}(B_u - B_t) = (y - B_t) \vee \max_{u \le t}(B_{t-u} - B_t)$$
$$= (y + \tilde{B}_t) \vee \max_{u \le t} \tilde{B}_u \stackrel{d}{=} (y + B_t) \vee M_t$$

where the last step follows from the fact that $\tilde{B}_u = B_{t-u} - B_t$ $(0 \le u \le t)$ is a Brownian motion.

It remains to show that

$$(y + B_t) \vee M_t \stackrel{d}{=} |y + B_t| \tag{5}$$

Given $b > 0$,

$$P\{(y + B_t) \vee M_t > b\} = P\{y + B_t > b\} + P\{y + B_t \le b, \ M_t > b\}$$

Let $\tau_b = \inf\{u \ge 0; B_u \ge b\}$. For the second term,

$$P\{y + B_t \le b, \ M_t > b\} = P\{y + (B_t - B_{\tau_b}) \le 0, \ \tau_b \le t\}$$
$$= P\{y - (B_t - B_{\tau_b}) \le 0, \ \tau_b \le t\} = P\{B_t \ge b + y, \ M_t > b\}$$
$$= P\{B_t \ge b + y\} = P\{y + B_t \le -b\}$$

where the second step follows from reflection, the fourth step follows from the relation $b + y \ge b$ and the last step follows from the equality in law $B_t \stackrel{d}{=} -B_t$.

Therefore,

$$P\{(y + B_t) \vee M_t > b\} = P\{y + B_t > b\} + P\{y + B_t \le -b\} = P\{|y + B_t| > b\}$$

So we have (5).

5.(a) For any $s, t \ge 0$

$$(B_{s+t}, M_{s+t}) = \Big(B_s + (B_{s+t} - B_s), \quad M_s \vee \max_{s \le u \le s+t} B_u\Big)$$
$$= \Big(B_s + (B_{s+t} - B_s), \quad M_s \vee \{B_s + \max_{u \le s+t}(B_{s+u} - B_s)\}\Big)$$

Notice that

$$\Big((B_{s+t} - B_s, \max_{u \le s+t}(B_{s+u} - B_s)\Big)$$

is independent of $\mathcal{F}_s = \sigma(B_u; \quad u \le s)$ and has the same distribution as (B_t, M_t). Thus, for any bounded function $u(x, y)$ on $\mathbf{R}^2$,

$$E\Big[u(B_{s+t}, M_t)|\mathcal{F}_s\Big] = Eu\big(x + B_t, y \vee \{x + M_t\}\big)\Big|_{(x,y)=(B_s,M_s)}$$

(b) Notic that

$$(B_{s+t}, I_{s+t}) = \left(B_s + (B_{s+t} - B_s), \quad tB_s + I_s + \int_0^t (B_{s+u} - B_s)du\right)$$

and that

$$\left(B_{s+t} - B_s, \quad \int_0^t (B_{s+u} - B_s)du\right)$$

is independent of $\mathcal{F}_s$ and has the same distribution as (B_t, I_t).

$$E\left[u(B_{s+t}, I_{s+t})\Big|\mathcal{F}_s\right] = Eu(x + B_t, tx + y + I_t)\Big|_{(x,y)=(B_s,I_s)} = \varphi_t(B_s, I_s)$$

(c). For any bounded $u(y)$

$$E\left[u(M_{s+t})|\mathcal{F}_s\right] = Eu(y \vee \{x + M_t\})\Big|_{(x,y)=(B_s,M_s)}$$

The right hand side not only depends on M_s, but also on B_s. The dependence on B_s disqualifies M_t as a Markov process.

Similarly, I_t is not Markov process.

Remark. This problem show a way of Markovianization of some non-Markovian processes. According to Problem #1, another way to make Markov process based on M_t is to consider $M_t - B_t$ instead.

8. Let $\tau = \inf\{t \geq 0; \quad |B_t| \geq x\}$. Then

$$P\left\{\max_{s\leq t} |B_s| \geq x\right\} = P\{\tau \leq t\}$$

Notice that

$$B_\tau = \frac{1}{2}\Big(B_\tau + (B_t - B_\tau)\Big) + \frac{1}{2}\Big(B_\tau - (B_t - B_\tau)\Big)$$

By triangle inequality,

$$x = |B_\tau| \leq \frac{1}{2}|B_\tau + (B_t - B_\tau)| + \frac{1}{2}|B_\tau - (B_t - B_\tau)|$$

Thus,

$$P\left\{\max_{s\leq t} |B_s| \geq x\right\} = P\left\{\tau \leq t, \ \frac{1}{2}|B_\tau + (B_t - B_\tau)| + \frac{1}{2}|B_\tau - (B_t - B_\tau)| \geq x\right\}$$
$$\leq P\left\{\tau \leq t, \ |B_\tau + (B_t - B_\tau)| \geq x\right\} + P\left\{\tau \leq t, \ |B_\tau - (B_t - B_\tau)| \geq x\right\}$$

For the first term on the right hand side

$$P\Big\{\tau \le t,\ |B_\tau + (B_t - B_\tau)| \ge x\Big\} = P\{\tau \le t,\ |S_t| \ge x\} = P\{|S_t| \ge x\}$$

As for the second term, by the idea of reflection,

$$P\Big\{\tau \le t,\ |B_\tau - (B_t - B_\tau)| \ge x\Big\} = P\Big\{\tau \le t,\ |B_\tau + (B_t - B_\tau)| \ge x\Big\} = P\{|S_t| \ge x\}$$

In summary,

$$P\Big\{\max_{s \le t} |B_s| \ge x\Big\} \le 2P\{|S_t| \ge x\}$$

11. Write $W_t = (B_1(t), B_2(t))$, where $B_1(t)$ and $B_2(t)$ are two independent Brownian motions with $B_1(0) = a$ and $B_2(0) = b$. Set

$$\tau = \inf\{t > 0;\quad B_2(t) = 0\}$$

The question is: $P^{(a,b)}\{B_1(\tau) > 0\} =?$

Let $F_\tau(t) = P\{\tau \le t\}$ be the distribution function of τ. We have

$$F_\tau(t) = P\{\max_{s \le t}(b + B_2(t)) \le 0\} = P\{\tau_{-b} \le t\} = P\{\tau_b \le t\}$$
$$= P\{\max_{s \le t} |B_s| \ge b\} = 2P\{B_t \ge b\} = 2\int_b^\infty p(t,x)dx$$

Notice that

$$\frac{\partial}{\partial t} p(t,x) = \frac{1}{2}\frac{\partial^2}{\partial x^2} p(t,x)$$

The density of τ is

$$f_\tau(x) = 2\int_b^\infty \frac{\partial}{\partial t} p(t,x)dx = \int_b^\infty \frac{\partial^2}{\partial x^2} p(t,x)dx = \frac{\partial}{\partial x} p(t,\infty) - \frac{\partial}{\partial x} p(t,b)$$
$$= -\frac{\partial}{\partial x} p(t,b) = \frac{b}{\sqrt{2\pi}} t^{-3/2} \exp\Big\{-\frac{b^2}{2t}\Big\} \qquad (t > 0)$$

By the independence between B_1 and B_2, τ is independent of B_1. By Fubini theorem, therefore,

$$P^{(a,b)}\{B_1(\tau) > 0\} = \int_0^\infty P^a\{B_1(t) > 0\} f_\tau(t)dt$$
$$= \int_0^\infty \Big(\int_{-a}^\infty \frac{1}{\sqrt{2\pi t}} \exp\Big\{-\frac{x^2}{2t}\Big\}dx\Big) \frac{b}{\sqrt{2\pi}} t^{-3/2} \exp\Big\{-\frac{b^2}{2t}\Big\}dt$$
$$= \frac{b}{2\pi} \int_{-a}^\infty \Big[\int_0^\infty t^{-2} \exp\Big\{-\frac{b^2 + x^2}{2t}\Big\}dt\Big] dx$$

Notice that

$$\int_0^\infty t^{-2}\exp\Big\{-\frac{b^2+x^2}{2t}\Big\}dt=\frac{2}{b^2+x^2}$$

we have that

$$P^{(a,b)}\{B_1(\tau)>0\}=\frac{b}{\pi}\int_{-a}^\infty\frac{1}{b^2+x^2}dx=\frac{1}{\pi}\arctan\Big(\frac{x}{b}\Big)\Big|_{-a}^\infty=\frac{1}{\pi}\Big\{\frac{\pi}{2}+\arctan\Big(\frac{a}{b}\Big)\Big\}$$

12. Clearly, (M_t, B_t) takes values in

$$D=\Big\{(b,x);\quad b>0 \text{ and } x\le b\Big\}$$

Set $\tau_b=\inf\{s\ge 0, B_s\ge b\}$. For any $(b,x)\in D$,

$$\begin{aligned}&P\{M_t\ge b,\ B_t\le x\}=P\{\tau_b\le t,\ b+(B_t-B_{\tau_b})\le x\}\\&=P\{\tau_b\le t,\ b-(B_t-B_{\tau_b})\le x\}=P\{M_t\ge b,\ B_t\ge 2b-x\}\\&=P\{B_t\ge 2b-x\}\end{aligned}$$

where the first step follows from reflection and the last step follows from the fact that $2b-x\ge b$ and therefore $\{B_t\ge 2b-x\}\subset\{M_t\ge b\}$.

Notice that $2b-x\ge b$. Therefore the right hand side is equal to

$$P\{B_t\ge 2b-x\}=\frac{1}{\sqrt{2\pi t}}\int_{2b-x}^\infty\exp\Big\{-\frac{u^2}{2t}\Big\}du$$

Thus, the joint distribution function of (M_t,B_t) is

$$P\{M_t\le b,\ B_t\le x\}=P\{B_t\le x\}-\frac{1}{\sqrt{2\pi t}}\int_{2b-x}^\infty\exp\Big\{-\frac{u^2}{2t}\Big\}du$$

Hence, the joint density of (M_t,B_t) is

$$f_t(b,x)=\frac{\partial^2}{\partial b\partial x}P\{M_t\le b,\ B_t\le x\}=\frac{2}{\sqrt{2\pi t}}\frac{2b-x}{t}\exp\Big\{-\frac{(2b-x)^2}{2t}\Big\}\qquad (b,x)\in D$$

15. Set

$$\tau_0=\tau_0(B(\cdot))=\inf\{s\ge 0;\quad B(s)=0\}$$

Then

$$\tilde{\xi}=t+\inf\{s\ge 0;\quad B(t+s)=0\}=t+\tau_0(B(\cdot+t))$$

By Markov property, for any $s \geq t$,

$$
\begin{aligned}
P\{\tilde{\xi}_t > s\} &= P\{\tau(B(\cdot + t)) > s - t\} = EP^{B(t)}\{\tau_0 > s - t\} \\
&= \int_{-\infty}^{\infty} P^x\{\tau_0 > s - t\} p(t, x) dx = \int_{-\infty}^{\infty} P\{\tau_{-x} > s - t\} p(t, x) dx \\
&= \int_{-\infty}^{\infty} P\{\tau_{|x|} > s - t\} p(t, x) dx = \int_{-\infty}^{\infty} P\{\max_{u \leq s-t} B(u) \leq |x|\} p(t, x) dx \\
&= \int_{-\infty}^{\infty} P\{|B(s - t)|) \leq |x|\} p(t, x) dx
\end{aligned}
$$

where the last step follows from reflection principle. By Fubini theorem, the right hand side is equal to

$$
P\{|B(t - s)| \leq |\widetilde{B}(t)|\} = P\Big\{|Y| \leq \sqrt{\frac{t}{s - t}}|X|\Big\}
$$

where $\widetilde{B}(t)$ is a Brownian motion independent of $B(t)$, X and Y are independent standard normals.

By the rotation invariance of (X, Y):

$$
\Big(X \cos\theta - Y \sin\theta, \quad X \sin\theta + Y \cos\theta\Big) \stackrel{d}{=} (X, Y)
$$

$$
P\Big\{|Y| \leq \sqrt{\frac{t}{s - t}}|X|\Big\} = \frac{4}{2\pi}\text{arctan}\frac{\sqrt{t}}{\sqrt{s - t}} = \frac{2}{\pi}\text{arcsin}\sqrt{\frac{t}{s}}
$$

In summary,

$$
P\{\tilde{\xi}_t > s\} = \frac{2}{\pi}\text{arcsin}\sqrt{\frac{t}{s}} \qquad s \geq t
$$

Or

$$
P\{\tilde{\xi}_t \leq s\} = 1 - \frac{2}{\pi}\text{arcsin}\sqrt{\frac{t}{s}} \qquad s \geq t
$$

The density of $\tilde{\xi}_t$ is

$$
f_{\tilde{\xi}_t}(s) = -\frac{2}{\pi}\frac{d}{ds}\text{arcsin}\sqrt{\frac{t}{s}} = \frac{1}{\pi}\sqrt{\frac{t}{s(s - t)}} \qquad s \geq t
$$

An alternative short-cut: For any $s > 0$, we adopt the notation in Section 6.5 with $\xi_s = \sup\{u \leq s;\ B_u = 0\}$. The crucial relation is that when $s > t$,

$$
P\{\tilde{\xi}_t > s\} = P\{B_u \neq 0 \ \text{ for every } t \leq u \leq s\} = P\{\xi_s \leq t\}
$$

By Theorem 6.20, p.79 with (s and t switching role), the right hand side is

$$
\frac{2}{\pi}\text{arcsin}\sqrt{\frac{t}{s}}
$$

Hence, the probability density of $\tilde{\xi}_t$ is

$$f_{\tilde{\xi}_t}(s) = -\frac{2}{\pi}\frac{d}{ds}\arcsin\sqrt{\frac{t}{s}} = \frac{1}{\pi}\sqrt{\frac{t}{s(s-t)}} \qquad s > t$$

16. By the fact (from Problem #1) that

$$\Big\{M_t - B_t;\quad t \geq 0\Big\} \stackrel{d}{=} \{|B_t|;\quad t \geq 0\}$$

we have

$$\eta_t = \sup\{s \leq t:\quad M_s - B_s > 0\} \stackrel{d}{=} \sup\{s \leq t:\quad |B_s| > 0\} = \sup\{s \leq t:\quad B_s \neq 0\} = \xi_t$$

Alternative solution. We can also do it without using the conclusion from Problem#1: For $0 < s < t$,

$$P\{\eta_t \leq s\} = P\{M_u - B_u > 0\ \forall u \in [s,t]\} = P\{M_s - B_u > 0\ \forall u \in [s,t]\}$$
$$= P\{(M_s - B_s) - \max_{u \leq t-s}(B_{s+u} - B_s) \geq 0\} = P\{|B_s| - \max_{u \leq t-s}(B_{s+u} - B_s) \geq 0\}$$

where the second eqality follows from the fact that on $\{M_u - B_u > 0\ \forall u \in [s,t]\}$, $M_u = M_s$ for $u \in [s,t]$; and last step follows from the identity in law $M_s - B_s \stackrel{d}{=} |B_s|$ and the independece between $\max_{u \leq t-s}(B_{s+u} - B_s)$ and $\{B_r \ \ r \leq s\}$.

Thus,

$$P\{\eta_t \leq s\} = P\{\max_{u \leq t-s}(B_{s+u} - B_s) \leq |B_s|\}$$
$$= P\{|\widetilde{B}_{t-s}| \leq |B_s|\} = \arcsin\sqrt{\frac{s}{t}}$$

where $\tilde{B}$ is a 1-dimensional Brownian motion independent of B and the second step follows from the identity in law

$$\max_{u \leq t-s}(B_{s+u} - B_s) \stackrel{d}{=} \max_{u \leq t-s} B_u \stackrel{d}{=} |B_{t-s}|$$

and the independence between $\max_{u \leq t-s}(B_{s+u} - B_s)$ and $|B_s|$.

Finally, the conclusion follows from Theorem 6.20 ((6.2)), p.79.