Homework # 4

Chapter 7

3. Need to prove that

$$\lim_{|x|\to\infty} E^x f(X_t)g(X_{t+s}) = 0$$

and $E^x f(X_t)g(X_{t+s})$ is contnuous in x. Indeed, by Markov property

$$E^x f(X_t)g(X_{t+s}) = E^x f(X_t)E^{X_t}g(X_s) = E^x h(X_t)$$

Hence, the conclusion follows from the facts that the function $h(x) = f(x)E^x g(X_t)$ is in $C_\infty(\mathbf{R}^d)$ and that X_t is Feller.

6. By the semi-group property,

$$P_{s+t}u(x) = P_s \circ P_t u(x) \qquad u \in \mathcal{B}_b(\mathbf{R^d})$$

Taking $u(x) = 1_C$ leads to the conclusion.

A probabilistic alternative:

$$\begin{aligned} P_{s+t}(x, C) &= P^x\{X_{s+t} \in C\} = E^x 1_{\{X_{s+t}\in C\}} \\ &= E^x\big(P^{X_s}\{X_t \in C\}\big) = E^x P_s(X_t, C) = \int P_s(x, dy)P_s(y, C) \end{aligned}$$

13. Let the random variable $\tau \sim \exp(\alpha)$ be independent of the Brownian motion B_t.

$$P_\tau \otimes P^x\{B_\tau \in C\} = \alpha \int_0^\infty P^x\{B_\tau \in C\}e^{-\alpha t}dt = \alpha U_\alpha 1_C(x)$$

Hence,

$$U_\alpha 1_C(x) = \alpha^{-1} P_\tau \otimes P^x\{B_\tau \in C\}$$

That is, $U_\alpha 1_C(x)$ is the α^{-1}-multiple of the probability that a Brownian motion starting at x visits the set C at an independent exponential time.

In addition

$$\lim_{\alpha\to 0^+} U_\alpha 1_C(x) = E^x\left\{\int_0^\infty 1_{\{B_t\in C\}}dt\right\} = E^x \mathrm{Leb}\{t \ge 0; \quad B_t \in C\}$$

Thus, the limit represents the the average total time that a Brownian motion (starting at x) stays in C

14. Assume that there is $u \in C_\infty(\mathbf{R}^d)$ such that $u \ge 0$ and that the set $\{x \in \mathbf{R}^d; \quad u(x) > 0\}$ has a positive Lebesgue measure. Assume that

$$\int_0^\infty P_t u(x)dt < \infty$$

for some $x \in \mathbf{R}^d$. We show that it must be that $d \geq 3$.

Indeed, for $t \geq 1$,

$$\begin{aligned} P_t u(x) &= \frac{1}{(2\pi t)^{d/2}} \int_{\mathbf{R}^d} u(y) \exp\Big\{ -\frac{|y-x|^2}{2t} \Big\} dy \\ &\geq \frac{1}{(2\pi t)^{d/2}} \int_{\mathbf{R}^d} u(y) \exp\Big\{ -\frac{|y-x|^2}{2} \Big\} dy \end{aligned}$$

Thus,

$$\int_0^\infty P_t u(x) dt \geq \int_1^\infty P_t u(x) dt \geq \Big(\int_{\mathbf{R}^d} u(y) \exp\Big\{ -\frac{|y-x|^2}{2} \Big\} dy \Big) \int_1^\infty \frac{1}{(2\pi t)^{d/2}} dt$$

By the assumption on $u(x)$,

$$\int_{\mathbf{R}^d} u(y) \exp\Big\{ -\frac{|y-x|^2}{2} \Big\} dy > 0$$

So we must have that

$$\int_1^\infty \frac{1}{(2\pi t)^{d/2}} dt < \infty$$

Therefore, we must have $d \geq 3$.

18. (a) Let $\mathcal{F}_t$ be the Brownian filtration, to which X_t is adaptive. Notice that X_t has independent and stationary increment: For any $s, t \geq 0$, $X_{s+t} - X_s = (t, B_{s+t} - B_s)$ is independent of $\mathcal{F}_s$ and $X_{s+t} - X_s \stackrel{d}{=} X_t$. For any bounded function $u(t, x)$

$$E[u(X_{s+t}) | \mathcal{F}_s] = E[u(s+t, B_{s+t}) | \mathcal{F}_s] = P_t u(X_s)$$

where $P_t u(s, x) = Eu\big((s, x) + X_t\big)$ $((s, x) \in \mathbf{R}^+ \times \mathbf{R}^d)$. So X_t is a Markov process. To establish the Feller property, let $u \in \mathcal{C}_\infty(\mathbf{R}^+ \times \mathbf{R}^d)$. We need to show that: (1) for any $t > 0$, $P_t u \in \mathcal{C}_\infty(\mathbf{R}^+ \times \mathbf{R}^d)$; (2).

$$\lim_{t \to 0^+} P_t u(s, x) = u(s, x) \qquad \text{uniformly over } (s, x) \in \mathbf{R}^+ \times \mathbf{R}^d$$

Indeed,

$$P_t u(s, x) = Eu(s+t, x+B_t) = \int_{\mathbf{R}^d} u(s+t, x+y) p_t(y) dy \tag{$*$}$$

All claim follows from some routine argument.

(b). The transition semi-group is defined by (*).

$$U_\alpha(s, x) = \int_0^\infty e^{-\alpha t} P_t u(s, x) dt \int_0^\infty e^{-\alpha t} \Big[\int_{\mathbf{R}^d} u(s+t, x+y) p_t(y) dy \Big] dt$$

As for the generator, let $u(t,x)$ be a function satisfying the assumption of Theorem 5.6. Then for any $(s,x)\in \mathbf{R}^+\times\mathbf{R}^d)$,

$$M_t = u(s+t, x+B_t) - \int_0^t \mathcal{L}u(s+r, x+B_r)dr$$

is a martingale, where

$$\mathcal{L}u(s,x) = \frac{\partial}{\partial s}u(s,x) + \frac{1}{2}\Delta u(s,x)$$

Consequently, $EM_t = EM_0$, or

$$P_t u(s,x) - u(s,x) = \int_0^t E\mathcal{L}u(s+r, x+B_r)dr = \int_0^t P_r\circ \mathcal{L}u(s,x)dr$$

By Feller property,

$$\lim_{t\to 0^+}\frac{P_t u(s,x) - u(s,x)}{t} = \lim_{t\to 0^+}\frac{1}{t}\int_0^t P_r\circ\mathcal{L}u(s,x)dr = \mathcal{L}u(s,x)$$

uniformly over (s,x). Hence, $\mathcal{L}$ is the generator.

(c). Applying Theorem 7.30 to $X_t=(t,B_t)$ we conclude that under $P^{(s,x)}$,

$$M_t = u(X_t) - \int_0^t \mathcal{L}u(X_s)ds$$

is a martingale— the same statement as Theorem 5.6.

19. By Theorem 7.30 (p.115), the process

$$M_t = u(X_t) - u(x) - \int_0^t Au(X_s)ds \quad t\ge 0$$

is a martingale under the law P^x. Clearly, M_t is continuous in t. By Doob's stopping rule, $E^xM_{t\wedge\sigma} = E^xM_0 = 0$ for any $t>0$.

$$E^x u(X_{t\wedge\sigma}) - u(x) = E^x\int_0^{t\wedge\sigma} Au(X_s)ds$$

Let $t\to\infty$. By the boundedness and continuity of u, the dominated converngence applies to the left hand so

$$\lim_{t\to\infty} E^x u(X_{t\wedge\sigma}) = E^x\lim_{t\to\infty}u(X_{t\wedge\sigma}) = E^x u(X_\sigma)$$

To apply dominated convergence to the right hand side, notice that

$$\left|\int_0^{t\wedge\sigma} Au(X_s)ds\right| \le \int_0^{t\wedge\sigma}|Au(X_s)|ds \le \|Au\|_\infty (t\wedge\sigma) \le \|Au\|_\infty\sigma$$

and that $E^x\sigma < \infty$. Hence

$$\lim_{t\to\infty} E^x \int_0^{t\wedge\sigma} Au(X_s)ds = E^x \lim_{t\to\infty} \int_0^{t\wedge\sigma} Au(X_s)ds = E^x \int_0^{\sigma} Au(X_s)ds$$

In summary,

$$E^x u(X_\sigma) - u(x) = E^x \int_0^{\sigma} Au(X_s)ds$$

Chapter 8.

2. We prove (8.3) has the unique solution

$$u(t,x) = E^x f(B_t) = P_t f(x)$$

under the assumption $f \in \mathcal{C}_\infty(\mathbf{R}^d)$.

To prove that $u(t,x) = E^x f(B_t)$ is a solution, all we need is

$$P_t f(x) = \frac{1}{2}\int_0^t \Delta\circ P_s f(x)ds$$

Given $\epsilon > 0$, by Lemma 7.10 (b), p.94, the function f_ϵ defined by

$$f_\epsilon(x) = \frac{1}{\epsilon}\int_0^\epsilon P_s f(x)ds$$

is in $\mathcal{D}(A)$. Applying Lemma 8.1 to f_ϵ leads to

$$P_t f_\epsilon(x) = \frac{1}{2}\int_0^t \Delta\circ P_s f_\epsilon(x)ds$$

Or, by semi-group property

$$\frac{1}{\epsilon}\int_0^\epsilon P_{t+s}f(x)ds = \frac{1}{2}\frac{1}{\epsilon}\int_0^\epsilon dr\int_0^t \Delta\circ P_{s+r}f(x)ds = \frac{1}{2}\frac{1}{\epsilon}\int_0^\epsilon P_r\Big(\int_0^t \Delta\circ P_s f ds\Big)(x)dr$$

Notice that the function

$$\varphi(x) \equiv \int_0^t \Delta\circ P_s f(x)ds$$

is in $C-\infty(\mathbf{R}^d)$, By Feller property, the right hand side tends to

$$\frac{1}{2}\int_0^t \Delta\circ P_s f(x)ds$$

as $\epsilon\to 0^+$. For the same reason, the left hand side goes to $P_t f(x)$ as $\epsilon\to 0^+$.

We now come to the uniqueness. Assume that $u(t,x)$ solves (8.3). It is easy to check that for each $\epsilon > 0$, the function $u_\epsilon(t,x)$ defined by

$$u_\epsilon(t,x) = \frac{1}{\epsilon}\int_0^\epsilon P_s(u(t,\cdot))(x)ds$$

solves (8.3) with f being replaced by f_ϵ. By the fact that $f_\epsilon \in \mathcal{D}(A)$ and by uniqueness,

$$u_\epsilon(t,x) = E^x f_\epsilon(B_t) = \frac{1}{\epsilon}\int_0^\epsilon E^x(P_s f)(B_t)ds = \frac{1}{\epsilon}\int_0^\epsilon P_{t+s}f(x)ds$$

where the last step follows from semi-group property. So we have

$$\frac{1}{\epsilon}\int_0^\epsilon P_s(u(t,\cdot))(x)ds = \frac{1}{\epsilon}\int_0^\epsilon P_{t+s}f(x)ds$$

Letting $\epsilon \to 0^+$. By Feller property

$$u(t,x) = P_t f(x) = E^x f(B_t)$$

6. Solving $u"(x) = 0$ leads to $u(x) = C_0 + C_1 x$. By $u(0) = a$ and $u(1) = b$, we have that $C_0 = a$ and $C_0 = b - a$. Hence the solution is $u(x) = a + (b-a)x$. We show that this agrees with the formula $u(x) = E^x f(B_{\tau_{[0,1]}})$, where $f(0) = a$ and $f(1) = b$. Indeed, for any $0 \le x \le 1$

$$\begin{aligned}
&E^x f(B_{\tau_{[0,1]}}) = aP^x\{B_{\tau_{[0,1]}} = 0\} + bP^x\{B_{\tau_{[0,1]}} = 1\} \\
&= aP\{B_{\tau_{[-x,1-x]}} = -x\} + bP\{B_{\tau_{[-x,1-x]}} = 1-x\} \\
&= a(1-x) + bx = a + (b-a)x
\end{aligned}$$