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HYPERBOLIC ANDERSON EQUATIONS WITH GENERAL TIME-INDEPENDENT GAUSSIAN NOISE: STRATONOVICH REGIME

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In this paper, we investigate the hyperbolic Anderson equation generated by a time-independent Gaussian noise with two objectives: The solvability and intermittency. First, we prove that Dalang's condition is necessary and sufficient for existence of the solution. Second, we establish the precise long time and high moment asymptotics for the solution under the usual homogeneity assumption of the covariance of the Gaussian noise. Our approach is fundamentally different from the ones existing in literature. The main contributions in our approach include the representation of Stratonovich moment under Laplace transform via the moments of the Brownian motions in Gaussian potentials and some large deviation skills developed in dealing effectively with the Stratonovich chaos expansion.

1. Introduction. At the core of the development of stochastic partial differential equations (SPDE) there are two major concerns: The degree of tolerance of the singularity brought by random noise and the impact of such singularity on the behaviors of the system. The former is to establish the solvability of the system under the best possible condition that controls the roughness of the noises. The latter is to understand the system disorder (the phenomena is also known as intermittency) caused by singularity of the random noise. When the noise is Gaussian, the system is often interpreted between the Skorohod and Stratonovich settings. The Skorohod solution is more accessible for quantitative analysis and is more tolerant to noise singularity, while the Stratonovich one is more physically relevant but harder for precise mathematical treatment.

To study magnetic impurities embedded in metals, the physicist Philip Warren Anderson ([1]) adds a multiplicative Gaussian noise to the heat equation. Due to its close links to other physical models such as KPZ equation ([26]), especially in the wake of the breakthrough of [20], the study of this equation has been rapidly developed. Today, the equation is known as parabolic Anderson model in literature. We refer the interested readers to [24] and the references therein for the general information on this subject.

With the heat operator being replaced by the wave operator, there are some compelling reasons for considering hyperbolic Anderson models. Instead of analyzing the rate of the change $\partial u/\partial t$ of the stochastic system, the set-up of the hyperbolic Anderson models are concerned with the acceleration of the system evolution (especially for the models relevant to the Newton's second law). Deterministic hyperbolic equations arise from acoustics, electromagnetism, fluid dynamics and many other fields (e.g. [14, 18]) and have been extensively studied until present. For the hyperbolic equations in a random environment, in particular, the hyperbolic Anderson model, we point out the publications [3], [4], [5], [6], [12], [13], [16] for an incomplete list of recent development on the study of hyperbolic Anderson models.

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Compared with the parabolic Anderson models, much less has been known for the hyperbolic Anderson models due to (partially, at least) the absence of Feynman-Kac formula that allows the representation of the parabolic solution in terms of Brownian motions.

In this paper we consider the hyperbolic Anderson equation

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(t,x) = \Delta u(t,x) + \dot{W}(x)u(t,x)\,, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d \\ u(0,x) = u_0(x) \ \text{ and } \ \frac{\partial u}{\partial t}(0,x) = u_1(x)\,, & x \in \mathbb{R}^d \end{cases} \tag{1.1}$$

run by a time-independent, mean zero and possibly generalized Gaussian noise $\dot{W}(x)$ with the covariance function

$$\text{Cov}\left(\dot{W}(x), \dot{W}(y)\right) = \gamma(x-y)\,, \qquad x, y \in \mathbb{R}^d\,. \tag{1.2}$$

As a covariance function the non-negative definiteness of $\gamma(\cdot)$ implies that it admits a spectral measure $\mu(d\xi)$ on $\mathbb{R}^d$ uniquely defined by the relation

$$\gamma(x) = \int_{\mathbb{R}^d} e^{i\xi\cdot x}\mu(d\xi)\,, \qquad x \in \mathbb{R}^d\,. \tag{1.3}$$

Throughout this work, we assume that $\gamma(\cdot) \geq 0$ and $d = 1,2,3$. The system is set up in Stratonovich regime in the sense that the product in (1.1) is interpreted as the ordinary (instead of Wick) one. The equation (1.1) will be approximated appropriately by classical wave equations run by the smoothed Gaussian noise $\dot{W}_\varepsilon(x)$. We shall provide the details of the construction of the solution in Section 2.

Our first concern is the condition to ensure the existence of solution. It is often formulated in terms of the integrability of the spectral measure $\mu(d\xi)$. In the Skorohod regime, where the product between $\dot{W}(x)$ and $u(t,x)$ in (1.1) is understood as Wick product, the condition ([4, Theorem 1.6], [13, Remark 3.4]) that (1.1) has a unique solution is

$$\int_{\mathbb{R}^d} \left(\frac{1}{1+|\xi|^2}\right)^{3/2} \mu(d\xi) < \infty\,. \tag{1.4}$$

Back to the Stratonovich regime and still in the time independent setting, Balan ([3]) recently proved that in the dimensions $d = 1,2$ Equation (1.1) has a solution if

$$\int_{\mathbb{R}^d} \left(\frac{1}{1+|\xi|^2}\right)^{1/2} \mu(d\xi) < \infty\,. \tag{1.5}$$

In the setting of time-space Gaussian noise, Chen, Deya, Song and Tindel ([12]) establish the existence/uniqueness under a condition comparable to (1.5).

Our first main result is to obtain the best condition for the existence of the solution, which is to remove the square root in (1.5). We can also allow the spatial dimension to be three as well.

THEOREM 1.1. *Let $d = 1,2,3$ and assume that $u_0(x) = 1$ and $u_1(x) = 0$ in (1.1).*

(i) *Under Dalang's condition*

$$\int_{\mathbb{R}^d} \frac{1}{1+|\xi|^2}\mu(d\xi) < \infty \tag{1.6}$$

the equation (1.1) has a solution in the sense of Definition 2.1 given in Section 2.

(ii) *If the equation (1.1) has a square integrable solution $u(t,x)$ that admits the Stratonovich expansion (see (2.10)) for some $t > 0$, then Dalang's condition (1.6) must be satisfied.*

Roughly speaking, the system (1.1) in Stratonovich regime can be viewed as a randomization of the deterministic wave equation

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(t,x) = \Delta u(t,x) + f(x)u(t,x)\,, & (t,x)\in\mathbb{R}^+\times\mathbb{R}^d \\ u(0,x)=u_0(x) \ \text{ and } \ \frac{\partial u}{\partial t}(0,x)=u_1(x)\,, & x\in\mathbb{R}^d \end{cases} \tag{1.7}$$

with a deterministic potential function $f(x)$ on $\mathbb{R}^d$. To this regard, it is hard not to notice the stochastic representation constructed by Dalang, Mueller and Tribe ([16]). We devote the subsection 3.3 below to address this link and to add some new elements to the representation theory for wave equations.

Our next topic is the intermittency of the equation (1.1). Here, the word "intermittency" refers to the phenomena (caused by the singularity of the noise) that the solution $u(t,x)$ (or $|u(t,x)|$) takes predominantly low or modest values in the space $\mathbb{R}^d$ with rare but endless exceptions of sudden and impulsive high peaks. The mathematical definition of intermittency is based on the asymptotic behaviors (see Remark 1.3 below for details) of the moments

$$\mathbb{E}u^p(t,x) \ \text{ and } \ \mathbb{E}|u(t,x)|^p$$

as $t\to\infty$ or as $p\to\infty$.

In the next theorem, we assume the homogeneity for the covariance structure:

$$\gamma(cx)=c^{-\alpha}\gamma(x)\,, \qquad x\in\mathbb{R}^d\,, \quad c>0 \tag{1.8}$$

for some $\alpha>0$. Taking $f(\lambda)=(1+\lambda^2)^{-1}$ and $v(d\xi)=\mu(d\xi)$ in [13, Lemma 3.10] yields

$$\int_{\mathbb{R}^d}\frac{1}{1+|\xi|^2}\mu(d\xi)=\alpha\mu\{\xi\in\mathbb{R}^d;\ |\xi|\le 1\}\int_0^\infty\frac{1}{1+\rho^2}\frac{d\rho}{\rho^{1-\alpha}}$$

as far as either of the above two sides is finite. This shows that under the homogeneity (1.8) on the noise covariance condition, Dalang's condition (1.6) becomes "$\alpha<2$". In addition (Remark 1.4, [13]), the fact that $\gamma(\cdot)$ is non-negative and non-negative definite (for being qualified as covariance function) requires that $\alpha\le d$. Further, the only setting where "$\alpha=d$" is allowed under $\alpha<2$ is when $\alpha=d=1$, or when $\gamma(\cdot)$ is a constant multiple of Dirac function (i.e., $\dot W$ is an 1-dimensional spatial white noise, see Corollary 1.4 below for intermittency in this case).

THEOREM 1.2. *Under the homogeneity condition (1.8) with $0<\alpha<2\wedge d$ or with $\alpha=d=1$ and under the initial condition $u_0(x)=1$ and $u_1(x)=0$, the following limits hold:*

$$\lim_{t\to\infty}t^{-\frac{4-\alpha}{3-\alpha}}\log\mathbb{E}u^p(t,x)=\frac{3-\alpha}{2}p^{\frac{4-\alpha}{3-\alpha}}\left(\frac{2\mathcal{M}^{1/2}}{4-\alpha}\right)^{\frac{4-\alpha}{3-\alpha}}, \qquad p=1,2,\cdots; \tag{1.9}$$

$$\lim_{p\to\infty}p^{-\frac{4-\alpha}{3-\alpha}}\log\mathbb{E}|u(t,x)|^p=\frac{3-\alpha}{2}t^{\frac{4-\alpha}{3-\alpha}}\left(\frac{2\mathcal{M}^{1/2}}{4-\alpha}\right)^{\frac{4-\alpha}{3-\alpha}}, \qquad \forall\, t>0\,, \tag{1.10}$$

where

$$\mathcal{M}=\sup_{g\in\mathcal{F}_d}\left\{\left(\int_{\mathbb{R}^d\times\mathbb{R}^d}\gamma(x-y)g^2(x)g^2(y)dxdy\right)^{1/2}-\int_{\mathbb{R}^d}|\nabla g(x)|^2dx\right\} \tag{1.11}$$

and

$$\mathcal{F}_d=\left\{g\in W^{1,2}(\mathbb{R}^d);\ \int_{\mathbb{R}^d}|g(x)|^2dx=1\right\},$$

where $W^{1,2}$ is the Sobolev space.

REMARK 1.3. Intermittency is defined with slight difference in literature. The most restrictive version requests the limit in (1.9) to exist for all $p=1,2,\cdots$ and to have a super-linear growth in p. Intermittency can also be defined similarly for fixed $t>0$. Therefore, (1.9) and (1.10) implies that under (1.8), the system (1.1) has an intermittent solution.

An interesting special case is when $\dot W(x)$ ($x\in\mathbb{R}$) is a white noise that symbols the derivative of a two sided Brownian motion $W(x)$ on $\mathbb{R}$. The corresponding covariance $\gamma(\cdot)=\delta_0(\cdot)$ is the Dirac delta function and the spectral measure $\mu(d\xi)=d\xi/(2\pi)$ is a multiple of the Lebesgue measure on $\mathbb{R}$. In this case by [8, Theorem C.4, p.307] (with $p=2$ and $\theta=1$), we have

$$\mathcal{M}=\frac{1}{4}\sqrt[3]{\frac{3}{2}}\,.$$

Thus we can write

COROLLARY 1.4. When $\dot W(x)$ ($x\in\mathbb{R}$) is an 1-dimensional white noise

$$\lim_{t\to\infty}t^{-3/2}\log\mathbb{E}u^p(t,x)=\frac{1}{2}\sqrt[4]{\frac{3}{4}}p^{3/2}\,,\qquad p=1,2,\cdots\,. \tag{1.12}$$

$$\lim_{p\to\infty}p^{-3/2}\log\mathbb{E}|u(t,x)|^p=\frac{1}{2}\sqrt[4]{\frac{3}{4}}t^{3/2}\,,\qquad \forall\, t>0\,. \tag{1.13}$$

In Skorohod regime ([4]), the high moment asymptotic theorem takes the same form as (1.10), while the long time asymptotic theorem takes the form

$$\lim_{t\to\infty}t^{-\frac{4-\alpha}{3-\alpha}}\log\mathbb{E}|u(t,x)|^p=\frac{3-\alpha}{2}p(p-1)^{\frac{1}{3-\alpha}}\left(\frac{2\mathcal{M}^{1/2}}{4-\alpha}\right)^{\frac{4-\alpha}{3-\alpha}} \tag{1.14}$$

for $p\geq 2$.

We now comment on the possibility of more general initial conditions. First notice that the expansion (2.10) in the next section is resulted from the iteration of the mild equation (2.1). Due to the fact that $G(t,x)\geq 0$ and $\gamma(\cdot)\geq 0$, the type of algebra carried out in Section 2 concludes that for any $p=1,2,\cdots$, $\mathbb{E}u^p(t,x)$ is monotonic in $u_0(t,x)$ (defined in (2.3)). By comparing $u_0(t,x)$ with 1, therefore, one can reduce the problem to the setting $u_0(x)=1$ and $u_1(x)=0$. In this way, the workable conditions on $u_0(x)$ and $u_1(x)$ can be the ones that produce the needed bounds of $u_0(t,x)$. We refer an interested reader to Proposition 2.6, [13] for some existing treatment.

We now mention some new ideas that are introduced in this paper. As usual, the solution can be formally written in terms of Stratonovich expansion (2.10). Therefore, the level of investigation is largely determined by our capability of handling the Stratonovich multiple integral $S_n\big(g_n(\cdot,t,x)\big)$ (see (2.12) for its definition) for fixed n and for large n as well. To this regard, the most significant observation made in this paper is the moment representation given in Theorem 3.3 that associates the study of $S_n\big(g_n(\cdot,t,x)\big)$ to the problem of Brownian motions in Gaussian potential. Another notable input is the algorithm development related to the Wick's formula (2.15), which is crucial to, among other things, the establishment of a moment inequality (Lemma 5.3) for the lower bound of the high moment asymptotics given in (1.10). Last but not least, some skills on large deviations and Laplacian transforms are developed for dealing with Stratonovich expansion.

Unfortunately, the idea of moment representation developed in this paper does not work, at least in its current form, in the setting of time-dependent Gaussian noise where $\dot W(x)$ in

(1.1) is replaced by a time-dependent Gaussian noise $\dot{W}(t,x)$. This subject remains widely open and challenged and we leave it to the future study.

Here is the organization of the paper. In next section (Section 2), we introduce the multiple Stratonovich integral and formally express the solution as Stratonovich expansion. In Section 3, we establish the Stratonovich integrability for the functions $g_n(\cdot,t,x)$, develop the Fubini theorem for the multiple Stratonovich integration and represent the Laplace transform of the multiple Stratonovich integral $S_n\big(g_n(\cdot,t,x)\big)$ in terms of Brownian motions in Gaussian potential. Section 4 and Section 5 are devoted to the proofs of Theorem 1.1 and 1.2, respectively. Some relevant results about the moment bound of Brownian intersection local times and about multiple Stratonovich integrals are provided in the appendix.

2. Stratonovich expansion and approximations.

As usual by the Duhamel principle the mathematical definition of the hyperbolic Anderson equation (1.1) will be the following mild form

$$u(t,x)=u_0(t,x)+\int_{\mathbb{R}^d}\left[\int_0^t G(t-s,x-y)u(s,y)ds\right]W(dy)\,, \tag{2.1}$$

where

(i) $G(t,x)$ is the fundamental solution defined by the deterministic wave equation

$$\begin{cases}\dfrac{\partial^2 G}{\partial t^2}(t,x)=\Delta G(t,x)\\[2ex] G(0,x)=0\ \text{ and }\ \dfrac{\partial G}{\partial t}(0,x)=\delta_0(x)\,, \qquad x\in\mathbb{R}^d\,.\end{cases} \tag{2.2}$$

(ii) $u_0(t,x)$ is the solution to the deterministic part of the equation (1.1):

$$u_0(t,x)=\int_{\mathbb{R}^d}\frac{\partial}{\partial t}G(t,x-y)u_0(y)dy+\int_{\mathbb{R}^d}G(t,x-y)u_1(y)dy\,. \tag{2.3}$$

Under the initial condition given in Theorem 1.1 and Theorem 1.2, $u_0(t,x)\equiv 1$;

(iii) the stochastic integral on the right hand side of (2.1) is interpreted as Stratonovich one (see discussion below for details).

2.1. *Green's function.*

The fundamental solution $G(t,x)$ associated with (2.2) plays a key role in determining the behavior of the system (2.1). Let us recall some basic facts. Taking Fourier transform in (2.2) we get the expression for the fundamental solution

$$\int_{\mathbb{R}^d}G(t,x)e^{i\xi\cdot x}dx=\frac{\sin(|\xi|t)}{|\xi|}\,,\qquad (t,\xi)\in\mathbb{R}^+\times\mathbb{R}^d \tag{2.4}$$

in its Fourier transform form. In the dimensions $d=1,2,3$, the fundamental solution $G(t,x)$ itself can be expressed explicitly as

$$G(t,x)=\begin{cases}\dfrac{1}{2}1_{\{|x|\le t\}} & d=1\\[2ex] \dfrac{1}{2\pi}\dfrac{1_{\{|x|\le t\}}}{\sqrt{t^2-|x|^2}} & d=2\\[2ex] \dfrac{1}{4\pi t}\sigma_t(dx) & d=3\,,\end{cases} \tag{2.5}$$

where $\sigma_t(dx)$ is the surface measure on the sphere $\{x \in \mathbb{R}^3;\ |x| = t\}$. We limit our attention to $d = 1, 2, 3$ in this work because the treatment developed here requires $G(t, x) \geq 0$. A scaling property we frequently use (especially in the proof of Theorem 1.2) is

$$G(t,x) = t^{-(d-1)} G(1, t^{-1}x)\,, \qquad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d\,. \tag{2.6}$$

2.2. *Stratonovich integral.* Before giving the definition of the mild solution we need to give a meaning to the Stratonovich integral appeared in (2.1). We shall do this by smoothing the noise as follows

$$\dot{W}_\varepsilon(x) = \int_{\mathbb{R}^d} \dot{W}(y) p_\varepsilon(y - x) dy\,, \qquad \varepsilon > 0, \quad x \in \mathbb{R}^d\,, \tag{2.7}$$

where $p_\varepsilon(x) = (2\pi\varepsilon)^{-d/2} \exp\left(-\frac{|x|^2}{2\varepsilon}\right)$ is the heat kernel [This specific mollifier $p_\varepsilon(\cdot)$ of the noise is not critical. Instead, one can pick the mollifier $h_\epsilon(x) = \epsilon^{-d} h(\epsilon^{-1}x)$ for any reasonably good probability density $h(x)$ and the results are independent of the choice of the mollifiers.] The covariance of $\dot{W}_\varepsilon(x)$ is

$$\mathbb{E}\left[\dot{W}_\varepsilon(x)\dot{W}_\varepsilon(x)\right] = \gamma_{2\varepsilon}(x - y)\,, \tag{2.8}$$

where $\gamma_\varepsilon(x) = \int_{\mathbb{R}^d} \gamma(z) p_\varepsilon(x - z) dz$. Given a random field $\Psi(x)$ ($x \in \mathbb{R}^d$) such that

$$\int_{\mathbb{R}^d} \Psi(x) \dot{W}_\varepsilon(x) dx \in \mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P}) \qquad \forall \varepsilon > 0\,,$$

We define the Stratonovich integral of $\{\Psi(x), x \in \mathbb{R}\}$ as

$$\int_{\mathbb{R}^d} \Psi(x) W(dx) \triangleq \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^d} \Psi(x) \dot{W}_\varepsilon(x) dx \tag{2.9}$$

whenever such limit exists in $\mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$. We can also use the convergence in probability in above definition. But as in most works on SPDE, $\mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$ norm is easier to deal with so that we choose the $\mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$ convergence throughout this work. Notice that this definition implicates that $u(t, x)$ as a solution to (2.1) is in $\mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$ for all $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^d$. After defining the Stratonovich integral, we can give the following definition about the solution.

DEFINITION 2.1. A random field $\{u(t, x)\,, t \geq 0\,, x \in \mathbb{R}^d\}$ is called a mild solution to (1.1) if $\int_0^t G(t - s, x - y) u(s, y) ds$ is well-defined and is Stratonovich integrable such that (2.1) is satisfied.

To prove Theorem 1.1, we shall use the Stratonovich expansion (see [22], [21] and references therein for the multiple Stratonovich integrals). Formally iterating (2.1) infinitely many times we have heuristically a solution candidate

$$u(t,x) = \sum_{n=0}^{\infty} S_n\big(g_n(\cdot, t, x)\big) \tag{2.10}$$

with $S_0\big(g_0(\cdot, t, x)\big) = 1$. Here is how the notation $S_n\big(g_n(\cdot, t, x)\big)$ is justified: The iteration procedure creates the recurrent relation

$$S_{n+1}\big(g_{n+1}(\cdot, t, x)\big) = \int_{\mathbb{R}^d} \left[\int_0^t G(t - s, x - y) S_n\big(g_n(\cdot, s, y)\big) ds\right] W(dy)\,. \tag{2.11}$$

Iterating this relation formally we have

$$
\begin{aligned}
(2.12)\quad & S_n\big(g_n(\cdot,t,x)\big) \\
&= \int_{(\mathbb{R}^d)^n}\left[\int_{[0,t]^n_<} d\mathbf{r}G(t-r_n,y_n-x)\cdots G(r_2-r_1,y_2-y_1)\right]W(dy_1)\cdots W(dy_n) \\
&= \int_{(\mathbb{R}^d)^n}\left[\int_{[0,t]^n_<} d\mathbf{s}\Big(\prod_{k=1}^n G(s_k-s_{k-1},x_k-x_{k-1})\Big)\right]W(dx_1)\cdots W(dx_n) \\
&= \int_{(\mathbb{R}^d)^n} g_n(x_1,\cdots,x_n,t,x)W(dx_1)\cdots W(dx_n) \quad \text{(say)},
\end{aligned}
$$

where $[0,t]^n_< := \{(s_1,\cdots,s_n)\in[0,t]^n$ satisfying $0<s_1<s_2<\cdots<s_n<t\}$, and the conventions $x_0=x$ and $s_0=0$ are adopted and the above second equality follows from the substitutions $s_k=t-r_{n-k+1}$ and $x_k=y_{n-k+1}-x$ $(k=1,\cdots,n)$.

Thus, the notation "$S_n\big(g_n(\cdot,t,x)\big)$" is reasonably introduced for a n-multiple Gaussian integral of the integrand

$$
(2.13)\qquad g_n(x_1,\cdots,x_n,t,x)=\int_{[0,t]^n_<}\Big(\prod_{k=1}^n G(s_k-s_{k-1},x_k-x_{k-1})\Big)ds_1\cdots ds_n\,,
$$

$(n=1,2,\cdots)$. In Section 3, the Stratonovich integrability of $g_n(\cdot,t,x)$ shall be rigorously established (see Theorem 3.8 and Theorem 6.2 for Stratonovich integrability of general kernels) and the Fubini's theorem posted in (2.11) shall be mathematically ratified (Remark 3.10).

The above argument gives us the impression that the existence of system (2.1) can be implied by the convergence of the random series defined by (2.10) in a certain appropriate form. This will be justified rigorously in Section 4 after we have more understanding of the multiple Stratonovich integral $S_n\big(g_n(\cdot,t,x)\big)$ with the specific kernel (2.13).

The multiple Stratonovich integration is defined as follows.

DEFINITION 2.2. Let $f:\mathbb{R}^n\to\mathbb{R}$ be measurable so that for every $\varepsilon>0$

$$
\int_{(\mathbb{R}^d)^n} f(x_1\cdots,x_n)\Big(\prod_{k=1}^n \dot W_\varepsilon(x_k)\Big)dx_1\cdots dx_n\in\mathcal{L}^2(\Omega,\mathcal{F},\mathbb{P})\,.
$$

Then we define the n-multiple Stratonovich integral of f as

$$
\begin{aligned}
(2.14)\qquad S_n(f) &:= \int_{(\mathbb{R}^d)^n} f(x_1\cdots,x_n)W(dx_1)\cdots W(dx_n) \\
&= \lim_{\varepsilon\to0^+}\int_{(\mathbb{R}^d)^n} f(x_1\cdots,x_n)\Big(\prod_{k=1}^n \dot W_\varepsilon(x_k)\Big)dx_1\cdots dx_n
\end{aligned}
$$

whenever the limit exists $\mathcal{L}^2(\Omega,\mathcal{F},\mathbb{P})$.

REMARK 2.3. Along with the set-up of our model, the Stratonovich integrand f is given as a measure in the dimension three ($d=3$). Indeed ([28]), Definition 2.2 can be extended to the setting of generalized functions f. A detail is provided near the end of this section for the construction needed in $d=3$.

The following lemma provides a convenient test of Stratonovich integrability that we shall use in this work.

LEMMA 2.4. *The n-multiple Stratonovich integral $S_n(f)$ exists if and only if the limit*

$$\lim_{\varepsilon,\varepsilon'\to 0^+}\mathbb{E}\Big\{\int_{(\mathbb{R}^d)^n} f(x_1,\cdots x_n)\Big(\prod_{k=1}^n \dot W_\varepsilon(x_k)\Big)dx_1\cdots dx_n\Big\}$$
$$\times\Big\{\int_{(\mathbb{R}^d)^n} f(x_1,\cdots x_n)\Big(\prod_{k=1}^n \dot W_{\varepsilon'}(x_k)\Big)dx_1\cdots dx_n\Big\}$$

exists

PROOF. The existence of the limit in (2.14) is another way to say that the family

$$\mathcal{Z}_\varepsilon=\int_{(\mathbb{R}^d)^n} f(x_1,\cdots x_n)\Big(\prod_{k=1}^n \dot W_\varepsilon(x_k)\Big)dx_1\cdots dx_n\,,\quad \varepsilon>0\,,$$

is a Cauchy sequence in $\mathcal{L}^2(\Omega,\mathcal{F},\mathbb{P})$ as $\varepsilon\to 0^+$, which is equivalent to the lemma. □

We refer to Theorem 6.2 for the exact conditions on f so that the multiple Stratonovich integral $S_n(f)$ exists in L^2.

Definition 2.2 can be extended to a random field $f(x_1,\cdots,x_n)$ in an obvious way. Most of the time in this paper, however, we deal with a deterministic integrand and demand some effective ways to compute the expectation of multiple Stratonovich integral of deterministic integrands. To this end let us recall an identity [27, p.201, Lemma 5.2.6] known as Wick's formula which states that

$$\begin{cases}\mathbb{E}\displaystyle\prod_{k=1}^{2n} g_k=\sum_{\mathcal{D}\in\Pi_n}\prod_{(j,k)\in\mathcal{D}}\mathbb{E}g_jg_k\\[2ex] \mathbb{E}\displaystyle\prod_{k=1}^{2n-1} g_k=0\,,\end{cases}\tag{2.15}$$

where $(g_1,\cdots,g_{2n})$ is a mean zero normal vector, and Π_n is the set of all pair partitions of $\{1,2,\cdots,2n\}$. As a side remark, $\#(\Pi_n)=\frac{(2n)!}{2^n n!}$. Applying (2.15) to $g_k=W_{\varepsilon_k}(x_k)$ in the case of deterministic integrand f, and taking the ε-limit, we have

$$\mathbb{E}\left[\int_{(\mathbb{R}^d)^{2n-1}} f(x_1,\cdots,x_{2n-1})W(dx_1)\cdots W(dx_{2n-1})\right]=0\tag{2.16}$$

and

$$\begin{aligned}&\mathbb{E}\left[\int_{(\mathbb{R}^d)^{2n}} f(x_1,\cdots,x_{2n})W(dx_1)\cdots W(dx_{2n})\right]\\ &\qquad=\sum_{\mathcal{D}\in\Pi_n}\int_{(\mathbb{R}^d)^{2n}}\Big(\prod_{(j,k)\in\mathcal{D}}\gamma(x_j-x_k)\Big)f(x_1,\cdots,x_{2n})dx_1\cdots dx_{2n}\end{aligned}\tag{2.17}$$

under the Stratonovich integrability on the left hand sides. In particular, the expectation of a $(2n)$-multiple Stratonovich integral is non-negative if the integrand is non-negative.

Since Dalang's condition (1.6) encompasses the cases where the covariance function $\gamma(\cdot)$ exists only as a generalized function (e.g., $\gamma(\cdot)=\delta_0(\cdot)$ in $d=1$), the meaning of the multiple integral on the right hand side of (2.17) needs to be clarified. Indeed, by (2.15)

$$\begin{aligned}&\mathbb{E}\int_{(\mathbb{R}^d)^{2n}} f(x_1,\cdots,x_{2n})\prod_{k=1}^{2n}\dot{W}_\epsilon(x_k)\\&=\int_{(\mathbb{R}^d)^{2n}}\Big(\prod_{(j,k)\in\mathcal{D}}\gamma_{2\epsilon}(x_j-x_k)\Big)f(x_1,\cdots,x_{2n})dx_1\cdots dx_{2n}\end{aligned}$$

where, we recall

$$\gamma_\epsilon(x)=\int_{\mathbb{R}^d}\gamma(y)p_\epsilon(x-y)dy\,,\qquad \epsilon>0,\quad x\in\mathbb{R}^d\,.$$

Inspired by (2.14), we therefore define

$$\begin{aligned}&\int_{(\mathbb{R}^d)^{2n}}\Big(\prod_{(j,k)\in\mathcal{D}}\gamma(x_j-x_k)\Big)f(x_1,\cdots,x_{2n})dx_1\cdots dx_{2n}\\&\triangleq\lim_{\epsilon\to0^+}\int_{(\mathbb{R}^d)^{2n}}\Big(\prod_{(j,k)\in\mathcal{D}}\gamma_{2\epsilon}(x_j-x_k)\Big)f(x_1,\cdots,x_{2n})dx_1\cdots dx_{2n}\end{aligned}\tag{2.18}$$

whenever the limit exists.

According to Theorem 6.2 and Remark 6.3, the $\mathcal{L}^2$-convergence in (2.14), Definition 2.2 implies the $\mathcal{L}^p$-convergence for any $p\in[2,\infty)$. Consequently, for any integers $l_1,\cdots,l_m\ge1$ and the l_j-multiple variate functions f_j $(1\le j\le m)$, the Stratonovich integrability of $f_1,\cdots,f_m$ implies the Stratonovich integrability of $f_1\otimes\cdots\otimes f_m$ and

$$S_{l_1+\cdots+l_m}(f_1\otimes\cdots\otimes f_m)=\prod_{j=1}^m S_{l_j}(f_j)\,.\tag{2.19}$$

According to (2.16), in particular,

$$\mathbb{E}\prod_{j=1}^m S_{l_j}(f_j)=0\quad\text{whenever } l_1+\cdots+l_m \text{ is odd}\,.\tag{2.20}$$

Given two Stratonovich integrable functions $f(x_1,\cdots,x_n)$ and $g(x_1,\cdots,x_n)$, by (2.17) and (2.19) (with $m=2$),

$$\begin{aligned}&\mathbb{E}S_n(f)S_n(g)\\&=\sum_{\mathcal{D}\in\Pi_n}\int_{(\mathbb{R}^d)^{2n}}dx_1\cdots dx_{2n}\Big(\prod_{(j,k)\in\mathcal{D}}\gamma(x_j-x_k)\Big)f(x_1,\cdots,x_n)g(x_{n+1},\cdots,x_{2n}).\end{aligned}\tag{2.21}$$

To end this section we take the chance to address an inconvenient fact from (2.5) where $G(t,x)$ is defined as a measure rather than a function in $d=3$ dimensional Euclidean space. In this case, we can combine $g_n(x_1,\cdots,x_n,t,x)$ and $dx_1\cdots dx_n$ together to have that

$$\begin{aligned}g_n(x_1,\cdots,x_n,t,x)dx_1\cdots dx_n&=\int_{[0,t]^n_<}\Big(\prod_{k=1}^n\frac{1}{4\pi(s_k-s_{k-1})}\sigma_{s_k-s_{k-1}}(x_{k-1},dx_k)\Big)ds_1\cdots ds_n\\&\triangleq\mu_n^{t,x}(dx_1\cdots dx_n)\end{aligned}$$

defines a measure on $(\mathbb{R}^3)^n$, where $\sigma_t(x,dy)$ represents the surface measure on the sphere $\{y\in\mathbb{R}^3;\ |y-x|=t\}$. For example, in defining $S_n\big(g_n(\cdot,t,x)\big)$ by (2.14) we use the convention

$$\int_{(\mathbb{R}^3)^n} g_n(x_1,\cdots,x_n,t,x)\bigg(\prod_{k=1}^n \dot W_\varepsilon(x_k)\bigg)dx_1\cdots dx_n=\int_{(\mathbb{R}^3)^n}\bigg(\prod_{k=1}^n \dot W_\varepsilon(x_k)\bigg)\mu_n^{t,x}(dx_1\cdots dx_n)\,.$$

It will be verified in the future that as $\varepsilon\downarrow 0+$, the above sequence converges in $L^2(\Omega,\mathcal{F},\mathbb{P})$ and the limit is denoted still by

$$\int_{(\mathbb{R}^3)^n} g_n(x_1,\cdots,x_n,t,x)W(dx_1)\cdots W(dx_n)$$

with

(2.22)

$$\mathbb{E}\bigg[\int_{(\mathbb{R}^3)^{2n}} g_n(x_1,\cdots,x_n,t,x)W(dx_1)\cdots W(dx_n)\bigg]^2$$
$$=\sum_{\mathcal{D}\in\Pi_n}\int_{(\mathbb{R}^3)^{2n}}\bigg(\prod_{(j,k)\in\mathcal{D}}\gamma(x_j-x_k)\bigg)\mu_n^{t,x}(dx_1\cdots dx_n)\mu_n^{t,x}(dx_{n+1},\cdots,dx_{2n})\,,$$

and the integral on the right hand side of (2.22) will be justified (Lemma 3.6) together with dimensions $d=1,2$ by the approximation procedure proposed in (2.18).

3. Stratonovich moments.

In the following discussion, $B(t),B_1(t),B_2(t),\cdots$ are independent d-dimensional Brownian motions. We assume independence between $\dot W$ and the Brownian motions and use the notation $\mathbb{E}_x$ for the expectation with respect to the Brownian motions with starting point x. We adopt the notation $\varepsilon=(\varepsilon_1,\cdots,\varepsilon_n)$ and $\epsilon'=(\epsilon_{n+1},\cdots,\epsilon_{2n})$ for $\varepsilon_1,\cdots,\varepsilon_{2n}>0$ and set

$$S_{n,\varepsilon}\big(g_n(\cdot,t,x)\big)=\int_{(\mathbb{R}^d)^n} g_n(x_1,\cdots,x_n,t,x)\bigg(\prod_{k=1}^n \dot W_{\varepsilon_k}(x_k)\bigg)dx_1\cdots dx_n\,. \tag{3.1}$$

For any pair partition $\mathcal{D}\in\Pi_n$, set

$$F^{\mathcal{D}}_{\epsilon,\epsilon'}(t_1,t_2)=\int_{(\mathbb{R}^d)^{2n}}dx_1\cdots dx_{2n}\bigg(\prod_{(j,k)\in\mathcal{D}}\gamma_{\epsilon_j+\epsilon_k}(x_j-x_k)\bigg)$$
$$\times\, g_n(x_1,\cdots,x_n,t_1,0)g_n(x_{n+1},\cdots,x_{2n},t_2,0)\,. \tag{3.2}$$

Again, $d=1,2,3$.

3.1. *Stratonovich moment representation.*

LEMMA 3.1. *Let $n=1,2,\cdots$. Under Dalang's condition (1.6),*

(i) For any $n\geq 1$, $\epsilon_1,\cdots,\epsilon_n>0$ and $\lambda>0$

$$\int_0^\infty e^{-\lambda t}S_{n,\epsilon}\big(g_n(\cdot,t,x)\big)dt$$
$$=\frac{\lambda}{2}\Big(\frac12\Big)^n\int_0^\infty\exp\Big\{-\frac{\lambda^2}{2}t\Big\}\mathbb{E}_x\int_{[0,t]^n_<}ds_1\cdots ds_n\prod_{k=1}^n\dot W_{\epsilon_k}\big(B(s_k)\big)\quad a.s. \tag{3.3}$$

(ii) For any $\lambda_1, \lambda_2 > 0$

$$\begin{aligned}(3.4)\qquad &\int_0^\infty\int_0^\infty e^{-\lambda_1 t_1-\lambda_2 t_2} F_{\epsilon,\epsilon'}^{\mathcal{D}}(t_1,t_2)dt_1dt_2\\ &\le \frac{\lambda_1\lambda_2}{4}\Big(\frac{1}{2}\Big)^{2n}\int_0^\infty\int_0^\infty dt_1dt_2\exp\Big\{-\frac{\lambda_1^2t_1+\lambda_2^2t_2}{2}\Big\}\\ &\quad\times\mathbb{E}_0\int_{[0,t_1]_<^n\times[0,t_2]_<^n} ds_1\cdots ds_{2n}\prod_{(j,k)\in\mathcal{D}}\gamma\big(B_{v(j)}(s_j)-B_{v(k)}(s_k)\big)\,,\end{aligned}$$

where the map v*:* $\{1,2,\cdots,2n\}\longrightarrow\{1,2\}$ *is defined as:* $v(k)=1$ *for* $1\le k\le n$ *and* $v(k)=2$ *for* $n+1\le k\le n$*.*
(iii) For any $\lambda_1,\lambda_2>0$

$$\begin{aligned}(3.5)\qquad &\lim_{\epsilon,\epsilon\to 0}\int_0^\infty\int_0^\infty e^{-\lambda_1 t_1-\lambda_2 t_2} F_{\epsilon,\epsilon'}^{\mathcal{D}}(t_1,t_2)dt_1dt_2\\ &= \frac{\lambda_1\lambda_2}{4}\Big(\frac{1}{2}\Big)^{2n}\int_0^\infty\int_0^\infty dt_1dt_2\exp\Big\{-\frac{\lambda_1^2t_1+\lambda_2^2t_2}{2}\Big\}\\ &\quad\times\mathbb{E}_0\int_{[0,t_1]_<^n\times[0,t_2]_<^n} ds_1\cdots ds_{2n}\prod_{(j,k)\in\mathcal{D}}\gamma\big(B_{v(j)}(s_j)-B_{v(k)}(s_k)\big)\,.\end{aligned}$$

REMARK 3.2. Under Dalang's condition (1.6), the intersection local times (Lemma A.1, [9])

$$\int_0^{t_1}\int_0^{t_2}\gamma\big(B(s)-B(r)\big)dsdr\quad\text{and}\quad\int_0^{t_1}\int_0^{t_2}\gamma\big(B_1(s)-B_2(r)\big)dsdr\,,\qquad t_1,t_2>0$$

are properly defined, so are the multiple time integral on the right hand sides of (3.4) and (3.5) in the spirit of Fubini's theorem. By Lemma 6.1, the moments of the intersection local times have (at most) polynomial increasing rate in t_1, t_2. Consequently, the right hand sides of (3.4) and (3.5) are finite for any $\lambda_1,\lambda_2>0$.

PROOF. The reason behind (3.3) is the simple fact that

$$(3.6)\qquad \int_0^\infty e^{-\lambda t}G(t,x)dt=\frac{1}{2}\int_0^\infty e^{-\lambda^2t/2}p(t,x)dt\,,\qquad x\in\mathbb{R}^d$$

for any $\lambda>0$, where $p(t,x)$ is the density of $B(t)$:

$$p(t,x)=\frac{1}{(2\pi t)^{d/2}}\exp\Big\{-\frac{|x|^2}{2t}\Big\},\qquad (t,x)\in\mathbb{R}^+\times\mathbb{R}^d\,.$$

Indeed, both sides have the same Fourier transform

$$\begin{aligned}\int_{\mathbb{R}^d}e^{i\xi\cdot x}\Big[\int_0^\infty e^{-\lambda t}G(t,x)dt\Big]dx&=\int_0^\infty e^{-\lambda t}\frac{\sin|\xi|t}{|\xi|}dt=\frac{1}{\lambda^2+|\xi|^2}\\ =\frac{1}{2}\int_0^\infty e^{-\lambda^2t/2}\exp\Big\{-\frac{1}{2}|\xi|^2t\Big\}dt&=\int_{\mathbb{R}^d}e^{i\xi\cdot x}\Big[\frac{1}{2}\int_0^\infty e^{-\lambda^2t/2}p(t,x)dt\Big]dx\end{aligned}$$

for every $\xi\in\mathbb{R}^d$.

Recall the identity (Lemma 2.2.7, p.39 in [8])

$$(3.7)\qquad \int_0^\infty e^{-\lambda t}\int_{[0,t]_<^n}ds_1\cdots ds_n\prod_{k=1}^n\varphi_k(s_k-s_{k-1})=\lambda^{-1}\prod_{k=1}^n\int_0^\infty\varphi_k(t)e^{-\lambda t}dt$$

with the convention $s_0 = 0$. Using it twice,

(3.8)

$$\int_0^\infty e^{-\lambda t} g_n(x_1, \cdots, x_n, t, x) dt$$
$$= \int_0^\infty dt e^{-\lambda t} \int_{[0,t]_<^n} ds_1 \cdots ds_n \Big(\prod_{k=1}^n G(s_k - s_{k-1}, x_k - x_{k-1}) \Big)$$
$$= \lambda^{-1} \prod_{k=1}^n \int_0^\infty e^{-\lambda t} G(t, x_k - x_{k-1}) dt = \Big(\frac{1}{2}\Big)^n \lambda^{-1} \prod_{k=1}^n \int_0^\infty e^{-\lambda^2 t/2} p(t, x_k - x_{k-1}) dt$$
$$= \frac{\lambda}{2} \Big(\frac{1}{2}\Big)^n \int_0^\infty dt e^{-\lambda^2 t/2} \int_{[0,t]_<^n} ds_1 \cdots ds_n \Big(\prod_{k=1}^n p(s_k - s_{k-1}, x_k - x_{k-1}) \Big).$$

Hence,

$$\int_0^\infty e^{-\lambda t} S_{n,\varepsilon}\big(g_n(\cdot, t, x)\big) dt$$
$$= \int_0^\infty dt e^{-\lambda t} \int_{(\mathbb{R}^d)^n} dx_1 \cdots dx_n g_n(x_1, \cdots, x_n, t, x) \Big(\prod_{k=1}^n \dot{W}_{\varepsilon_k}(x_k) \Big)$$
$$= \frac{\lambda}{2} \Big(\frac{1}{2}\Big)^n \int_0^\infty dt \exp\Big\{ -\frac{\lambda^2}{2} t \Big\} \int_{[0,t]_<^n} ds_1 \cdots ds_n$$
$$\times \int_{(\mathbb{R}^d)^n} dx_1 \cdots dx_n \Big(\prod_{k=1}^n p(s_k - s_{k-1}, x_k - x_{k-1}) \Big) \Big(\prod_{k=1}^n \dot{W}_{\varepsilon_k}(x_k) \Big).$$

Given $(s_1, \cdots, s_n) \in [0,t]_<^n$, the random vector $\big(B(s_1), \cdots, B(s_n)\big)$ has the joint density

$$f_{s_1, \cdots, s_n}(x_1, \cdots, x_n) \triangleq \prod_{k=1}^n p(s_k - s_{k-1}, x_k - x_{k-1}).$$

So we have

$$\int_{(\mathbb{R}^d)^n} d\mathbf{x} \Big(\prod_{k=1}^n p(s_k - s_{k-1}, x_k - x_{k-1}) \Big) \Big(\prod_{k=1}^n \dot{W}_{\varepsilon_k}(x_k) \Big) = \mathbb{E}_x \prod_{k=1}^n \dot{W}_{\varepsilon_k}\big(B(s_k)\big).$$

This completes the proof (3.3).

By (3.8) we have

$$\begin{aligned}
&\int_0^\infty\int_0^\infty e^{-\lambda_1 t_1-\lambda_2 t_2}F^{\mathcal{D}}_{\epsilon,\epsilon'}(t_1,t_2)dt_1dt_2\\
&=\frac{\lambda_1\lambda_2}{4}\Big(\frac{1}{2}\Big)^{2n}\int_0^\infty\int_0^\infty dt_1dt_2\exp\Big\{-\frac{\lambda_1^2t_1+\lambda_2^2t_2}{2}\Big\}\int_{[0,t_1]^n_<\times[0,t_2]^n_<}ds_1\cdots ds_{2n}\\
&\qquad\times\int_{(\mathbb{R}^d)^{2n}}dx_1\cdots dx_{2n}\bigg(\prod_{(j,k)\in\mathcal{D}}\gamma_{\epsilon_j+\epsilon_k}(x_j-x_k)\bigg)\\
&\qquad\times\bigg(p(s_1,x_1)\prod_{k=2}^n p(s_k-s_{k-1},x_k-x_{k-1})\bigg)\\
&\qquad\times\bigg(p(s_{n+1},x_{n+1})\prod_{k=n+2}^{2n}p(s_k-s_{k-1},x_k-x_{k-1})\bigg).
\end{aligned}$$

For fixed $(s_1,\cdots,s_{2n})$, the function

$$\begin{aligned}
&f(x_1,\cdots,x_{2n})\\
&=\bigg(p(s_1,x_1)\prod_{k=2}^n p(s_k-s_{k-1},x_k-x_{k-1})\bigg)\bigg(p(s_{n+1},x_{n+1})\prod_{k=n+2}^{2n}p(s_k-s_{k-1},x_k-x_{k-1})\bigg)
\end{aligned}$$

is the density of the random vector $\big(B_1(s_1),\cdots,B_1(s_n);B_2(s_{n+1}),\cdots B_2(s_{2n})\big)$. We have

$$\begin{aligned}
&\int_0^\infty\int_0^\infty e^{-\lambda_1 t_1-\lambda_2 t_2}F^{\mathcal{D}}_{\epsilon,\epsilon'}(t_1,t_2)dt_1dt_2\\
&=\frac{\lambda_1\lambda_2}{4}\Big(\frac{1}{2}\Big)^{2n}\int_0^\infty\int_0^\infty dt_1dt_2\exp\Big\{-\frac{\lambda_1^2t_1+\lambda_2^2t_2}{2}\Big\}\int_{[0,t_1]^n_<\times[0,t_2]^n_<}ds_1\cdots ds_{2n}\\
&\qquad\times\mathbb{E}_0\prod_{(j,k)\in\mathcal{D}}\gamma_{\epsilon_j+\epsilon_k}\big(B_{v(j)}(s_j)-B_{v(k)}(s_k)\big).
\end{aligned}$$

By Fourier transform

$$\begin{aligned}
&\mathbb{E}_0\prod_{(j,k)\in\mathcal{D}}\gamma_{\epsilon_j+\epsilon_k}\big(B_{\nu(j)}(s_j)-B_{v(k)}(s_k)\big)\\
&=\int_{(\mathbb{R}^d)^n}\bigg(\prod_{(j,k)\in\mathcal{D}}\mu(d\xi_{j,k})\bigg)\exp\bigg\{-\sum_{(j,k)\in\mathcal{D}}\frac{\epsilon_j+\epsilon_k}{2}|\xi_{j,k}|^2\bigg\}\\
&\qquad\times\mathbb{E}_0\exp\bigg\{i\sum_{(j,k)\in\mathcal{D}}\xi_{j,k}\cdot\big(B_{v(j)}(s_j)-B_{v(k)}(s_k)\big)\bigg\}\\
&=\int_{(\mathbb{R}^d)^n}\bigg(\prod_{(j,k)\in\mathcal{D}}\mu(d\xi_{j,k})\bigg)\exp\bigg\{-\sum_{(j,k)\in\mathcal{D}}\frac{\epsilon_j+\epsilon_k}{2}|\xi_{j,k}|^2\bigg\}\\
&\qquad\times\exp\bigg\{-\frac{1}{2}\mathrm{Var}\bigg(\sum_{(j,k)\in\mathcal{D}}\xi_{j,k}\cdot\big(B_{v(j)}(s_j)-B_{v(k)}(s_k)\big)\bigg)\bigg\}.
\end{aligned}$$

We have

$$\int_0^\infty\int_0^\infty e^{-\lambda_1 t_1-\lambda_2 t_2}F^{\mathcal{D}}_{\epsilon,\epsilon'}(t_1,t_2)dt_1dt_2 \tag{3.9}$$

$$= \frac{\lambda_1\lambda_2}{4}\Big(\frac{1}{2}\Big)^{2n}\int_0^\infty\int_0^\infty dt_1dt_2\exp\Big\{-\frac{\lambda_1^2t_1+\lambda_2^2t_2}{2}\Big\}\int_{[0,t_1]_<^n\times[0,t_2]_<^n}ds_1\cdots ds_{2n}$$
$$\times\int_{(\mathbb{R}^d)^n}\Big(\prod_{(j,k)\in\mathcal{D}}\mu(d\xi_{j,k})\Big)\exp\Big\{-\sum_{(j,k)\in\mathcal{D}}\frac{\epsilon_j+\epsilon_k}{2}|\xi_{j,k}|^2\Big\}$$
$$\times\exp\Big\{-\frac{1}{2}\mathrm{Var}\Big(\sum_{(j,k)\in\mathcal{D}}\xi_{j,k}\cdot\big(B_{v(j)}(s_j)-B_{v(k)}(s_k)\big)\Big)\Big\}.$$

Therefore,

$$\int_0^\infty\int_0^\infty e^{-\lambda_1t_1-\lambda_2t_2}F^{\mathcal{D}}_{\epsilon,\epsilon'}(t_1,t_2)dt_1dt_2$$
$$\le\frac{\lambda_1\lambda_2}{4}\Big(\frac{1}{2}\Big)^{2n}\int_0^\infty\int_0^\infty dt_1dt_2\exp\Big\{-\frac{\lambda_1^2t_1+\lambda_2^2t_2}{2}\Big\}\int_{[0,t_1]_<^n\times[0,t_2]_<^n}ds_1\cdots ds_{2n}$$
$$\times\int_{(\mathbb{R}^d)^n}\Big(\prod_{(j,k)\in\mathcal{D}}\mu(d\xi_{j,k})\Big)\exp\Big\{-\frac{1}{2}\mathrm{Var}\Big(\sum_{(j,k)\in\mathcal{D}}\xi_{j,k}\cdot\big(B_{v(j)}(s_j)-B_{v(k)}(s_k)\big)\Big)\Big\}$$
$$=\frac{\lambda_1\lambda_2}{4}\Big(\frac{1}{2}\Big)^{2n}\int_0^\infty\int_0^\infty dt_1dt_2\exp\Big\{-\frac{\lambda_1^2t_1+\lambda_2^2t_2}{2}\Big\}\mathbb{E}_0\int_{[0,t_1]_<^n\times[0,t_2]_<^n}ds_1\cdots ds_{2n}$$
$$\times\prod_{(j,k)\in\mathcal{D}}\gamma\big(B_{v(j)}(s_j)-B_{v(k)}(s_k)\big).$$

We have proved (3.4). Finally, taking limit in (3.9)

$$\lim_{\epsilon,\epsilon'\to0}\int_0^\infty\int_0^\infty e^{-\lambda_1t_1-\lambda_2t_2}F^{\mathcal{D}}_{\epsilon,\epsilon'}(t_1,t_2)dt_1dt_2$$
$$=\frac{\lambda_1\lambda_2}{4}\Big(\frac{1}{2}\Big)^{2n}\int_0^\infty\int_0^\infty dt_1dt_2\exp\Big\{-\frac{\lambda_1^2t_1+\lambda_2^2t_2}{2}\Big\}\int_{[0,t_1]_<^n\times[0,t_2]_<^n}ds_1\cdots ds_{2n}$$
$$\times\int_{(\mathbb{R}^d)^n}\Big(\prod_{(j,k)\in\mathcal{D}}\mu(d\xi_{j,k})\Big)\exp\Big\{-\frac{1}{2}\mathrm{Var}\Big(\sum_{(j,k)\in\mathcal{D}}\xi_{j,k}\cdot\big(B_{v(j)}(s_j)-B_{v(k)}(s_k)\big)\Big)\Big\}$$
$$=\frac{\lambda_1\lambda_2}{4}\Big(\frac{1}{2}\Big)^{2n}\int_0^\infty\int_0^\infty dt_1dt_2\exp\Big\{-\frac{\lambda_1^2t_1+\lambda_2^2t_2}{2}\Big\}$$
$$\times\mathbb{E}_0\int_{[0,t_1]_<^n\times[0,t_2]_<^n}ds_1\cdots ds_{2n}\prod_{(j,k)\in\mathcal{D}}\gamma\big(B_{v(j)}(s_j)-B_{v(k)}(s_k)\big).$$

This proves (3.5). □

THEOREM 3.3. *Under Dalang's condition (1.6), the function $g_n(\cdot,t,x)$ defined in (2.13) is Stratonovich integrable in the sense of Definition 2.2. Furthermore,*

(3.10)
$$\int_0^\infty e^{-\lambda t}S_n\big(g_n(\cdot,t,x)\big)dt=\frac{1}{n!}\frac{\lambda}{2}\Big(\frac{1}{2}\Big)^n\int_0^\infty\exp\Big\{-\frac{\lambda^2}{2}t\Big\}\mathbb{E}_x\Big[\int_0^t\dot W\big(B(s)\big)ds\Big]^ndt$$

almost surely for any $\lambda>0$.

PROOF. We first explain the time integral appearing on the right hand side of (3.10). It is defined as

$$\int_0^t \dot W\big(B(s)\big)ds \triangleq \lim_{\varepsilon\to 0^+}\int_0^t \dot W_\varepsilon\big(B(s)\big)ds \quad \text{ in } \mathcal{L}^2(\Omega,\mathcal{F},\mathbb{P}_x\otimes\mathbb{P}),$$

where the existence of the limit on the right hand is established in Lemma A.1, [9] under Dalang's condition (1.6). Conditioning on the Brownian motion B, it is a mean zero normal random variable with the variance

$$\int_0^t\int_0^t \gamma(B_s-B_r)dsdr$$

whose distribution does not depend on the starting point x of the Brownian motion. So we have

$$\mathbb{E}\otimes\mathbb{E}_x\bigg[\int_0^t \dot W\big(B(s)\big)ds\bigg]^n = \begin{cases} \frac{n!}{2^{n/2}(n/2)!}\mathbb{E}_0\bigg[\int_0^t\int_0^t\gamma(B_s-B_r)dsdr\bigg]^{\frac{n}{2}} & \text{when } n \text{ is even};\\ 0 & \text{when } n \text{ is odd}.\end{cases} \tag{3.11}$$

The above n-th moment is finite ((6.1), Lemma 6.1 below) for all $n=1,2,\cdots$. Consequently, the quenched moment

$$\mathbb{E}_x\bigg[\int_0^t \dot W\big(B(s)\big)ds\bigg]^n$$

exists almost surely. In addition, the bound provided in (6.1) in the Lemma 6.1 below makes the right hand side of (3.10) well-defined for any $\lambda>0$.

Taking $\epsilon_1=\cdots=\epsilon_{2n}=\delta$ in (3.3), we have

$$\int_0^\infty e^{-\lambda t}S_{n,\epsilon}\big(g_n(\cdot,t,x)\big)dt = \frac{\lambda}{2}\Big(\frac{1}{2}\Big)^n\frac{1}{n!}\int_0^\infty \exp\Big\{-\frac{\lambda^2}{2}t\Big\}\mathbb{E}_x\bigg[\int_0^t \dot W_\delta\big(B(s)\big)ds\bigg]^n dt\,.$$

We now let $\delta\to 0^+$ on both sides. Notice that

$$\lim_{\delta\to 0^+}\mathbb{E}_x\bigg[\int_0^t \dot W_\delta\big(B(s)\big)ds\bigg]^n = \mathbb{E}_x\bigg[\int_0^t \dot W\big(B(s)\big)ds\bigg]^n \quad \text{in } \mathcal{L}^2(\Omega,\mathcal{F},\mathbb{P})\,.$$

In addition, by Cauchy-Schwartz and Jensen inequalities

$$\begin{aligned}
&\mathbb{E}\bigg\{\mathbb{E}_x\bigg[\int_0^t \dot W_\delta\big(B(s)\big)ds\bigg]^n\bigg\}^2 \le \mathbb{E}_0\otimes\mathbb{E}\bigg[\int_0^t \dot W_\delta\big(B(s)\big)ds\bigg]^{2n}\\
&\le \int_{\mathbb{R}^d} p_\delta(y)\mathbb{E}_0\otimes\mathbb{E}\bigg[\int_0^t \dot W\big(y+B(s)\big)ds\bigg]^{2n}dy = \mathbb{E}_0\otimes\mathbb{E}\bigg[\int_0^t \dot W\big(B(s)\big)ds\bigg]^{2n}\\
&= \mathbb{E}_0\bigg[\int_0^t\int_0^t \gamma\big(B(s)-B(r)\big)dsdr\bigg]^n.
\end{aligned}$$

By Lemma 6.1, the right hand side has at most a polynomial increasing rate in t. By the dominated convergence theorem, we have

$$\lim_{\delta\to 0^+}\int_0^\infty \exp\Big\{-\frac{\lambda^2}{2}t\Big\}\mathbb{E}_x\bigg[\int_0^t \dot W_\delta\big(B(s)\big)ds\bigg]^n dt = \int_0^\infty \exp\Big\{-\frac{\lambda^2}{2}t\Big\}\mathbb{E}_x\bigg[\int_0^t \dot W\big(B(s)\big)ds\bigg]^n dt$$

in $\mathcal{L}^2(\Omega,\mathcal{F},\mathbb{P})$.

The Stratonovich integrability of $g_n(\cdot,t,x)$ shall be established in Theorem 3.8 below to make sense of left hand side of (3.10). By stationarity in x, all we need is the following convergence

$$\lim_{\delta\to 0^+}\int_0^\infty e^{-\lambda t}S_{n,\epsilon}\big(g_n(\cdot,t,0)\big)dt=\int_0^\infty e^{-\lambda t}S_n\big(g_n(\cdot,t,0)\big)dt\quad\text{in } \mathcal{L}^2(\Omega,\mathcal{F},\mathbb{P}).$$

This is given in Part (ii), Theorem 3.8. □

COROLLARY 3.4. Assume Dalang's condition (1.6). Let $p\ge 1$ and $n\ge 1$ be any integers. Given $\lambda_1,\cdots,\lambda_p>0$,

$$\begin{aligned}(3.12)\qquad &\int_{(\mathbb{R}^+)^p}dt_1\cdots dt_p\exp\Big\{-\sum_{j=1}^p\lambda_jt_j\Big\}\sum_{l_1+\cdots+l_p=2n}\mathbb{E}\prod_{j=1}^p S_{l_j}\big(g_{l_j}(\cdot,t_j,0)\big)\\ &=\Big(\frac{1}{2}\Big)^{3n}\frac{1}{n!}\Big(\prod_{j=1}^p\frac{\lambda_j}{2}\Big)\int_{(\mathbb{R}^+)^p}dt_1\cdots dt_p\exp\Big\{-\frac{1}{2}\sum_{j=1}^p\lambda_j^2t_j\Big\}\\ &\qquad\times\mathbb{E}_0\Big[\sum_{j,k=1}^p\int_0^{t_j}\int_0^{t_k}\gamma\big(B_j(s)-B_k(r)\big)dsdr\Big]^n,\end{aligned}$$

where $B_1(t),\cdots,B_p(t)$ are independent d-dimensional Brownian motions starting at 0.

PROOF. By Theorem 3.3,

$$\begin{aligned}&\int_{(\mathbb{R}^+)^p}dt_1\cdots dt_p\exp\Big\{-\sum_{j=1}^p\lambda_jt_j\Big\}\sum_{l_1+\cdots+l_p=2n}\prod_{j=1}^p S_{l_j}\big(g_{l_j}(\cdot,t,0)\big)\\ &=\sum_{l_1+\cdots+l_p=2n}\prod_{j=1}^p\int_0^\infty e^{-\lambda_jt}S_{l_j}\big(g_{l_j}(\cdot,t_j,0)\big)dt\\ &=\sum_{l_1+\cdots+l_p=2n}\Big(\prod_{j=1}^p\frac{\lambda_j}{2}\Big(\frac{1}{2}\Big)^{l_j}\Big)\prod_{j=1}^p\frac{1}{l_j!}\int_0^\infty dte^{-\lambda_j^2t/2}\mathbb{E}_0\Big[\int_0^t\dot W\big(B(s)\big)ds\Big]^{l_j}\\ &=\Big(\frac{1}{2}\Big)^{2n}\Big(\prod_{j=1}^p\frac{\lambda_j}{2}\Big)\int_{(\mathbb{R}^+)^p}dt_1\cdots dt_p\exp\Big\{-\frac{1}{2}\sum_{j=1}^p\lambda_j^2t_j\Big\}\frac{1}{(2n)!}\mathbb{E}_0\Big[\sum_{j=1}^p\int_0^{t_j}\dot W\big(B_j(s)\big)ds\Big]^{2n},\end{aligned}$$

where the last step follows from Newton's multi-nominal formula. By the fact that conditioning on the Brownian motions,

$$\sum_{j=1}^p\int_0^{t_j}\dot W\big(B_j(s)\big)ds$$

is normal with zero mean and the variance

$$\sum_{j,k=1}^p\int_0^{t_j}\int_0^{t_k}\gamma\big(B_j(s)-B_k(r)\big)dsdr$$

we have

$$(3.13)\qquad \mathbb{E}\Big[\sum_{j=1}^p\int_0^{t_j}\dot W\big(B_j(s)\big)ds\Big]^{2n}=\frac{(2n)!}{2^nn!}\Big[\sum_{j,k=1}^p\int_0^{t_j}\int_0^{t_k}\gamma\big(B_j(s)-B_k(r)\big)dsdr\Big]^n.$$

Thus, we have proved (3.12). □

COROLLARY 3.5. (1) For any $\lambda_1, \lambda_2 > 0$

$$\int_0^\infty \int_0^\infty dt_1 dt_2 e^{-\lambda_1 t_1 - \lambda_2 t_2} \mathbb{E} S_n\big(g_n(\cdot, t_1, 0)\big) S_n\big(g_n(\cdot, t_2, 0)\big) \tag{3.14}$$

$$= \Big(\frac{1}{4}\Big)^n \frac{\lambda_1 \lambda_2}{4} \frac{1}{(n!)^2} \int_0^\infty \int_0^\infty dt_1 dt_2 \exp\Big\{ - \frac{\lambda_1^2 t_1 + \lambda_2 t_2}{2} \Big\}$$

$$\times \mathbb{E} \otimes \mathbb{E}_0 \Big[\int_0^{t_1} \dot{W}\big(B_1(s) ds\Big]^n \Big[\int_0^{t_2} \dot{W}\big(B_2(s)\big) ds\Big]^n .$$

(2) For any $\epsilon = (\epsilon_1, \cdots, \epsilon_n)$ and $\epsilon' = (\epsilon_{n+1}, \cdots, \epsilon_{2n})$

$$\int_0^\infty \int_0^\infty dt_1 dt_2 e^{-\lambda_1 t_1 - \lambda_2 t_2} \mathbb{E} S_{n,\epsilon}\big(g_n(\cdot, t_1, 0)\big) S_{n,\epsilon'}\big(g_n(\cdot, t_2, 0)\big) \tag{3.15}$$

$$\le \int_0^\infty \int_0^\infty dt_1 dt_2 e^{-\lambda_1 t_1 - \lambda_2 t_2} \mathbb{E} S_n\big(g_n(\cdot, t_1, 0)\big) S_n\big(g_n(\cdot, t_2, 0)\big) .$$

PROOF. (3.14) is a direct consequence of Theorem 3.3. By the definition of $S_{n,\epsilon}\big((g_n(\cdot, t, x)\big)$ given in (3.1),

$$\mathbb{E} S_{n,\epsilon}\big(g_n(\cdot, t_1, 0)\big) S_{n,\epsilon'}\big(g_n(\cdot, t_2, 0)\big)$$

$$= \mathbb{E} \int_{(\mathbb{R}^d)^{2n}} dx_1 \cdots dx_{2n} g_n(x_1, \cdots, x_n, t_1, 0) g_n(x_{n+1}, \cdots, x_{2n}, t_2, 0) \prod_{k=1}^{2n} \dot{W}_{\epsilon_k}(x_k)$$

$$= \sum_{\mathcal{D} \in \Pi_n} \int_{(\mathbb{R}^d)^{2n}} dx_1 \cdots dx_{2n} \Big(\prod_{(j,k) \in \mathcal{D}} \gamma_{\epsilon_j + \epsilon_k}(x_j - x_k) \Big)$$

$$\times g_n(x_1, \cdots, x_n, t_1, 0) g_n(x_{n+1}, \cdots, x_{2n}, t_2, 0)$$

$$= \sum_{\mathcal{D} \in \Pi_n} F^{\mathcal{D}}_{\epsilon, \epsilon'}(t_1, t_2) ,$$

where the second equality follows from the Wick's formula (2.15) with $g_k = \dot{W}_{\epsilon_k}(x_k)$ and $F^{\mathcal{D}}_{\epsilon,\epsilon'}(t_1, t_2)$ is given in (3.2). By (3.4), we see

$$\int_0^\infty \int_0^\infty dt_1 dt_2 e^{-\lambda_1 t_1 - \lambda_2 t_2} \mathbb{E} S_{n,\epsilon}\big(g_n(\cdot, t_1, 0)\big) S_{n,\epsilon'}\big(g_n(\cdot, t_2, 0)\big)$$

$$\le \frac{\lambda_1 \lambda_2}{4} \Big(\frac{1}{2}\Big)^{2n} \int_0^\infty \int_0^\infty dt_1 dt_2 \exp\Big\{ - \frac{\lambda_1^2 t_1 + \lambda_2^2 t_2}{2} \Big\}$$

$$\times \mathbb{E}_0 \sum_{\mathcal{D} \in \Pi_n} \int_{[0,t_1]^n_< \times [0,t_2]^n_<} ds_1 \cdots ds_{2n} \prod_{(j,k) \in \mathcal{D}} \gamma\big(B_{v(j)}(s_j) - B_{v(k)}(s_k)\big) .$$

For any permutation σ on $\{1, \cdots, 2n\}$ with $\sigma(\{1, \cdots, n\}) = \{1, \cdots, n\}$ and $\sigma(\{n + 1, \cdots, 2n\}) = \{n+1, \cdots, 2n\}$

$$\sum_{\mathcal{D} \in \Pi_n} \int_{[0,t_1]^n_< \times [0,t_2]^n_<} ds_1 \cdots ds_{2n} \prod_{(j,k) \in \mathcal{D}} \gamma\big(B_{v(j)}(s_{\sigma(j)}) - B_{v(k)}(s_{\sigma(k)})\big)$$

$$= \sum_{\mathcal{D} \in \Pi_n} \int_{[0,t_1]^n_< \times [0,t_2]^n_<} ds_1 \cdots ds_{2n} \prod_{(j,k) \in \mathcal{D}} \gamma\big(B_{v(j)}(s_j) - B_{v(k)}(s_k)\big) .$$

Therefore

$$\sum_{\mathcal{D}\in\Pi_n}\int_{[0,t_1]^n_<\times[0,t_2]^n_<} ds_1\cdots ds_{2n}\prod_{(j,k)\in\mathcal{D}}\gamma\big(B_{v(j)}(s_j)-B_{v(k)}(s_k)\big)$$
$$=\frac{1}{(n!)^2}\sum_{\mathcal{D}\in\Pi_n}\int_{[0,t_1]^n\times[0,t_2]^n} ds_1\cdots ds_{2n}\prod_{(j,k)\in\mathcal{D}}\gamma\big(B_{v(j)}(s_j)-B_{v(k)}(s_k)\big).$$

A crucial observation is that

$$\int_{[0,t_1]^n\times[0,t_2]^n} ds_1\cdots ds_{2n}\prod_{(j,k)\in\mathcal{D}}\gamma\big(B_{v(j)}(s_j)-B_{v(k)}(s_k)\big)$$
$$=\prod_{(j,k)\in\mathcal{D}}\int_0^{t_{v(j)}}\int_0^{t_{v(k)}}\gamma\big(B_{v(j)}(s)-B_{v(k)}(r)\big)\big)dsdr.$$

Applying the Wick's formula (2.15) conditionally on the Brownian motions to the $2n$-dimensional normal vector

$$\Big(\overbrace{\int_0^{t_1}\dot W\big(B_1(s)\big)ds,\cdots,\int_0^{t_1}\dot W\big(B_1(s)\big)ds,}^{n}\ \overbrace{\int_0^{t_2}\dot W\big(B_2(s)\big)ds,\cdots,\int_0^{t_2}\dot W\big(B_2(s)\big)ds}^{n}\Big)$$

the right hand side is equal to

$$\mathbb{E}\Big[\int_0^{t_1}\dot W\big(B_1(s)\big)ds\Big]^n\Big[\int_0^{t_2}\dot W\big(B_2(s)\big)ds\Big]^n.$$

In summary

$$\int_0^\infty\int_0^\infty dt_1dt_2e^{-\lambda_1t_1-\lambda_2t_2}\mathbb{E}S_{n,\epsilon}\big(g_n(\cdot,t_1,0)\big)S_{n,\epsilon'}\big(g_n(\cdot,t_2,0)\big)$$
$$\le\frac{\lambda_1\lambda_2}{4}\Big(\frac{1}{2}\Big)^{2n}\frac{1}{(n!)^2}\int_0^\infty\int_0^\infty dt_1dt_2\exp\Big\{-\frac{\lambda_1^2t_1+\lambda_2^2t_2}{2}\Big\}$$
$$\times\mathbb{E}_0\otimes\mathbb{E}\Big[\int_0^{t_1}\dot W\big(B_1(s)\big)ds\Big]^n\Big[\int_0^{t_2}\dot W\big(B_2(s)\big)ds\Big]^n.$$

Finally, (3.15) follows from (3.14). □

3.2. *Stratonovich integrability and Fubini's theorem.* Recall that the function $F^{\mathcal{D}}_{\epsilon,\epsilon'}(t_1,t_2)$ is defined in (3.2).

LEMMA 3.6. *Under Dalang's condition (1.6), the limit*

$$\lim_{\epsilon,\epsilon'\to 0}F^{\mathcal{D}}_{\epsilon,\epsilon'}(t_1,t_2) \tag{3.16}$$

exists for any $n\ge 1$, $t_1,t_2>0$ and any pair partition $\mathcal{D}\in\Pi_n$. Further, the limiting function is continuous in t_1,t_2.

REMARK 3.7. In view of (2.18), Lemma 3.6 justifies the definition

$$\int_{(\mathbb{R}^d)^{2n}}dx_1\cdots dx_{2n}\Big(\prod_{(j,k)\in\mathcal{D}}\gamma(x_j-x_k)\Big)g_n(x_1,\cdots,x_n,t_1,x)g_n(x_{n+1},\cdots,x_{2n},t_2,0)$$
$$\triangleq\lim_{\epsilon\to 0^+}\int_{(\mathbb{R}^d)^{2n}}dx_1\cdots dx_{2n}\Big(\prod_{(j,k)\in\mathcal{D}}\gamma_\epsilon(x_j-x_k)\Big)g_n(x_1,\cdots,x_n,t_1,x)g_n(x_{n+1},\cdots,x_{2n},t_2,0).$$

PROOF. Clearly, $F^{\mathcal{D}}_{\epsilon,\epsilon'}(t_1,t_2)$ is non-negative, non-decreasing and continuous on $\mathbb{R}^+\times\mathbb{R}^+$. By (3.5), Lemma 3.1, the limit

$$\lim_{\epsilon,\epsilon'\to 0}\int_0^\infty\int_0^\infty e^{-\lambda_1 t_1-\lambda_2 t_2}F^{\mathcal{D}}_{\epsilon,\epsilon'}(t_1,t_2)dt_1dt_2$$

exists for any $\lambda_1,\lambda_2>0$.

By continuity theorem for Laplace transform [25, Theorem 5.2.2], therefore, the function $F^{\mathcal{D}}_{\epsilon}(t_1,t_2)$ weakly converges to a non-negative, non-decreasing and right continuous function $F^{\mathcal{D}}(t_1,t_2)$ on $(\mathbb{R}^+)^2$, i.e.,

$$\lim_{\epsilon,\epsilon'\to 0}F^{\mathcal{D}}_{\epsilon,\epsilon'}(t_1,t_2)=F^{\mathcal{D}}(t_1,t_2)$$

for any continuous point (t_1,t_2) of $F^{\mathcal{D}}$ and

(3.17)
$$\lim_{\epsilon,\epsilon'\to 0}\int_0^\infty\int_0^\infty e^{-\lambda_1 t_1-\lambda_2 t_2}F^{\mathcal{D}}_{\epsilon,\epsilon'}(t_1,t_2)dt_1dt_2=\int_0^\infty\int_0^\infty e^{-\lambda_1 t_1-\lambda_2 t_2}F^{\mathcal{D}}(t_1,t_2)dt_1dt_2\,.$$

(Actually, Theorem 5.22, [25] is stated for probability measures on $(\mathbb{R}^+)^d$. The case of general measures on $(\mathbb{R}^+)^d$ can be derived as in the proof of [19, Theorem 2a, Section 1, Chapter XIII]. Although this theorem only considers measures on $\mathbb{R}^+$ its extension to $(\mathbb{R}^+)^2$ is routine).

To establish the existence for the limit in (3.16) and therefore to complete the proof, all we need is to show that $F^{\mathcal{D}}(t_1,t_2)$ is continuous on $(\mathbb{R}^+)^2$ so

$$\lim_{\epsilon,\epsilon'\to 0}F^{\mathcal{D}}_{\epsilon,\epsilon'}(t_1,t_2)=F^{\mathcal{D}}(t_1,t_2)\,,\qquad \forall t_1,t_2>0\,. \tag{3.18}$$

We shall do it by establishing

$$\lim_{\delta_1,\delta_2\to 0^+}\sup_{\epsilon,\epsilon'}\left\{F^{\mathcal{D}}_{\epsilon,\epsilon'}(t_1,t_2)-F^{\mathcal{D}}_{\epsilon,\epsilon'}(t_1-\delta_1,t_2-\delta_2)\right\}=0\,. \tag{3.19}$$

Write

$$\begin{aligned}&F^{\mathcal{D}}_{\epsilon,\epsilon'}(t_1,t_2)\\&=\int_{(\mathbb{R}^d)^{2n}}dx_1\cdots dx_{2n}\bigg(\prod_{(j,k)\in\mathcal{D}}\gamma_{\epsilon_j+\epsilon_k}(x_j-x_k)\bigg)\int_{[0,t_1]^n_<\times[0,t_2]^n_<}ds_1\cdots ds_{2n}\\&\qquad\times\bigg(G(s_1,x_1)\prod_{l=2}^{n}G(s_l-s_{l-1},x_l-x_{l-1}\bigg)\\&\qquad\quad\times\bigg(G(s_{n+1},x_{n+1})\prod_{k=n+2}^{2n}G(s_l-s_{l-1},x_k-x_{k-1}\bigg)\\&=\mathcal{E}_{\epsilon,\epsilon'}([0,t_1]^n_<\times[0,t_2]^n_<)\qquad(\text{say})\,.\end{aligned}$$

To prove (3.19), all we need is

$$\lim_{\delta_1,\delta_2\to 0^+}\sup_{\epsilon,\epsilon'}\mathcal{E}_{\epsilon,\epsilon'}\Big(\big\{[0,t_1]^n_<\times[0,t_2]^n_<\big\}\setminus\big\{[0,t_1-\delta_1]^n_<\times[0,t_2-\delta_1]^n_<\big\}\Big)=0\,.$$

By the extension $G(t,x)=0$ for $t<0$, we can extend $\mathcal{E}_{\epsilon,\epsilon'}(\cdot)$ from a measure on $(\mathbb{R}^+)^n_<\times(\mathbb{R}^+)^n_<$ to a measure on $(\mathbb{R}^+)^n\times(\mathbb{R}^+)^n$ in an obvious way. Applying the general set identity

$$(A_1\times A_2)\setminus(B_1\times B_2)=\big(A_1\times(A_2\setminus B_2)\big)\cap\big((A_1\setminus B_1)\times A_2\big)$$

with $B_1 \subseteq A_1$ and $B_2 \subseteq A_2$, we have

$$\begin{aligned}&\big\{[0,t_1]^n_< \times [0,t_2]^n_<\big\} \setminus \big\{[0,t_1-\delta_1]^n_< \times [0,t_2-\delta_1]^n_<\big\}\\ &\quad= \Big([0,t_1]^n_< \times \big\{[0,t_2]^n_< \setminus [0,t_2-\delta_2]^n_<\big\}\Big) \cup \Big(\big\{[0,t_1]^n_< \setminus [0,t_1-\delta_1]^n_<\big\} \times [0,t_2]^n_<\Big)\\ &\quad\subseteq \Big([0,t_1]^n_< \times \big\{[0,t_2]^{n-1}_< \times [t_2-\delta_2,t_2]\big\}\Big) \cup \Big(\big\{[0,t_1]^{n-1}_< \times [t_1-\delta_1,t_1]\big\} \times [0,t_2]^n_<\Big),\end{aligned}$$

where the second step follows from the relations

$$[0,t_i]^n_< \setminus [0,t_i-\delta_i]^n_< \subseteq [0,t_i]^{n-1}_< \times [t_i-\delta_i,t_i] \qquad i=1,2.$$

Indeed, "$(s_1,\cdots,s_n) \in [0,t_1]^n_< \setminus [0,t_1-\delta_1]^n_<$" means "$0 \le s_1 < \cdots < s_n \le t_1$" and "$s_k > t_1-\delta_1$ for some $1 \le k \le n$". Since $s_n \ge s_k > t_1-\delta_1$, we have $(s_1,\cdots,s_n) \in [0,t_1]^{n-1}_< \times [t_1-\delta_1,\ t_1]$.

Therefore, the problem is further reduced to

$$\lim_{\delta\to 0^+} \sup_{\epsilon,\epsilon'} \mathcal{E}_{\epsilon,\epsilon'}\Big([0,t_1]^n_< \times [0,t_2]^{n-1}_< \times [t_2-\delta,t_2]\Big) = 0 \tag{3.20}$$

and

$$\lim_{\delta\to 0^+} \sup_{\epsilon,\epsilon'} \mathcal{E}_{\epsilon,\epsilon'}\Big([0,t_1]^{n-1}_< \times [t_1-\delta,t_1] \times [0,t_2]^n_<\Big) = 0\,. \tag{3.21}$$

Due to similarity, we only prove (3.20). By Fubini's theorem

$$\begin{aligned}&\mathcal{E}_{\epsilon,\epsilon'}\Big([0,t_1]^n_< \times [0,t_2]^{n-1}_< \times [t_2-\delta,t_2]\Big)\\ &= \int_{(\mathbb{R}^d)^{2n-1}} dx_1\cdots dx_{2n-1}\Big(\prod_{(j,k)\in\mathcal{D}'} \gamma_{\epsilon_j+\epsilon_k}(x_j-x_k)\Big)\int_{[0,t_1]^n_<\times[0,t_2]^{n-1}_<} ds_1\cdots ds_{2n-1}\\ &\qquad\times\Big(G(s_1,x_1)\prod_{l=2}^n G(s_l-s_{l-1},x_l-x_{l-1})\Big)\Big(G(s_{n+1},x_{n+1})\prod_{k=n+2}^{2n-1} G(s_l-s_{l-1},x_k-x_{k-1})\Big)\\ &\qquad\times\int_{\mathbb{R}^d} dx_{2n}\gamma_{\epsilon_{j_0}+\epsilon_{2n}}(x_{2n}-x_{j_0})\int_{t_2-\delta}^{t_2} G(s_{2n}-s_{2n-1},x_{2n}-x_{2n-1})ds_{2n}\,,\end{aligned}$$

where $1 \le j_0 \le 2n-1$ satisfies $(j_0,2n) \in \mathcal{D}$ and where $\mathcal{D}' \in \Pi_{n-1}$ is given by $\mathcal{D}' = \mathcal{D} \setminus (j_0,2n)$.

By Fourier transform and Fubini's theorem

$$\begin{aligned}&\int_{\mathbb{R}^d} dx_{2n}\gamma_{\epsilon_{j_0}+\epsilon_{2n}}(x_{2n}-x_{j_0})\int_{t_2-\delta}^{t_2} G(s_{2n}-s_{2n-1},x_{2n}-x_{2n-1})ds_{2n}\\ &\quad= \int_{\mathbb{R}^d} dx_{2n}\gamma_{\epsilon_{j_0}+\epsilon_{2n}}(x_{2n}-x_{j_0})\int_{0\vee(t_2-s_{2n-1}-\delta)}^{t_2-s_{2n-1}} G(s,x_{2n}-x_{2n-1})ds\\ &\quad= \int_{\mathbb{R}^d} \mu(d\xi)\exp\Big\{-\frac{\epsilon_{j_0}+\epsilon_{2n}}{2}|\xi|^2\Big\}\int_{0\vee(t_2-s_{2n-1}-\delta)}^{t_2-s_{2n-1}} ds\\ &\qquad\times\int_{\mathbb{R}^d}\exp\big\{i\xi\cdot(x_{2n}-x_{j_0})\big\}G(s,x_{2n}-x_{2n-1})dx_{2n}\,.\end{aligned}$$

Using (2.4), the right hand side is equal to

$$\int_{\mathbb{R}^d}\mu(d\xi)\exp\Big\{-\frac{\epsilon_{j_0}+\epsilon_{2n}}{2}|\xi|^2+i\xi\cdot(x_{2n-1}-x_{j_0})\Big\}\int_{0\vee(t_2-s_{2n-1}-\delta)}^{t_2-s_{2n-1}}ds$$
$$\times\int_{\mathbb{R}^d}\exp\big\{i\xi\cdot(x_{2n}-x_{2n-1})\big\}G(s,x_{2n}-x_{2n-1})dx_{2n}$$
$$=\int_{\mathbb{R}^d}\mu(d\xi)\exp\Big\{-\frac{\epsilon_{j_0}+\epsilon_{2n}}{2}|\xi|^2+i\xi\cdot(x_{2n-1}-x_{j_0})\Big\}\int_{0\vee(t_2-s_{2n-1}-\delta)}^{t_2-s_{2n-1}}\frac{\sin(|\xi|s)}{|\xi|}ds$$
$$\le\int_{\mathbb{R}^d}\mu(d\xi)\bigg|\int_{0\vee(t_2-s_{2n-1}-\delta)}^{t_2-s_{2n-1}}\frac{\sin(|\xi|s)}{|\xi|}ds\bigg|.$$

Notice that

$$\int_{0\vee(t_2-s_{2n-1}-\delta)}^{t_2-s_{2n-1}}\frac{\sin(|\xi|s)}{|\xi|}ds=\frac{\cos\big(0\vee(t_2-s_{2n-1}-\delta)|\xi|-\cos(t_2-s_{2n-1})|\xi|}{|\xi|^2}$$
$$=\frac{2}{|\xi|^2}\sin\frac{|\xi|\big((t_2-s_{2n-1})-0\vee(t_2-s_{2n-1}-\delta)\big)}{2}$$
$$\times\sin\frac{|\xi|\big((t_2-s_{2n-1})+0\vee(t_2-s_{2n-1}-\delta)\big)}{2}.$$

By the bounds $0\le(t_2-s_{2n-1})-0\vee(t_2-s_{2n-1}-\delta)\le\delta$ and $|\sin\theta|\le|\theta|$

$$\int_{\mathbb{R}^d}\mu(d\xi)\bigg|\int_{0\vee(t_2-s_{2n-1}-\delta)}^{t_2-s_{2n-1}}\frac{\sin(|\xi|s)}{|\xi|}ds\bigg|\le 4t_2\delta\mu(|\xi|\le N)+2\int_{\{|\xi|\ge N\}}\frac{1}{|\xi|^2}\mu(d\xi)$$

for any $N>0$.

In summary, there is a $\beta(\delta)>0$ independent of (ϵ,ϵ') such that

$$\int_{\mathbb{R}^d}dx_{2n}\gamma_{\epsilon_{j_0}+\epsilon_{2n}}(x_{2n}-x_{j_0})\int_{t_2-\delta}^{t_2}G(s_{2n}-s_{2n-1},x_{2n}-x_{2n-1})ds_{2n}\le\beta(\delta)$$

and that $\beta(\delta)\to0$ as $\delta\to0^+$. Consequently,

$$\mathcal{E}_{\epsilon,\epsilon'}\Big([0,t_1]^n_<\times[0,t_2]^{n-1}_<\times[t_2-\delta,t_2]\Big)\le\beta(\delta)\Lambda_{\epsilon,\epsilon'}(t_1,t_2),$$

where

$$\Lambda_{\epsilon,\epsilon'}(t_1,t_2)=\int_{(\mathbb{R}^d)^{2n-1}}dx_1\cdots dx_{2n-1}\bigg(\prod_{(j,k)\in\mathcal{D}'}\gamma_{\epsilon_j+\epsilon_k}(x_j-x_k)\bigg)\int_{[0,t_1]^n_<\times[0,t_2]^{n-1}_<}ds_1\cdots ds_{2n-1}$$
$$\times\bigg(G(s_1,x_1)\prod_{l=2}^nG(s_l-s_{l-1},x_l-x_{l-1})\bigg)\bigg(G(s_{n+1},x_{n+1})\prod_{k=n+2}^{2n-1}G(s_l-s_{l-1},x_k-x_{k-1}\bigg).$$

To establish (3.20) and therefore to complete the proof, it suffices to show that

$$\sup_{\epsilon,\epsilon'}\Lambda_{\epsilon,\epsilon'}(t_1,t_2)<\infty. \tag{3.22}$$

Indeed, by a computation similar to the one used for (3.4)

$$\int_0^\infty\int_0^\infty e^{-t-\tilde{t}}\Lambda_{\epsilon,\epsilon'}(t,\tilde{t})dtd\tilde{t}$$
$$\leq \Big(\frac{1}{2}\Big)^{2n+1}\int_0^\infty\int_0^\infty dtd\tilde{t}\exp\Big\{-\frac{t+\tilde{t}}{2}\Big\}\mathbb{E}_0\int_{[0,t]^n_<\times[0,\tilde{t}]^{n-1}_<} ds_1\cdots ds_{2n-1}$$
$$\times\prod_{(j,k)\in\mathcal{D}'}\gamma\big(B_{v(j)}(s_j)-B_{v(k)}(s_k)\big)$$

for any ϵ,ϵ'. The above right hand side is finite by the fact (Lemma 6.1) that the moments of Brownian intersection local times have polynomial increasing rates in time.

Finally, by non-negativity and monotonicity of $\Lambda_{\epsilon,\epsilon'}(t,\tilde{t})$ in t and $\tilde{t}$, (3.20) follows from the bound

$$\sup_{\epsilon,\epsilon'}\Lambda_{\epsilon,\epsilon'}(t_1,t_2)\leq\exp\{t_1+t_2\}\sup_{\epsilon,\epsilon'}\int_0^\infty\int_0^\infty e^{-t-\tilde{t}}\Lambda_{\epsilon,\epsilon'}(t,\tilde{t})dtd\tilde{t}<\infty\,.$$

This completes the proof. □

Keep in mind that the proof of Theorem 3.3 depends on the Stratonovich integrability of $g_n(\cdot,t,x)$ and the $\mathcal{L}^2$-convergence of the Laplace transform

$$\int_0^\infty e^{-\lambda t}S_{n,\epsilon}\big(g_n(\cdot,t,x)\big)dt \quad\text{as } \epsilon\to 0$$

that are installed in the following:

THEOREM 3.8. *Under Dalang's condition (1.6),*

(i) *the $\mathcal{L}^2$-limit*

$$\lim_{\epsilon_1,\cdots,\epsilon_n\to 0^+}\int_{(\mathbb{R}^d)^n} g_n(x_1,\cdots,x_n,t,x)\Big(\prod_{k=1}^n\dot{W}_{\epsilon_k}(x_k)\Big)dx_1\cdots dx_n \tag{3.23}$$

exists for any $n\geq 1$ and $(t,x)\in\mathbb{R}^+\times\mathbb{R}^d$. Consequently, $g_n(\cdot,t,x)$ is integrable in the sense of Definition 2.2 and the limit in (3.23) is $S_n\big(g_n(\cdot,t,x)\big)$.

(ii) *for any $\lambda>0$,*

$$\lim_{\epsilon\to 0}\int_0^\infty e^{-\lambda t}S_{n,\epsilon}\big(g_n(\cdot,t,x)\big)dt=\int_0^\infty e^{-\lambda t}S_n\big(g_n(\cdot,t,x)\big)dt \quad \textit{in } \mathcal{L}^2(\Omega,\mathcal{F},\mathbb{P})\,. \tag{3.24}$$

PROOF. By Lemma 2.4, all we need is to show

$$\lim_{\epsilon,\epsilon'\to 0^+}\mathbb{E}S_{n,\epsilon}\big(g_n(\cdot,t,x)\big)S_{n,\epsilon'}\big(g_n(\cdot,t,x)\big)\,, \tag{3.25}$$

exists, where $S_{n,\epsilon}\big(g_n(\cdot,t,x)\big)$ is defined in (3.1) and where $\epsilon'=(\epsilon_{n+1},\cdots,\epsilon_{2n})$. We have

$$\mathbb{E}S_{n,\epsilon}\big(g_n(\cdot,t,x)\big)S_{n,\epsilon'}\big(g_n(\cdot,t,x)\big)$$
$$=\int_{(\mathbb{R}^d)^{2n}}dx_1\cdots dx_{2n}g_n(x_1,\cdots,x_n,t,0)g_n(x_{n+1},\cdots,x_{2n},t,0)\mathbb{E}\Big(\prod_{k=1}^{2n}\dot{W}_{\epsilon_k}(x_k)\Big)$$
$$=\sum_{\mathcal{D}\in\Pi_n}\int_{(\mathbb{R}^d)^{2n}}dx_1\cdots dx_{2n}\Big(\prod_{(j,k)\in\mathcal{D}}\gamma_{\epsilon_j+\epsilon_k}(x_j-x_k)\Big)g_n(x_1,\cdots,x_n,t,0)g_n(x_{n+1},\cdots,x_{2n},t,0)$$
$$=\sum_{\mathcal{D}\in\Pi_n}F^{\mathcal{D}}_{\epsilon,\epsilon'}(t_1,t_2)\,,$$

where the second step follows from the Wick's formula (2.15) with $g_k = \dot W_{\epsilon_k}(x_k)$ ($k = 1, \cdots 2n$). Therefore, the existence of the limit in (3.25) follows from Lemma 3.6.

We now come to Part (ii). Notice that

$$\begin{aligned}
&\mathbb{E}\Big[\int_0^\infty e^{-\lambda t} S_{n,\epsilon}\big(g_n(\cdot,t,x)\big)dt - \int_0^\infty e^{-\lambda t} S_n\big(g_n(\cdot,t,x)\big)dt\Big]^2 \\
&\quad = \int_0^\infty\int_0^\infty e^{-\lambda(t_1+t_2)}\mathbb{E}S_{n,\epsilon}\big(g_n(\cdot,t_1,0)\big)S_{n,\epsilon}\big(g_n(\cdot,t_2,0)\big)dt_1dt_2 \\
&\qquad - 2\int_0^\infty\int_0^\infty e^{-\lambda(t_1+t_2)}\mathbb{E}S_{n,\epsilon}\big(g_n(\cdot,t_1,0)\big)S_n\big(g_n(\cdot,t_2,0)\big)dt_1dt_2 \\
&\qquad + \int_0^\infty\int_0^\infty e^{-\lambda(t_1+t_2)}\mathbb{E}S_n\big(g_n(\cdot,t_1,0)\big)S_n\big(g_n(\cdot,t_2,0)\big)dt_1dt_2\,.
\end{aligned}$$

For the first term,

$$\begin{aligned}
&\int_0^\infty\int_0^\infty dt_1dt_2 e^{-\lambda(t_1+t_2)}\mathbb{E}S_{n,\epsilon}\big(g_n(\cdot,t_1,0)\big)S_{n,\epsilon}\big(g_n(\cdot,t_2,0)\big)dt_1dt_2 \\
&\quad = \sum_{\mathcal{D}\in\Pi_n}\int_0^\infty\int_0^\infty e^{-\lambda(t_1+t_2)}F^{\mathcal{D}}_{\epsilon,\epsilon}(t_1,t_2)dt_1dt_2\,.
\end{aligned}$$

In view of Remark 3.7, the function $F^{\mathcal{D}}(t_1,t_2)$ appearing in (3.18) is identified as

$$\begin{aligned}
F^{\mathcal{D}}(t_1,t_2) = &\int_{(\mathbb{R}^d)^{2n}} dx_1\cdots dx_{2n}\Big(\prod_{(j,k)\in\mathcal{D}}\gamma(x_j-x_k)\Big) \\
&\times g_n(x_1,\cdots,x_n,t_1,0)g_n(x_{n+1},\cdots,x_{2n},t_2,0)\,.
\end{aligned}$$

By (3.17) with $\lambda_1=\lambda_2=\lambda$, therefore,

$$\begin{aligned}
&\lim_{\epsilon\to 0}\int_0^\infty\int_0^\infty dt_1dt_2e^{-\lambda(t_1+t_2)}\mathbb{E}S_{n,\epsilon}\big(g_n(\cdot,t_1,0)\big)S_{n,\epsilon}\big(g_n(\cdot,t_2,0)\big)dt_1dt_2 \\
&\quad = \sum_{\mathcal{D}\in\Pi_n}\int_0^\infty\int_0^\infty dt_1dt_2e^{-\lambda(t_1+t_2)}\int_{(\mathbb{R}^d)^{2n}}dx_1\cdots dx_{2n}\Big(\prod_{(j,k)\in\mathcal{D}}\gamma(x_j-x_k)\Big) \\
&\quad \times g_n(x_1,\cdots,x_n,t_1,0)g_n(x_{n+1},\cdots,x_{2n},t_2,0) \\
&\quad = \int_0^\infty\int_0^\infty dt_1dt_2e^{-\lambda(t_1+t_2)}\mathbb{E}S_n\big(g_n(\cdot,t_2,0)\big)S_n\big(g_n(\cdot,t_2,0)\big)dt_1dt_2\,,
\end{aligned}$$

where the last step follows from Stratonovich integrability stated in Part (i) and the identity in (2.21).

Using Part (i),

$$\mathbb{E}S_{n,\epsilon}\big(g_n(\cdot,t_1,0)\big)S_n\big(g_n(\cdot,t_2,0\big) = \lim_{\epsilon'\to 0}\mathbb{E}S_{n,\epsilon}\big(g_n(\cdot,t_1,0)\big)S_{n,\epsilon'}\big(g_n(\cdot,t_2,0)\big)\,.$$

By the fact that $\mathbb{E}S_{n,\epsilon}\big(g_n(\cdot,t_1,0)\big)S_n\big(g_n(\cdot,t_2,0)\big)\geq 0$ and by Fatou's lemma,

$$\begin{aligned}\liminf_{\epsilon\to 0}\int_0^\infty\int_0^\infty & dt_1dt_2e^{-\lambda(t_1+t_2)}\mathbb{E}S_{n,\epsilon}\big(g_n(\cdot,t_1,0)\big)S_n\big(g_n(\cdot,t_2,0)\big)\\ &\geq\int_0^\infty\int_0^\infty dt_1dt_2e^{-\lambda(t_1+t_2)}\liminf_{\epsilon\to 0}\mathbb{E}S_{n,\epsilon}\big(g_n(\cdot,t_1,0)\big)S_n\big(g_n(\cdot,t_2,0)\big)\\ &=\int_0^\infty\int_0^\infty dt_1dt_2e^{-\lambda(t_1+t_2)}\lim_{\epsilon,\epsilon'\to 0}\mathbb{E}S_{n,\epsilon}\big(g_n(\cdot,t_1,0)\big)S_{n,\epsilon'}\big(g_n(\cdot,t_2,0)\big)\\ &=\int_0^\infty\int_0^\infty dt_1dt_2e^{-\lambda(t_1+t_2)}\mathbb{E}S_n\big(g_n(\cdot,t_1,0)\big)S_n\big(g_n(\cdot,t_2,0)\big)\,,\end{aligned}$$

where we have used Part (i) in the last two steps.

Summarizing our argument,

$$\lim_{\epsilon\to 0}\mathbb{E}\bigg[\int_0^\infty e^{-\lambda t}S_{n,\epsilon}\big(g_n(\cdot,t,0)\big)dt-\int_0^\infty e^{-\lambda t}S_n\big(g_n(\cdot,t,0)\big)dt\bigg]^2=0\,.$$

This competes the proof. □

We now establish Fubini's theorem for the multiple Stratonovich integral with the integrand $g_n(\cdot,t,x)$.

LEMMA 3.9. *Under Dalang's condition (1.6), we have*

(3.26)

$$\lim_{\varepsilon_2,\cdots,\varepsilon_n\to 0^+}S_{n,\varepsilon}\big(g_n(\cdot,t,x)\big)=\int_{\mathbb{R}^d}\bigg(\int_0^t G(t-s,y-x)S_{n-1}\big(g_{n-1}(\cdot,s,y)\big)ds\bigg)\dot{W}_{\varepsilon_1}(y)dy$$

and

(3.27)

$$\lim_{\varepsilon_1\to 0^+}\int_{\mathbb{R}^d}\bigg(\int_0^t G(t-s,y-x)S_{n-1}\big(g_{n-1}(\cdot,s,y)\big)ds\bigg)\dot{W}_{\varepsilon_1}(y)dy=S_n\big(g_n(\cdot,t,x)\big)\,,$$

where the limits are taken in $\mathcal{L}^p(\Omega,\mathcal{F},\mathbb{P})$ for any $p\geq 1$.

REMARK 3.10. The identity (3.27) mathematically confirms the relation (2.11).

PROOF. Part (i) in Theorem 3.8 shows that $S_{n,\epsilon}\big(g_n(\cdot,t,x)\big)$ $\mathcal{L}^2$-converges to $S_n\big(g_n(\cdot,t,x)\big)$ as $\epsilon=(\epsilon_1,\cdots,\epsilon_n)\to 0$. By Theorem 6.2 this convergence also holds to L^p for any $p\geq 1$. Thus, the set

$$\{|S_{n,\epsilon}\big(g_n(\cdot,t,x)\big)|^p,\varepsilon_i\in(0,1],i=2,\cdots,d\}$$

is bounded in L^2 for any $p\geq 1$ (and for any fixed $\varepsilon_1>0$), and hence it is uniformly integrable. Therefore, by [2, p. 297, Theorem 7.5.4] all we need for establishing (3.26) is to prove it with the convergence in $\mathcal{L}^1(\Omega,\mathcal{F},\mathbb{P})$ instead. By the Fubini theorem

$$S_{n,\varepsilon}\big(g_n(\cdot,t,x)\big)=\int_{\mathbb{R}^d}\bigg(\int_0^t G(t-s,y-x)S_{n-1,\tilde{\varepsilon}}\big(g_{n-1}(\cdot,s,y)\big)ds\bigg)\dot{W}_{\varepsilon_1}(y)dy\,,$$

where

$$\begin{aligned}S_{n-1,\tilde{\varepsilon}}\big(g_{n-1}(\cdot,t,x)\big)=&\int_{(\mathbb{R}^d)^{n-1}}\int_{[0,t]_<^{n-1}}ds_1\cdots ds_{n-1}\\ &\times\bigg(\prod_{k=1}^{n-1}G(s_k-s_{k-1},x_k-x_{k-1})\bigg)\prod_{k=1}^{n-1}\dot{W}_{\varepsilon_{k+1}}(x_k)dx_k\end{aligned}$$

with notation $\tilde\varepsilon=(\varepsilon_2,\cdots,\varepsilon_n)$. So we have

$$S_{n,\varepsilon}\big(g_n(\cdot,t,x)\big)-\int_{\mathbb{R}^d}\left(\int_0^t G(t-s,y-x)S_{n-1}\big(g_{n-1}(\cdot,s,y)\big)ds\right)\dot W_{\varepsilon_1}(y)dy$$
$$=\int_0^t ds\int_{\mathbb{R}^d}\Big[S_{n-1,\tilde\varepsilon}\big(g_{n-1}(\cdot,s,y)\big)-S_{n-1}\big(g_{n-1}(\cdot,s,y)\big)\Big]W_{\varepsilon_1}(y)G(t-s,y-x)dy\,.$$

By the Cauchy-Schwartz inequality

$$\mathbb{E}\left|S_{n,\varepsilon}\big(g_n(\cdot,t,x)\big)-\int_{\mathbb{R}^d}\left(\int_0^t G(t-s,y-x)S_{n-1}\big(g_{n-1}(\cdot,s,y)\big)ds\right)\dot W_{\varepsilon_1}(y)dy\right|$$
$$\le\left\{\mathbb{E}\int_0^t ds\int_{\mathbb{R}^d}\Big[S_{n-1,\tilde\varepsilon}\big(g_{n-1}(\cdot,s,y)\big)-S_{n-1}\big(g_{n-1}(\cdot,s,y)\big)\Big]^2G(t-s,y-x)dy\right\}^{1/2}$$
$$\times\left\{\mathbb{E}\int_0^t ds\int_{\mathbb{R}^d}|\dot W_{\varepsilon_1}(y)|^2G(t-s,y-x)dy\right\}^{1/2}$$
$$=\left\{\mathbb{E}\int_0^t ds\mathbb{E}\Big[S_{n-1,\tilde\varepsilon}\big(g_{n-1}(\cdot,s,0)\big)-S_{n-1}\big(g_{n-1}(\cdot,s,0)\big)\Big]^2\int_{\mathbb{R}^d}G(t-s,y-x)dy\right\}^{1/2}$$
$$\times\left\{\mathbb{E}|\dot W_{\varepsilon_1}(0)|^2\right\}^{1/2}\left\{\int_0^t ds\int_{\mathbb{R}^d}G(t-s,y-x)dy\right\}^{1/2},$$

where the last step follows from the facts that $\mathbb{E}|\dot W_{\varepsilon_1}(y)|^2=\mathbb{E}|\dot W_{\varepsilon_1}(0)|^2$ and

$$\mathbb{E}\Big[S_{n-1,\tilde\varepsilon}\big(g_{n-1}(\cdot,s,y)\big)-S_{n-1}\big(g_{n-1}(\cdot,s,y)\big)\Big]^2=\mathbb{E}\Big[S_{n-1,\tilde\varepsilon}\big(g_{n-1}(\cdot,s,0)\big)-S_{n-1}\big(g_{n-1}(\cdot,s,0)\big)\Big]^2.$$

Further,

$$\int_{\mathbb{R}^d}G(t-s,y-x)dy=t-s$$

We have the bound

$$\mathbb{E}\left|S_{n,\varepsilon}\big(g_n(\cdot,t,x)\big)-\int_{\mathbb{R}^d}\left(\int_0^t G(t-s,y-x)S_{n-1}\big(g_{n-1}(\cdot,s,y)\big)ds\right)\dot W_{\varepsilon_1}(y)dy\right|$$
$$\le\frac{1}{2}t^{3/2}\left\{\mathbb{E}|\dot W_{\varepsilon_1}(0)|^2\right\}^{1/2}\left\{\int_0^t\mathbb{E}\Big[S_{n-1,\tilde\varepsilon}\big(g_{n-1}(\cdot,s,0)\big)-S_{n-1}\big(g_{n-1}(\cdot,s,0)\big)\Big]^2ds\right\}^{1/2}.$$

By Part (i) of Theorem 3.8 (with n being replaced by $n-1$),

$$\lim_{\varepsilon_2,\cdots,\varepsilon_n\to0^+}\mathbb{E}\Big[S_{n-1,\tilde\varepsilon}\big(g_{n-1}(\cdot,s,0)\big)-S_{n-1}\big(g_{n-1}(\cdot,s,0)\Big]^2=0\,,\qquad 0\le s\le t\,.$$

In addition

$$\mathbb{E}\Big[S_{n-1,\tilde\varepsilon}\big(g_{n-1}(\cdot,s,0)\big)-S_{n-1}\big(g_{n-1}(\cdot,s,0)\big)\Big]^2$$
$$\le\mathbb{E}\big[S_{n-1,\tilde\varepsilon}\big(g_{n-1}(\cdot,s,0)\big)\big]^2+\mathbb{E}\big[S_{n-1}\big(g_{n-1}(\cdot,s,0)\big)\big]^2$$
$$\le\mathbb{E}\big[S_{n-1,\tilde\varepsilon}\big(g_{n-1}(\cdot,t,0)\big)\big]^2+\mathbb{E}\big[S_{n-1}\big(g_{n-1}(\cdot,t,0)\big)\big]^2.$$

By dominated convergence we see

$$\lim_{\varepsilon_2,\cdots,\varepsilon_n\to0^+}\int_0^t\mathbb{E}\Big[S_{n-1,\tilde\varepsilon}\big(g_{n-1}(\cdot,s,0)\big)-S_{n-1}\big(g_{n-1}(\cdot,s,0)\big)\Big]^2ds=0\,.$$

This proves the (3.26). Finally, (3.27) follows from (3.26) and Theorem 3.8. □

3.3. *Link to Dalang-Mueller-Tribe's work.* The discussion in this sub-section does not contribute to the proof of the main theorems in this paper. Rather, it helps the interested reader to better understand the true nature of Stratonovich solution and provide a new representation to the Laplace transform of the deterministic system (1.7) for possible future investigation.

Let $N(t)$ ($t\ge 0$) be a Poisson process with parameter 1 and $\{\tau_k\}_{k\ge 1}$ be the jumping times of $N(t)$ with definition $\tau_0=0$. The stochastic process X_t ($t\ge 0$) is defined as follows: First, $\{X_{\tau_k}\}_{k\ge 1}$ is a random sequence whose finite-dimensional distribution of $(X_{\tau_1},\cdots,X_{\tau_k})$ has the conditional distribution (conditioning on $\{\tau_1,\cdots,\tau_n\}$)

$$\Big(\prod_{k=1}^n(\tau_k-\tau_{k-1})^{-1}G(\tau_k-\tau_{k-1},x_k-x_{k-1})\Big)dx_1\cdots dx_n\,.$$

Set $X_{\tau_0}=X_0=x$. The process X_t is defined as the linear interpolation of $\{X_{\tau_k}\}_{k\ge 0}$.

Dalang, Mueller and Tribe (Theorem 3.2, [16]) prove that the function

$$u(t,x)=e^t\mathbb{E}_x\left[u_0(t-\tau_{N(t)},X_{\tau_{N(t)}})\prod_{k=1}^{N(t)}(\tau_k-\tau_{k-1})f(X_{\tau_k})\right] \tag{3.28}$$

solves the wave equation (1.7), where $u_0(t,x)$ appears in (2.1). For the purpose of comparison, we consider the case when $u_0(t,x)=1$ and write

$$\begin{aligned}u(t,x)&=\sum_{n=0}^\infty e^t\mathbb{P}\{N(t)=n\}\mathbb{E}_x\left[\prod_{k=1}^n(\tau_k-\tau_{k-1})f(X_{\tau_k})\Big|N(t)=n\right]\\&=\sum_{n=0}^\infty\frac{t^n}{n!}\mathbb{E}_x\left[\prod_{k=1}^n(\tau_k-\tau_{k-1})f(X_{\tau_k})\Big|N(t)=n\right].\end{aligned}$$

By the classic fact that conditioning on $\{N(t)=n\}$, the n-dimensional vector $(\tau_1,\cdots,\tau_n)$ is uniformly distribution on $[0,t]^n_<$,

$$\begin{aligned}u(t,x)&=\sum_{n=0}^\infty\int_{[0,t]^n_<}ds_1\cdots ds_n\int_{(\mathbb{R}^d)^n}dx_1\cdots dx_n\Big(\prod_{k=1}^nG(s_k-s_{k-1},x_k-x_{k-1})\Big)\prod_{k=1}^nf(x_k)\\&=\sum_{n=0}^\infty\int_{(\mathbb{R}^d)^n}dx_1\cdots dx_ng_n(x_1,\cdots,x_n,t,x)\prod_{k=1}^nf(x_k)\end{aligned} \tag{3.29}$$

with the convention $s_0=0$ and $x_0=x$. Comparing this with (2.2) and (2.12) we see the deterministic root of stochastic model (1.1) in the Stratonovich setting.

Similar to (3.10), the same computation leads to

$$\begin{aligned}&\int_0^\infty e^{-\lambda t}\int_{(\mathbb{R}^d)^n}dx_1\cdots dx_ng_n(x_1,\cdots,x_n,t,x)\prod_{k=1}^nf(x_k)\\&\qquad=\frac{1}{n!}\frac{\lambda}{2}\Big(\frac12\Big)^n\int_0^\infty\exp\Big\{-\frac{\lambda^2}{2}t\Big\}\mathbb{E}_x\left[\int_0^tf\big(B(s)\big)ds\right]^ndt\,.\end{aligned} \tag{3.30}$$

Summing both sides over n, we obtain the following representation

$$\int_0^\infty e^{-\lambda t}u(t,x)dt=\frac{\lambda}{2}\int_0^\infty\exp\Big\{-\frac{\lambda^2}{2}t\Big\}\mathbb{E}_x\exp\Big\{\frac12\int_0^tf\big(B(s)\big)ds\Big\}dt \tag{3.31}$$

in the sense that finiteness of one side leads to finiteness of the other side, and to the equality.

The classic semi-group theory (see, e.g., Section 4.1, [8]) claims an asymptotically linear growth of the logarithmic exponential moment

$$\log\mathbb{E}_x\exp\left\{\frac{1}{2}\int_0^t f\big(B(s)\big)ds\right\}\qquad (t\to\infty)$$

for a class of functions f. In this case, the right hand side of (3.31) is finite for large λ.

On the other hand, the representation (3.31) unlikely makes sense for the stochastic wave equation (1.1). Under the assumption in Theorem 1.2, we have ([9])

$$\log\mathbb{E}_x\exp\left\{\frac{1}{2}\int_0^t \dot W\big(B(s)\big)ds\right\}\sim C(\gamma)t(\log t)^{\frac{2}{4-\alpha}}\qquad a.s.$$

for some constant $C(\gamma)>0$ as $t\to\infty$. So (3.31) almost surely blows up for any $\lambda>0$ when $f(\cdot)$ is replaced by $\dot W(\cdot)$ (or, when the deterministic system (1.7) is replaced by our model (1.1))

4. Proof of Theorem 1.1.

PROOF OF THEOREM 1.1. To show that the Stratonovich expansion (2.15) converges in $\mathcal{L}^2(\Omega,\mathcal{F},\mathbb{P})$, by the triangle inequality and by the fact that $u(t,x)$ (if defined) is stationary in x, all we need is

$$\sum_n\left\{\mathbb{E}\big[S_n\big(g_n(\cdot,t,0)\big)\big]^2\right\}^{1/2}<\infty,\qquad\forall t>0. \tag{4.1}$$

The procedure starts at Corollary 3.5. By the Cauchy-Schwartz inequality

$$\begin{aligned}
&\mathbb{E}\otimes\mathbb{E}_0\left[\int_0^{t_1}\dot W\big(B_1(s)ds\right]^n\left[\int_0^{t_2}\dot W\big(B_2(s)\big)ds\right]^n\\
&\le\left\{\mathbb{E}\otimes\mathbb{E}_0\left[\int_0^{t_1}\dot W\big(B(s)ds\right]^{2n}\right\}^{1/2}\left\{\mathbb{E}\otimes\mathbb{E}_0\left[\int_0^{t_2}\dot W\big(B(s)ds\right]^{2n}\right\}^{1/2}\\
&=\left\{\mathbb{E}_0\left[\int_0^{t_1}\!\!\int_0^{t_1}\gamma\big(B(s)-B(r)\big)dsdr\right]^n\right\}^{1/2}\left\{\mathbb{E}_0\left[\int_0^{t_2}\!\!\int_0^{t_2}\gamma\big(B(s)-B(r)\big)dsdr\right]^n\right\}^{1/2}.
\end{aligned}$$

Let $t>0$ be fixed. Taking $\lambda_1=\lambda_2=nt^{-1}$ in (3.14), Corollary 3.5:

$$\begin{aligned}
&\int_0^\infty\!\!\int_0^\infty dt_1dt_2\exp\left\{-\frac{n}{t}(t_1+t_2)\right\}\mathbb{E}\left[S_n\big(g_n(\cdot,t_1,0)\big)S_n\big(g_n(\cdot,t_2,0)\big)\right]\\
&\le\frac{(2n)!}{(n!)^2}\Big(\frac{n}{2t}\Big)^2\Big(\frac{1}{2}\Big)^{3n}\int_0^\infty\!\!\int_0^\infty dt_1dt_2\exp\left\{-\frac{n^2}{2t^2}(t_1+t_2)\right\}\\
&\quad\times\left\{\mathbb{E}_0\left[\int_0^{t_1}\!\!\int_0^{t_1}\gamma\big(B(s)-B(r)\big)dsdr\right]^n\right\}^{1/2}\left\{\mathbb{E}_0\left[\int_0^{t_2}\!\!\int_0^{t_2}\gamma\big(B(s)-B(r)\big)dsdr\right]^n\right\}^{1/2}\\
&=\frac{(2n)!}{(n!)^2}\Big(\frac{n}{2t}\Big)^2\Big(\frac{1}{2}\Big)^{3n}\left\{\int_0^\infty d\tilde t\exp\left\{-\frac{n^2}{2t^2}\tilde t\right\}\left(\mathbb{E}_0\left[\int_0^{\tilde t}\!\!\int_0^{\tilde t}\gamma\big(B(s)-B(r)\big)dsdr\right]^n\right)^{1/2}\right\}^2.
\end{aligned}$$

Recall ((1.5), Theorem 1.1, [11]) that under Dalang's condition (1.6), the limit

$$\lim_{\tilde t\to\infty}\frac{1}{\tilde t}\log\mathbb{E}_0\exp\left\{\frac{1}{\tilde t}\int_0^{\tilde t}\!\!\int_0^{\tilde t}\gamma(B(s)-B(r))dsdr\right\}$$

exists and is finite. This means there is a constant C such that

$$\mathbb{E}_0 \exp\left\{\frac{1}{\tilde{t}}\int_0^{\tilde{t}}\int_0^{\tilde{t}}\gamma(B(s)-B(r))dsdr\right\} \le \exp\{C\tilde{t}\}\,.$$

By the relation

$$\frac{1}{n!\tilde{t}^n}\mathbb{E}_0\left[\int_0^{\tilde{t}}\int_0^{\tilde{t}}\gamma(B(s)-B(r))dsdr\right]^n \le \mathbb{E}_0 \exp\left\{\frac{1}{\tilde{t}}\int_0^{\tilde{t}}\int_0^{\tilde{t}}\gamma(B(s)-B(r))dsdr\right\}$$

for any $\tilde{t}>0$, we have the bound that is uniform in $\tilde{t}$ and n:

$$\mathbb{E}_0\left[\int_0^{\tilde{t}}\int_0^{\tilde{t}}\gamma\big(B(s)-B(r)\big)dsdr\right]^n \le n!\tilde{t}^n\exp\{C\tilde{t}\}\,.$$

Hence,

$$\int_0^\infty \exp\Big\{-\frac{n^2}{2t^2}\tilde{t}\Big\}\bigg(\mathbb{E}_0\bigg[\int_0^{\tilde{t}}\int_0^{\tilde{t}}\gamma\big(B(s)-B(r)\big)dsdr\bigg]^n\bigg)^{1/2}d\tilde{t}$$
$$\le (n!)^{1/2}\int_0^\infty \exp\Big\{-\frac{n^2}{4t^2}\tilde{t}\Big\}\tilde{t}^{n/2}d\tilde{t} = (n!)^{1/2}\Big(\frac{4t^2}{n^2}\Big)^{n+1}\Gamma\Big(\frac{n}{2}+1\Big)\,.$$

Thus, by the Stirling formula we get the bound

(4.2)
$$\int_0^\infty\int_0^\infty dt_1dt_2\exp\Big\{-\frac{n}{t}(t_1+t_2)\Big\}\mathbb{E}\left[S_n\big(g_n(\cdot,t_1,0)\big)S_n\big(g_n(\cdot,t_2,0)\big)\right] \le \frac{C^n}{n!}t^{2n+4}\,.$$

By the fact that the moment

$$\mathbb{E}\left[S_n\big(g_n(\cdot,t_1,0)\big)S_n\big(g_n(\cdot,t_2,0)\big)\right]$$

is non-negative and non-decreasing in t_1 and t_2 we have

$$\int_0^\infty\int_0^\infty dt_1dt_2\exp\Big\{-\frac{n}{t}(t_1+t_2)\Big\}\mathbb{E}\left[S_n\big(g_n(\cdot,t_1,0)\big)S_n\big(g_n(\cdot,t_2,0)\big)\right]dt_1dt_2$$
$$\ge \mathbb{E}\left[S_n\big(g_n(\cdot,t,0)\big)\right]^2\int_t^\infty\int_t^\infty dt_1dt_2\exp\Big\{-\frac{n}{t}(t_1+t_2)\Big\}dt_1dt_2$$
$$= \frac{t^2}{n^2}e^{-2n}\mathbb{E}\left[S_n\big(g_n(\cdot,t,0)\big)\right]^2\,.$$

Comparing this with (4.2) we get the bound

$$\mathbb{E}\left[S_n\big(g_n(\cdot,t,0)\big)\right]^2 \le \frac{C_2^n}{n!}t^{2n+2}\,,\qquad n=1,2,\cdots \tag{4.3}$$

This leads to (4.1) and therefore to the $\mathcal{L}^2$-convergence of the Stratonovich expansion in (2.10).

In view of (3.15), the bound (4.3) remains true for $\mathbb{E}\left[S_{n,\epsilon}\big(g_n(\cdot,t,0)\big)\right]^2$ for any $\epsilon=(\epsilon_1,\cdots,\epsilon_n)$, i.e.,

$$\mathbb{E}\left[S_{n,\epsilon}\big(g_n(\cdot,t,0)\big)\right]^2 \le \frac{C_2^n}{n!}t^{2n+2}\,,\qquad n=1,2,\cdots$$

for any $t>0$. Let $\epsilon_2,\cdots,\epsilon_n\to 0^+$ on the left hand side. By (3.26), Lemma 3.9,

$$\mathbb{E}\left|\int_{\mathbb{R}^d}\bigg(\int_0^t G(t-s,y-x)S_{n-1}\big(g_{n-1}(\cdot,s,y)\big)ds\bigg)\dot{W}_{\varepsilon_1}(y)dy\right|^2 \le \frac{C_2^n}{n!}t^{2n+2} \tag{4.4}$$

for any $n = 1, 2\cdots$, any $t > 0$ and any $\epsilon_1 > 0$.

To show that $\{u(t,x)\}$ is a solution in the sense of Definition 2.1 and therefore to complete the proof of Part (i) of Theorem 1.1, we only need to show

(1) For any $t > 0$ and $x \in \mathbb{R}^d$, the random field $V(y) \equiv \int_0^t G_{t-s}(x-y)u(s,y)ds$ is Stratonovich integrable, or

$$\lim_{\epsilon_1 \to 0^+} \int_{\mathbb{R}^d} \Big(\int_0^t G_{t-s}(x-y)u(s,y)ds \Big) W_{\varepsilon_1}(y)dy = \int_{\mathbb{R}^d} \Big(\int_0^t G_{t-s}(x-y)u(s,y)ds \Big) W(dy)$$

in $\mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$.

(2) Equation (2.1) is satisfied with $u_0(t,x) = 1$.

Because of (4.3) and (4.4), to show (1) and (2) one has only to show that for all fixed $n \ge 1$, $\int_{\mathbb{R}^d} \big(\int_0^t G_{t-s}(x-y) S_{n-1}\big(g_{n-1}(\cdot,s,y)\big) ds \big) W_\varepsilon(y)dy$ converges to $S_n\big(g_n(\cdot,t,x)\big)$ in $\mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$. This is done in Lemma 3.9, Equation (3.27).

To prove Part (ii) of Theorem 1.1, all we need is to show that Dalang's condition is necessary for

$$\mathbb{E}\big[S_2\big(g_2(\cdot,t,0)\big)\big]^2 < \infty$$

with any $t > 0$. Indeed,

$$\begin{aligned}
&\mathbb{E}\big[S_2\big(g_2(\cdot,t,0)\big)\big]^2 \\
&= \sum_{\mathcal{D}\in\Pi_2} \int_{(\mathbb{R}^d)^4} dx_1dx_2dx_3dx_4 \Big(\prod_{\mathcal{D}\in\Pi_2} \gamma(x_j - x_k) \Big) g_2(x_1,x_2,t,0) g_2(x_3,x_4,t,0) \\
&\ge \int_{(\mathbb{R}^d)^4} dx_1dx_2dx_3dx_4 \gamma(x_1-x_2)\gamma(x_3-x_4) g_2(x_1,x_2,t,0) g_2(x_3,x_4,t,0) \\
&= \Big(\int_{(\mathbb{R}^d)^2} \gamma(x_2-x_1) g(x_1,x_2,t,0) dx_1dx_2 \Big)^2
\end{aligned}$$

and

$$\begin{aligned}
&\int_{(\mathbb{R}^d)^2} \gamma(x_2-x_1) g(x_1,x_2,t,0) dx_1dx_2 \\
&= \int_{[0,t]^2_<} ds_1ds_2 \int_{(\mathbb{R}^d)^2} \gamma(x_2-x_1) G(s_1,x_1) G(s_2-s_1, x_2-x_1) dx_1dx_2 \\
&= \int_{[0,t]^2_<} \Big(\int_{\mathbb{R}^d} G(s_1,x)dx \Big) \Big(\int_{\mathbb{R}^d} \gamma(x) G(s_2-s_1,x)dx \Big) ds_1ds_2 \\
&= \int_{[0,t]^2_<} s_1 \Big[\int_{\mathbb{R}^d} \frac{\sin(|\xi|(s_2-s_1))}{|\xi|} \mu(d\xi) \Big] ds_1 = \int_{\mathbb{R}^d} \frac{\mu(d\xi)}{|\xi|} \int_0^t s_1 \Big[\int_0^{t-s_1} \sin(|\xi|s_2) ds_2 \Big] ds_1 \\
&= \int_0^t s_1 \Big[\int_{\mathbb{R}^d} \frac{1-\cos\big(|\xi|(t-s_1)\big)}{|\xi|^2} \mu(d\xi) \Big] ds_1 \,.
\end{aligned}$$

Clearly, the finiteness on the right hand side leads to Dalang's condition (1.6). □

REMARK 4.1. By the moment bound (4.3) and by the expansion (2.10), we get the moment bound

$$\mathbb{E}u^2(t,x) \le Ce^{C_2t^2} \tag{4.5}$$

Using the equality (3.12) instead of (3.14) and slightly modifying the estimation in this section, one can extend (4.5) to the $\mathcal{L}_p$-bound

$$\mathbb{E}|u(t,x)|^p \le Ce^{C_p t^2} \qquad p=1,2,\cdots \tag{4.6}$$

Notice that $\frac{4-\alpha}{3-\alpha} \le 2$ for $0 \le \alpha \le 2$ and that the equality holds if and only if $\alpha = 2$. It shows that without assuming homogeneity (1.8) one still can have some moment bound for the solution that is weaker than what is offered in Theorem 1.2 (where $\alpha < 2$).

5. Proof of Theorem 1.2. From the expansion (2.10) and the stationarity of the Stratonovich moment in x, a formal algebra leads to

$$\mathbb{E}u^p(t,x) = \sum_{n=0}^{\infty} \sum_{l_1+\cdots+l_p=n} \mathbb{E}\prod_{j=1}^p S_{l_j}\big(g_{l_j}(\cdot,t,0)\big) = \sum_{n=0}^{\infty} \sum_{l_1+\cdots+l_p=2n} \mathbb{E}\prod_{j=1}^p S_{l_j}\big(g_{l_j}(\cdot,t,0)\big)\,,$$

where the second equality follows from the fact ((2.20)) that

$$\mathbb{E}\prod_{j=1}^p S_{l_j}\big(g_{l_j}(\cdot,t,0)\big) = 0$$

whenever $l_1+\cdots+l_p$ is odd. Moreover, the expansion for $\mathbb{E}u^p(t,x)$ appears as a positive series. Consequently $\mathbb{E}u^p(1,x) > 0$.

Mathematically, under Dalang's condition (1.6) the Stratonovich expansion (2.10) converges in $\mathcal{L}^p(\Omega,\mathcal{F},\mathbb{P})$ for any $p > 0$. Indeed, it is enough to exam this for all even numbers p. This follows from the estimate

$$\mathbb{E}\bigg|\sum_{n=N+1}^{N+m} S_n\big(g_n(\cdot,t,0)\big)\bigg|^p \le \sum_{n:2n\ge N+1} \sum_{l_1+\cdots+l_p=2n} \mathbb{E}\prod_{j=1}^p S_{l_j}\big(g_{l_j}(\cdot,t,0)\big)\,.$$

Therefore, the claimed $\mathcal{L}^p$-convergence relies on the fact

$$\sum_{n=0}^{\infty} \sum_{l_1+\cdots+l_p=2n} \mathbb{E}\prod_{j=1}^p S_{l_j}\big(g_{l_j}(\cdot,t,0)\big) < \infty$$

which appears as a direct consequence of (5.1) and (5.3) below.

By (1.8) and (2.6), in addition, one can verify that

$$\sum_{l_1+\cdots+l_p=2n} \mathbb{E}\prod_{j=1}^p S_{l_j}\big(g_{l_j}(\cdot,t,0)\big) = t^{(4-\alpha)n} \sum_{l_1+\cdots+l_p=2n} \mathbb{E}\prod_{j=1}^p S_{l_j}\big(g_{l_j}(\cdot,1,0)\big)\,, \qquad \forall t>0\,. \tag{5.1}$$

Therefore, for each $p = 1,2,\cdots$,

$$\mathbb{E}u^p(t,x) = \sum_{n=0}^{\infty} t^{(4-\alpha)n}\bigg(\sum_{l_1+\cdots+l_p=2n} \mathbb{E}\prod_{j=1}^p S_{l_j}\big(g_{l_j}(\cdot,1,0)\big)\bigg) \tag{5.2}$$

whenever the series on the right hand side converges.

PROOF OF THEOREM 1.2. First, we claim

$$\lim_{n\to\infty}\frac{1}{n}\log(n!)^{3-\alpha}\bigg(\sum_{l_1+\cdots+l_p=2n} \mathbb{E}\prod_{j=1}^p S_{l_j}\big(g_{l_j}(\cdot,1,0)\big)\bigg) = \log\Big(\frac{1}{2}\Big)^{3-\alpha} p^{4-\alpha}\Big(\frac{2\mathcal{M}^{1/2}}{4-\alpha}\Big)^{4-\alpha} \tag{5.3}$$

for each integer $p \geq 1$. In next subsections we shall prove the upper bound part of this claim in (5.15) and the lower bound part in (5.22).

After we established (5.3) the proof of (1.9) is easy and can be seen through the following computation: From (5.2) and then (5.3) it follows

(5.4)
$$\begin{aligned}\lim_{t\to\infty} t^{-\frac{4-\alpha}{3-\alpha}} \log u^p(t,x) &= \lim_{t\to\infty} t^{-\frac{4-\alpha}{3-\alpha}} \log \sum_{n=0}^{\infty} t^{(4-\alpha)n} \Big(\sum_{l_1+\cdots+l_p=2n} \mathbb{E} \prod_{j=1}^{p} S_{l_j}\big(g_{l_j}(\cdot,1,0)\big)\Big) \\ &= \lim_{t\to\infty} t^{-\frac{4-\alpha}{3-\alpha}} \log \sum_{n=0}^{\infty} \frac{t^{(4-\alpha)n}}{(n!)^{3-\alpha}} \left(\left(\frac{1}{2}\right)^{3-\alpha} p^{4-\alpha} \left(\frac{2\mathcal{M}^{1/2}}{4-\alpha}\right)^{4-\alpha}\right)^{n} \\ &= \frac{3-\alpha}{2} p^{\frac{4-\alpha}{3-\alpha}} \left(\frac{2\mathcal{M}^{1/2}}{4-\alpha}\right)^{\frac{4-\alpha}{3-\alpha}},\end{aligned}$$

where the last step follows from the following elementary fact of the asymptotics of the Mittag-Leffler function (Lemma A.3, [4]):

$$\lim_{b\to\infty} b^{-1/\gamma} \log \sum_{n=0}^{\infty} \frac{\theta^n b^n}{(n!)^{\gamma}} = \gamma \theta^{1/\gamma}, \qquad \theta > 0 \tag{5.5}$$

with $\gamma = 3-\alpha$ and with $b = t^{4-\alpha}$.

The proof for the upper bound of (1.10) is given in (5.7) of Lemma 5.1; and the lower bound is established in (5.27). □

5.1. *Upper bounds of (5.3) and (1.10).*

LEMMA 5.1. *Under the condition in Theorem 1.2, we have the following statements.*

(1) *for any $\lambda_1, \cdots, \lambda_p > 0$ and $p = 1, 2, \cdots$*

(5.6)
$$\begin{aligned}&\limsup_{n\to\infty} \frac{1}{n} \log \frac{1}{n!} \int_{(\mathbb{R}^+)^p} dt_1 \cdots dt_p \exp\Big\{ -\sum_{j=1}^{p} \lambda_j t_j \Big\} \Big(\sum_{l_1+\cdots+l_p=2n} \mathbb{E} \prod_{j=1}^{p} S_{l_j}\big(g_{l_j}(\cdot,t_j,0)\big)\Big) \\ &\qquad \leq \log 2\mathcal{M}^{\frac{4-\alpha}{2}} + \frac{4-\alpha}{2} \sum_{j=1}^{p} \frac{\lambda_j^{-2} \log \lambda_j^{-2}}{\lambda_1^{-2}+\cdots+\lambda_p^{-2}}.\end{aligned}$$

(2) *for any $t > 0$,*

$$\limsup_{p\to\infty} p^{-\frac{4-\alpha}{3-\alpha}} \log \mathbb{E}|u(t,0)|^p \leq \frac{3-\alpha}{2} t^{\frac{4-\alpha}{3-\alpha}} \left(\frac{2\mathcal{M}^{1/2}}{4-\alpha}\right)^{\frac{4-\alpha}{3-\alpha}}. \tag{5.7}$$

PROOF. The proof starts with the moment representation in Corollary 3.4. On the right hand side of (3.12), we perform the estimation by Fourier transform

$$\begin{aligned}
&\sum_{j,k=1}^p \int_0^{t_j}\int_0^{t_k} \gamma\big(B_j(s)-B_k(r)\big)dsdr \\
&\quad = \int_{\mathbb{R}^d}\mu(d\xi)\bigg\vert\sum_{j=1}^p\int_0^{t_j} e^{i\xi\cdot B_j(s)}ds\bigg\vert^2 \\
&\quad = (t_1+\cdots+t_p)^2\int_{\mathbb{R}^d}\mu(d\xi)\bigg\vert\sum_{j=1}^p\frac{t_j}{t_1+\cdots+t_p}\frac{1}{t_j}\int_0^{t_j} e^{i\xi\cdot B_j(s)}ds\bigg\vert^2 \\
&\quad \le (t_1+\cdots+t_p)\sum_{j=1}^p t_j\int_{\mathbb{R}^d}\mu(d\xi)\bigg\vert\frac{1}{t_j}\int_0^{t_j} e^{i\xi\cdot B_j(s)}ds\bigg\vert^2 \\
&\quad = (t_1+\cdots+t_p)\sum_{j=1}^p \frac{1}{t_j}\int_0^{t_j}\int_0^{t_j}\gamma\big(B_j(s)-B_j(r)\big)dsdr \\
&\quad \overset{d}{=} (t_1+\cdots+t_p)\sum_{j=1}^p t_j^{\frac{2-\alpha}{2}}\int_0^1\int_0^1\gamma\big(B_j(s)-B_j(r)\big)dsdr\,.
\end{aligned}$$

The advantage of the above inequality is to replace the sum of dependent quantities by the sum of independent ones, where the last step follows from scaling

$$\int_0^{t_j}\int_0^{t_j}\gamma\big(B_j(s)-B_j(r)\big)dsdr \overset{d}{=} t_j^{\frac{4-\alpha}{2}}\int_0^1\int_0^1\gamma\big(B_j(s)-B_j(r)\big)dsdr\,,\qquad j=1,\cdots,p$$

and the independence of the Brownian motions.

Combining the above result with (3.12) gives

$$\begin{aligned}
&\int_{(\mathbb{R}^+)^p} dt_1\cdots dt_p \exp\Big\{-\sum_{j=1}^p\lambda_j t_j\Big\}\sum_{l_1+\cdots+l_p=2n}\mathbb{E}\prod_{j=1}^p S_{l_j}\big(g_{l_j}(\cdot,t_j,0)\big) \\
&\quad \le \Big(\frac{1}{2}\Big)^{3n}\frac{1}{n!}\bigg(\prod_{j=1}^p\frac{\lambda_j}{2}\bigg)\int_{(\mathbb{R}^+)^p} dt_1\cdots dt_p\exp\Big\{-\frac{1}{2}\sum_{j=1}^p\lambda_j^2 t_j\Big\}(t_1+\cdots+t_p)^n \\
&\qquad \times\mathbb{E}_0\bigg[\sum_{j=1}^p t_j^{\frac{2-\alpha}{2}}\int_0^1\int_0^1\gamma\big(B_j(s)-B_j(r)\big)dsdr\bigg]^n \\
&\quad = \bigg(\prod_{j=1}^p\frac{\lambda_j}{2}\bigg)\Big(\frac{1}{2}\Big)^{3n}\sum_{l_1+\cdots+l_p=n}\frac{1}{l_1!\cdots l_p!}\bigg\{\prod_{j=1}^p\mathbb{E}_0\bigg[\int_0^1\int_0^1\gamma\big(B(s)-B(r)\big)dsdr\bigg]^{l_j}\bigg\}
\end{aligned}$$

(5.8)

$$\times\int_{(\mathbb{R}^+)^p} dt_1\cdots dt_p(t_1+\cdots+t_p)^n\exp\Big\{-\frac{1}{2}\sum_{j=1}^p\lambda_j^2 t_j\Big\}\prod_{j=1}^p t_j^{\frac{2-\alpha}{2}l_j}\,.$$

By [11, Theorem 1.1], we see

$$\begin{aligned}
&\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}_0\exp\left\{\left(\int_0^t\int_0^t\gamma\big(B(s)-B(r)\big)dsdr\right)^{1/2}\right\}\\
&=\sup_{g\in\mathcal{F}_d}\left\{\left(\int_{\mathbb{R}^d\times\mathbb{R}^d}\gamma(x-y)g^2(x)g^2(y)dxdy\right)^{1/2}-\frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2dx\right\}\\
&=2^{\frac{\alpha}{4-\alpha}}\mathcal{M}\,,
\end{aligned}$$

where the last equality follows from the one-to-one map $\mathcal{F}_d\longrightarrow\mathcal{F}_d$ defined as

$$g(\cdot)\mapsto 2^{\frac{d}{4-\alpha}}g\left(2^{\frac{2}{4-\alpha}}(\cdot)\right)\qquad g\in\mathcal{F}_d.$$

By (1.8) and the Brownian scaling,

$$\int_0^t\int_0^t\gamma\big(B(s)-B(r)\big)dsdr\overset{d}{=}t^{\frac{4-\alpha}{2}}\int_0^1\int_0^1\gamma\big(B(s)-B(r)\big)dsdr$$

we can rewrite it as

$$\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}_0\exp\left\{t^{\frac{4-\alpha}{4}}\left(\int_0^1\int_0^1\gamma\big(B(s)-B(r)\big)dsdr\right)^{1/2}\right\}=2^{\frac{\alpha}{4-\alpha}}\mathcal{M}\,. \tag{5.9}$$

On the other hand, by Taylor's expansion and the positivity of $\gamma(\cdot)$, we have

$$\begin{aligned}
&\frac{1}{(2n)!}t^{\frac{4-\alpha}{2}n}\mathbb{E}_0\left[\int_0^1\int_0^1\gamma\big(B(s)-B(r)\big)dsdr\right]^n\\
&\qquad\le\mathbb{E}_0\exp\left\{t^{\frac{4-\alpha}{4}}\left(\int_0^1\int_0^1\gamma\big(B(s)-B(r)\big)dsdr\right)^{1/2}\right\}\\
&\qquad=\exp\left\{\big(1+o(1)\big)2^{\frac{\alpha}{4-\alpha}}\mathcal{M}t\right\}\qquad(t\to\infty)\,.
\end{aligned}$$

For a fixed $\theta>0$, taking $t=\theta n$

$$\begin{aligned}
&\mathbb{E}_0\left[\int_0^1\int_0^1\gamma\big(B(s)-B(r)\big)dsdr\right]^n\\
&\le(2n)!(\theta n)^{-\frac{4-\alpha}{2}n}\exp\left\{\big(1+o(1)\big)2^{\frac{2}{2-\alpha}}\mathcal{M}(\theta n)\right\}\\
&=\big(1+o(1)\big)^n(n!)^{\alpha/2}4^n\theta^{\frac{4-\alpha}{2}n}\exp\left\{-\frac{4-\alpha}{2}n+\big(1+o(1)\big)2^{\frac{2}{2-\alpha}}\mathcal{M}\theta n\right\}
\end{aligned}$$

as $n\to\infty$, where the last step follows from Stirling's formula. Thus,

$$\begin{aligned}
&\limsup_{n\to\infty}\frac{1}{n}\log(n!)^{-\alpha/2}\mathbb{E}_0\left[\int_0^1\int_0^1\gamma\big(B(s)-B(r)\big)dsdr\right]^n\\
&\qquad\le-\frac{4-\alpha}{2}+\log 4+2^{\frac{\alpha}{4-\alpha}}\mathcal{M}\theta-\frac{4-\alpha}{2}\log\theta\,.
\end{aligned}$$

Picking the minimizer

$$\theta=2^{-\frac{\alpha}{4-\alpha}}\frac{4-\alpha}{2\mathcal{M}}$$

yields

$$\limsup_{n\to\infty}\frac{1}{n}\log(n!)^{-\alpha/2}\mathbb{E}_0\Big[\int_0^1\int_0^1\gamma\big(B(s)-B(r)\big)dsdr\Big]^n\le\log 2^4\Big(\frac{\mathcal{M}}{4-\alpha}\Big)^{\frac{4-\alpha}{2}}.$$

Suggested by a referee, we provide the following instructive exposition on how we come up with the idea $t=\theta n$. Comparing the large deviations in (5.5) (with $b=t^{\frac{4-\alpha}{4}}$ and $\gamma=\frac{4}{4-\alpha}$) and in (5.9), by the Taylor's expansion we believe that the asymptotic pattern

$$\frac{1}{n!}\mathbb{E}_0\Big[\int_0^1\int_0^1\gamma\big(B(s)-B(r)\big)dsdr\Big]^{n/2}\approx\frac{C^n}{(n!)^{\frac{4-\alpha}{4}}}\qquad(n\to\infty)$$

The rest is to find the dominant term(s) in the limit (from (5.5))

$$\lim_{t\to\infty}\frac{1}{t}\log\sum_{n=0}^\infty C^n\Big(\frac{t^n}{n!}\Big)^{\frac{4-\alpha}{4}}=\frac{4}{4-\alpha}C^{\frac{4-\alpha}{4}}.$$

It is easy to see (by considering Poisson distribution, for example) that the domination occurs at $n\approx C^{\frac{4}{4-\alpha}}t$. It therefore justifies the choice $t=\theta n$ with $\theta=C^{-\frac{4}{4-\alpha}}$.

Returning to our proof, we conclude that for any given $\delta>0$ there is $C_\delta>0$ such that

$$\mathbb{E}_0\Big[\int_0^1\int_0^1\gamma\big(B(s)-B(r)\big)dsdr\Big]^l\le C_\delta(l!)^{\alpha/2}\bigg((1+\delta)2^4\Big(\frac{\mathcal{M}}{4-\alpha}\Big)^{\frac{4-\alpha}{2}}\bigg)^l,\qquad l=1,2,\cdots$$

Substituting this bound into (5.8) gives

$$\int_{(\mathbb{R}^+)^p}dt_1\cdots dt_p\exp\Big\{-\sum_{j=1}^p\lambda_jt_j\Big\}\sum_{l_1+\cdots+l_p=2n}\mathbb{E}\prod_{j=1}^pS_{l_j}\big(g_{l_j}(\cdot,t_j,0)\big)$$
$$\le\bigg(\prod_{j=1}^p\frac{C_\delta\lambda_j}{2}\bigg)\bigg(2(1+\delta)\Big(\frac{\mathcal{M}}{4-\alpha}\Big)^{\frac{4-\alpha}{2}}\bigg)^n\sum_{l_1+\cdots+l_p=n}\bigg(\prod_{j=1}^p(l_j!)^{-\frac{2-\alpha}{2}}\bigg)$$
$$\times\int_{(\mathbb{R}^+)^p}dt_1\cdots dt_p(t_1+\cdots+t_p)^n\exp\Big\{-\frac{1}{2}\sum_{j=1}^p\lambda_j^2t_j\Big\}\prod_{j=1}^pt_j^{\frac{2-\alpha}{2}l_j}.\tag{5.10}$$

For each $(l_1,\cdots,l_p)$, we can write the above multiple integral as

$$\int_{(\mathbb{R}^+)^p}dt_1\cdots dt_p(t_1+\cdots+t_p)^n\exp\Big\{-\frac{1}{2}\sum_{j=1}^p\lambda_j^2t_j\Big\}\prod_{j=1}^pt_j^{\frac{2-\alpha}{2}l_j}$$
$$=\sum_{k_1+\cdots+k_p=n}\frac{n!}{k_1!\cdots k_p!}\int_{(\mathbb{R}^+)^p}dt_1\cdots dt_p\bigg(\prod_{j=1}^pt_j^{k_j+\frac{2-\alpha}{2}l_j}\bigg)\exp\Big\{-\frac{1}{2}\sum_{j=1}^p\lambda_j^2t_j\Big\}$$
$$=\sum_{k_1+\cdots+k_p=n}\frac{n!}{k_1!\cdots k_p!}\prod_{j=1}^p\Big(\frac{2}{\lambda_j^2}\Big)^{k_j+\frac{2-\alpha}{2}l_j+1}\Gamma\Big(k_j+\frac{2-\alpha}{2}l_j+1\Big).$$

In the sequel, we shall use the Stirling formula of the following form:

$$n^ne^{-n}\le\Gamma(n+1)\le n^{n+1}e^{-n+1},\qquad n=1,2,\cdots$$

By using this type of Stirling's formula and by routine simplification

$$\sum_{l_1+\cdots+l_p=n}\Big(\prod_{j=1}^p (l_j!)^{-\frac{2-\alpha}{2}}\Big)\int_{(\mathbb{R}^+)^p} dt_1\cdots dt_p(t_1+\cdots+t_p)^n\exp\Big\{-\frac{1}{2}\sum_{j=1}^p\lambda_j^2 t_j\Big\}\prod_{j=1}^p t_j^{\frac{2-\alpha}{2}l_j}$$

$$\le n!C^p\Big(\frac{2-\alpha}{2}\Big)^{\frac{2-\alpha}{2}n}\Big(2\sum_{j=1}^p\frac{1}{\lambda_j^2}\Big)^{\frac{4-\alpha}{2}n}$$

$$\times\sum_{\substack{k_1+\cdots+k_p=n\\ l_1+\cdots+l_p=n}}\prod_{j=1}^p\Big(\frac{k_j+\frac{2-\alpha}{2}l_j}{k_j}\Big)^{k_j}\Big(\frac{k_j+\frac{2-\alpha}{2}l_j}{\frac{2-\alpha}{2}l_j}\Big)^{\frac{2-\alpha}{2}l_j}\theta_j^{k_j+\frac{2-\alpha}{2}l_j},$$

where $C>0$ is a constant independent of n and p, and

$$\theta_j=\Big(\frac{1}{\lambda_1^2}+\cdots+\frac{1}{\lambda_p^2}\Big)^{-1}\frac{1}{\lambda_j^2},\qquad j=1,\cdots,p.$$

It is straightforward to check that the Lagrange problem

$$\max\Big\{\prod_{j=1}^p\Big(\frac{x_j+y_j}{x_j}\Big)^{x_j}\Big(\frac{x_j+y_j}{y_j}\Big)^{y_j}\theta_j^{x_j+y_j};\quad x_1+\cdots+x_p=n,$$

$$\text{and}\quad y_1+\cdots+y_p=\frac{2-\alpha}{2}n,x_1,\cdots,x_p,y_1,\cdots,y_p>0\Big\}$$

has the solution

$$x_j=\theta_j n\quad\text{and}\quad y_j=\frac{2-\alpha}{2}\theta_j n,\qquad j=1,\cdots,p.$$

Therefore, since $\sum_{j=1}^p\theta_j=1$

$$\prod_{j=1}^p\Big(\frac{k_j+\frac{2-\alpha}{2}l_j}{k_j}\Big)^{k_j}\Big(\frac{k_j+\frac{2-\alpha}{2}l_j}{\frac{2-\alpha}{2}l_j}\Big)^{\frac{2-\alpha}{2}l_j}\theta_j^{k_j+\frac{2-\alpha}{2}l_j}$$

$$\le\prod_{j=1}^p\Big(\frac{4-\alpha}{2}\Big)^{\theta_j n}\Big(\frac{4-\alpha}{2-\alpha}\Big)^{\frac{2-\alpha}{2}\theta_j n}\theta_j^{\frac{4-\alpha}{2}\theta_j n}$$

$$=\Big(\frac{4-\alpha}{2}\Big)^n\Big(\frac{4-\alpha}{2-\alpha}\Big)^{\frac{2-\alpha}{2}n}\prod_{j=1}^p\theta_j^{\frac{4-\alpha}{2}\theta_j n}=\Big(\frac{4-\alpha}{2}\Big)^{\frac{4-\alpha}{2}n}\Big(\frac{2}{2-\alpha}\Big)^{\frac{2-\alpha}{2}n}\prod_{j=1}^p\theta_j^{\frac{4-\alpha}{2}\theta_j n}$$

uniformly over $l_1,\cdots,l_p;k_1,\cdots,k_p$.

Summarizing our steps since (5.10) and noticing

$$\sum_{l_1+\cdots+l_p=n}1=\binom{n+p-1}{p-1}$$

we have the bound

$$\int_{(\mathbb{R}^+)^p}dt_1\cdots dt_p\exp\Big\{-\sum_{j=1}^p\lambda_j t_j\Big\}\sum_{l_1+\cdots+l_p=2n}\mathbb{E}\prod_{j=1}^p S_{l_j}\big(g_{l_j}(\cdot,t_j,0)\big)\tag{5.11}$$

$$\le C^p n!\binom{n+p-1}{p-1}^2\Big(\prod_{j=1}^p\frac{C_\delta\lambda_j}{2}\Big)\Big(2(1+\delta)\Big(\frac{\mathcal{M}}{4-\alpha}\Big)^{\frac{4-\alpha}{2}}\Big)^n$$

$$\times\left((4-\alpha)\sum_{j=1}^p\frac{1}{\lambda_j^2}\right)^{\frac{4-\alpha}{2}n}\prod_{j=1}^p\theta_j^{\frac{4-\alpha}{2}\theta_j n}.$$

This leads to (5.6) as $\delta>0$ can be made arbitrarily small.

The bound (5.11) can also be used to the proof of (5.7). To see this we can allow p tends to infinity only along integer points. This does not compromise the claim there by the following interpolation argument: For any real and large $p\geq 1$, let $\langle p/2\rangle$ be the smallest integer larger than or equal to $p/2$. Then, by Hölder's inequality we have

$$\Big\{\mathbb{E}|u(t,0)|^p\Big\}^{1/p}\leq\Big\{\mathbb{E}u^{2\langle p/2\rangle}(t,0)\Big\}^{\frac{1}{2\langle p/2\rangle}}.$$

Thus, it suffices to show (5.7) along the positive integers p and with $\mathbb{E}u^p(t,0)$ instead of $\mathbb{E}|u(t,0)|^p$.

By monotonicity of $g_n(\cdot,t,0)$ in t,

$$\begin{aligned}\sum_{l_1+\cdots+l_p=2n}\mathbb{E}\prod_{j=1}^p S_{l_j}\big(g_{l_j}(\cdot,t_j,0)\big)&\geq\sum_{l_1+\cdots+l_p=2n}\mathbb{E}\prod_{j=1}^p S_{l_j}\Big(g_{l_j}(\cdot,\min_{1\leq j\leq p}t_j,0)\Big)\\&=\Big(\sum_{l_1+\cdots+l_p=2n}\mathbb{E}\prod_{j=1}^p S_{l_j}\big(g_{l_j}(\cdot,1,0)\big)\Big)\Big(\min_{1\leq j\leq p}t_j\Big)^{(4-\alpha)n},\end{aligned}$$

where the last step follows from (5.1). Thus,

$$\begin{aligned}&\int_{(\mathbb{R}^+)^p}dt_1\cdots dt_p\exp\Big\{-\sum_{j=1}^p t_j\Big\}\sum_{l_1+\cdots+l_p=2n}\mathbb{E}\prod_{j=1}^p S_{l_j}\big(g_{l_j}(,\cdot,t_j,0)\big)\\&\quad\geq\Big(\sum_{l_1+\cdots+l_p=2n}\mathbb{E}\prod_{j=1}^p S_{l_j}\big(g_{l_j}(\cdot,1,0)\big)\Big)\int_{(\mathbb{R}^+)^p}dt_1\cdots dt_p\exp\Big\{-\sum_{j=1}^p t_j\Big\}\Big(\min_{1\leq j\leq p}t_j\Big)^{(4-\alpha)n}.\end{aligned}$$

By the fact that given i.i.d. exponential times $\tau_1,\cdots,\tau_p$ of parameter 1, $\min\limits_{1\leq j\leq p}\tau_j$ is an exponential time with parameter p,

$$\begin{aligned}\text{(5.12)}\qquad\int_{(\mathbb{R}^+)^p}dt_1\cdots dt_p\exp\Big\{-\sum_{j=1}^p t_j\Big\}\Big(\min_{1\leq j\leq p}t_j\Big)^{(4-\alpha)n}&=p\int_0^\infty e^{-pt}t^{(4-\alpha)n}dt\\&=p^{-(4-\alpha)n}\Gamma\Big(1+(4-\alpha)n\Big).\end{aligned}$$

In summary, we have

(5.13)

$$\begin{aligned}&\sum_{l_1+\cdots+l_p=2n}\mathbb{E}\prod_{j=1}^p S_{l_j}\big(g_{l_j}(\cdot,1,0)\big)\\&\quad\leq\frac{p^{(4-\alpha)n}}{\Gamma\big(1+(4-\alpha)n\big)}\int_{(\mathbb{R}^+)^p}dt_1\cdots dt_p\exp\Big\{-\sum_{j=1}^p t_j\Big\}\sum_{l_1+\cdots+l_p=2n}\mathbb{E}\prod_{j=1}^p S_{l_j}\big(g_{l_j}(,\cdot,t_j,0)\big)\\&\quad\leq\Big(\frac{CC_\delta}{2}\Big)^p\binom{n+p-1}{p-1}^2\frac{n!p^{(4-\alpha)n}}{\Gamma\big(1+(4-\alpha)n\big)}\Big(2(1+\delta)\mathcal{M}^{\frac{4-\alpha}{2}}\Big)^n,\end{aligned}$$

where the second step follows directly from the bound (5.11) with $\lambda_1=\cdots=\lambda_p=1$.

Using (5.1) and (5.2) we then have

$$\begin{aligned}\mathbb{E}u^p(t,0) &\le \Big(\frac{CC_\delta}{2}\Big)^p\sum_{n=0}^\infty\binom{n+p-1}{p-1}^2\frac{n!(pt)^{(4-\alpha)n}}{\Gamma\big(1+(4-\alpha)n\big)}\Big(2(1+\delta)\mathcal{M}^{\frac{4-\alpha}{2}}\Big)^n\\ &\le \Big(\frac{CC_\delta}{2}\frac{\theta}{\theta-1}\Big)^{2p}\sum_{n=0}^\infty\frac{n!(pt)^{(4-\alpha)n}}{\Gamma\big(1+(4-\alpha)n\big)}\Big(2\theta^2(1+\delta)\mathcal{M}^{\frac{4-\alpha}{2}}\Big)^n,\end{aligned}$$

where $\theta>1$ is arbitrary, and the second step follows from the estimate

$$\theta^{-n}\binom{n+p-1}{p-1}\le\sum_{k=0}^\infty\theta^{-k}\binom{k+p-1}{p-1}=\Big(\frac{\theta}{\theta-1}\Big)^p.$$

By the Stirling formula, $\Gamma\big(1+(4-\alpha)n\big)$ is replaceable by

$$(n!)^{4-\alpha}(4-\alpha)^{(4-\alpha)n}.$$

By the asymptotics of the Mittag-Leffler function (5.5) with $\gamma=3-\alpha$ and $b=p^{4-\alpha}$, and θ being replaced by $t^{4-\alpha}\bigg(2\theta^2(1+\delta)\Big(\frac{\mathcal{M}^{1/2}}{4-\alpha}\Big)^{4-\alpha}\bigg)$, we have

$$\begin{aligned}&\limsup_{p\to\infty}p^{-\frac{4-\alpha}{3-\alpha}}\log\mathbb{E}u^p(t,0)\\ &\quad\le\lim_{p\to\infty}p^{-\frac{4-\alpha}{3-\alpha}}\log\sum_{n=0}^\infty\frac{(pt)^{(4-\alpha)n}}{(n!)^{3-\alpha}}\bigg(2\theta^2(1+\delta)\Big(\frac{\mathcal{M}^{1/2}}{4-\alpha}\Big)^{4-\alpha}\bigg)^n\\ &\quad=(3-\alpha)t^{\frac{4-\alpha}{3-\alpha}}\bigg(2\theta^2(1+\delta)\Big(\frac{\mathcal{M}^{1/2}}{4-\alpha}\Big)^{4-\alpha}\bigg)^{\frac{1}{3-\alpha}}.\end{aligned}$$

Letting $\delta\to0^+$ and $\theta\to1^+$ on the right hand side gives (5.7). □

We end this subsection by the following statement: First, taking $\lambda_1=\cdots=\lambda_p=1$ in (5.6) leads to

$$\begin{aligned}&\limsup_{n\to\infty}\frac1n\log\frac{1}{n!}\int_{(\mathbb{R}^+)^p}dt_1\cdots dt_p\exp\Big\{-\sum_{j=1}^p t_j\Big\}\bigg(\sum_{l_1+\cdots+l_p=2n}\mathbb{E}\prod_{j=1}^p S_{l_j}\big(g_{l_j}(\cdot,t_j,0)\big)\bigg)\\ &\quad\le\log 2\mathcal{M}^{\frac{4-\alpha}{2}}.\end{aligned}\tag{5.14}$$

Second, applying (5.13) to the setting of fixed integer $p\ge1$ yields the upper bound of (5.3)

$$\begin{aligned}&\limsup_{n\to\infty}\frac1n\log(n!)^{3-\alpha}\bigg(\sum_{l_1+\cdots+l_p=2n}\mathbb{E}\prod_{j=1}^p S_{l_j}\big(g_{l_j}(\cdot,1,0)\big)\bigg)\\ &\quad\le\log\Big(\frac12\Big)^{3-\alpha}p^{4-\alpha}\Big(\frac{2\mathcal{M}^{1/2}}{4-\alpha}\Big)^{4-\alpha}.\end{aligned}\tag{5.15}$$

5.2. *Lower bound for (5.3).* In this subsection we start by the lower bound correspondent to (5.14).

LEMMA 5.2. *Under the condition in Theorem 1.2, we have*

$$\liminf_{n\to\infty}\frac{1}{n}\log\frac{1}{n!}\int_{(\mathbb{R}^+)^p}dt_1\cdots dt_p\exp\Big\{-\sum_{j=1}^p t_j\Big\}\sum_{l_1+\cdots+l_p=2n}\mathbb{E}\prod_{j=1}^p S_{l_j}\big(g_{l_j}(\cdot,t_j,0)\big)$$

$$\geq\log 2\mathcal{M}^{\frac{4-\alpha}{2}} \tag{5.16}$$

for $p=1,2,\cdots$.

PROOF. Notice

$$\begin{aligned}\sum_{j,k=1}^p\int_0^{t_j}\int_0^{t_k}\gamma\big(B_j(s)-B_k(r)\big)dsdr &= \int_{\mathbb{R}^d}\mu(d\xi)\Big|\sum_{j=1}^p\int_0^{t_j}e^{i\xi\cdot B_j(s)}ds\Big|^2\\ &\geq\Big[\int_{\mathbb{R}^d}\mu(d\xi)f(\xi)\Big(\sum_{j=1}^p\int_0^{t_j}e^{i\xi\cdot B_j(s)}ds\Big)\Big]^2\\ &=\Big[\sum_{j=1}^p\int_{\mathbb{R}^d}\mu(d\xi)f(\xi)\Big(\int_0^{t_j}e^{i\xi\cdot B_j(s)}ds\Big)\Big]^2\end{aligned}$$

for any non-negative $f(\xi)$ with

$$\int_{\mathbb{R}^d}|f(\xi)|^2\mu(d\xi)=1\,. \tag{5.17}$$

Therefore, we have

$$\begin{aligned}&\mathbb{E}_0\Big[\sum_{j,k=1}^p\int_0^{t_j}\int_0^{t_k}\gamma\big(B_j(s)-B_k(r)\big)dsdr\Big]^n\geq\mathbb{E}_0\Big[\sum_{j=1}^p\int_{\mathbb{R}^d}\mu(d\xi)f(\xi)\Big(\int_0^{t_j}e^{i\xi\cdot B_j(s)}ds\Big)\Big]^{2n}\\ &\quad=\sum_{l_1+\cdots+l_p=2n}\frac{(2n)!}{l_1!\cdots l_p!}\prod_{j=1}^p\mathbb{E}_0\Big[\int_{\mathbb{R}^d}\mu(d\xi)f(\xi)\Big(\int_0^{t_j}e^{i\xi\cdot B_j(s)}ds\Big)\Big]^{l_j}\\ &\quad=(2n)!\sum_{l_1+\cdots+l_p=2n}\prod_{j=1}^p\int_{(\mathbb{R}^d)^{l_j}}\mu^{\otimes l_j}(d\xi)\Big(\prod_{k=1}^{l_j}f(\xi_k)\Big)\int_{[0,t_j]_<^{l_j}}d\mathbf{s}\prod_{k=1}^{l_j}\exp\Big\{-\frac{s_k-s_{k-1}}{2}\Big|\sum_{i=k}^{l_j}\xi_i\Big|^2\Big\}.\end{aligned}$$

Taking $\lambda_1=\cdots=\lambda_p=1$ in Corollary 3.4 and inserting the above computation into the obtained expression yield

(5.18)

$$\begin{aligned}&\int_{(\mathbb{R}^+)^p}dt_1\cdots dt_p\exp\Big\{-\sum_{j=1}^p t_j\Big\}\sum_{l_1+\cdots+l_p=2n}\mathbb{E}\prod_{j=1}^p S_{l_j}\big(g_{l_j}(\cdot,t_j,0)\big)\\ &\quad\geq\Big(\frac{1}{2}\Big)^p\Big(\frac{1}{2}\Big)^{3n}\frac{(2n)!}{n!}\int_{(\mathbb{R}^+)^p}dt_1\cdots dt_p\exp\Big\{-\frac{1}{2}\sum_{j=1}^p t_j\Big\}\\ &\qquad\times\sum_{l_1+\cdots+l_p=2n}\prod_{j=1}^p\int_{(\mathbb{R}^d)^{l_j}}\mu^{\otimes l_j}(d\xi)\Big(\prod_{k=1}^{l_j}f(\xi_k)\Big)\int_{[0,t_j]_<^{l_j}}d\mathbf{s}\prod_{k=1}^{l_j}\exp\Big\{-\frac{s_k-s_{k-1}}{2}\Big|\sum_{i=k}^{l_j}\xi_i\Big|^2\Big\}\\ &\quad\geq\Big(\frac{1}{2}\Big)^{p+3n}\frac{(2n)!}{n!}\sum_{l_1+\cdots+l_p=2n}\prod_{j=1}^p\int_{(\mathbb{R}^d)^{l_j}}\mu^{\otimes l_j}(d\xi)\Big(\prod_{k=1}^{l_j}f(\xi_k)\Big)\end{aligned}$$

$$\times\int_0^\infty dte^{-t/2}\int_{[0,t]_<^{l_j}} d\mathbf{s}\prod_{k=1}^{l_j}\exp\Big\{-\frac{s_k-s_{k-1}}{2}\Big\vert\sum_{i=k}^{l_j}\xi_i\Big\vert^2\Big\}$$

$$=\Big(\frac{1}{2}\Big)^{p+3n}\frac{(2n)!}{n!}\sum_{l_1+\cdots+l_p=2n}\prod_{j=1}^p\int_{(\mathbb{R}^d)^{l_j}}\mu^{\otimes l_j}(d\xi)\Big(\prod_{k=1}^{l_j}f(\xi_k)\Big)\prod_{k=1}^{l_j}\int_0^\infty e^{-t/2}\exp\Big\{-\frac{t}{2}\Big\vert\sum_{i=k}^{l_j}\xi_i\Big\vert^2\Big\}dt$$

$$=\Big(\frac{1}{2}\Big)^{p+3n}\frac{(2n)!}{n!}\sum_{l_1+\cdots+l_p=2n}\prod_{j=1}^p\int_{(\mathbb{R}^d)^{l_j}}\mu^{\otimes l_j}(d\xi)\Big(\prod_{k=1}^{l_j}f(\xi_k)\Big)\prod_{k=1}^{l_j}\Big\{\frac{1}{2}+\frac{1}{2}\Big\vert\sum_{i=k}^{l_j}\xi_i\Big\vert^2\Big\}^{-1}$$

$$=\Big(\frac{1}{2}\Big)^{p+n}\frac{(2n)!}{n!}\sum_{l_1+\cdots+l_p=2n}\prod_{j=1}^p\int_{(\mathbb{R}^d)^{l_j}}\mu^{\otimes l_j}(d\xi)\Big(\prod_{k=1}^{l_j}f(\xi_k)\Big)\prod_{k=1}^{l_j}\Big\{1+\Big\vert\sum_{i=k}^{l_j}\xi_i\Big\vert^2\Big\}^{-1}.$$

By the computation in [7, (3.7)-(3.9)]

$$\liminf_{n\to\infty}\frac{1}{n}\log\int_{(\mathbb{R}^d)^n}\mu^{\otimes n}(d\xi)\Big(\prod_{k=1}^n f(\xi_k)\Big)\prod_{k=1}^n\Big\{1+\Big\vert\sum_{i=k}^n\xi_i\Big\vert^2\Big\}^{-1} \tag{5.19}$$
$$\geq\log\sup_{\|\varphi\|_2=1}\int_{\mathbb{R}^d}\mu(d\xi)f(\xi)\Big[\int_{\mathbb{R}^d}d\eta\frac{\varphi(\eta)\varphi(\eta+\xi)}{\sqrt{(1+|\eta|^2)(1+|\xi+\eta|^2)}}\Big]$$
$$\stackrel{\Delta}{=}\log\rho(f).$$

For a given $\delta>0$, therefore, there is $C_\delta>0$ such that

$$\int_{(\mathbb{R}^d)^l}\mu(d\xi)\Big(\prod_{k=1}^l f(\xi_k)\Big)\prod_{k=1}^l\Big\{1+\Big\vert\sum_{i=k}^l\xi_i\Big\vert^2\Big\}^{-1}\geq C_\delta^{-1}\Big((1-\delta)\rho(f)\Big)^l,\qquad l=1,2,\cdots$$

Therefore,

$$\prod_{j=1}^p\int_{(\mathbb{R}^d)^{l_j}}\mu^{\otimes l_j}(d\xi)\Big(\prod_{k=1}^{l_j}f(\xi_k)\Big)\prod_{k=1}^{l_j}\Big\{1+\Big\vert\sum_{i=k}^{l_j}\xi_i\Big\vert^2\Big\}^{-1}\geq C_\delta^{-p}\Big((1-\delta)\rho(f)\Big)^{2n}$$

for any $l_1,\cdots,l_p\geq 0$ with $l_1+\cdots+l_p=2n$. In addition, by Stirling's formula,

$$\Big(\frac{1}{2}\Big)^{p+n}\frac{(2n)!}{n!}=\big(1+o(1)\big)^n 2^n n!\qquad(n\to\infty).$$

Together with (5.18), one has

$$\liminf_{n\to\infty}\frac{1}{n}\log\frac{1}{n!}\int_{(\mathbb{R}^+)^p}dt_1\cdots dt_p\exp\Big\{-\sum_{j=1}^p t_j\Big\}\sum_{l_1+\cdots+l_p=2n}\mathbb{E}\prod_{j=1}^p S_{l_j}\big(g_{l_j}(\cdot,t_j,0)\big)$$
$$\geq\log 2\Big((1-\delta)\rho(f)\Big)^2.$$

Letting $\delta\to 0^+$ and taking supremum over all non-negative functions f satisfying (5.17) on the right hand side, we have

$$\liminf_{n\to\infty}\frac{1}{n}\log\frac{1}{n!}\int_{(\mathbb{R}^+)^p}dt_1\cdots dt_p\exp\Big\{-\sum_{j=1}^p t_j\Big\}\sum_{l_1+\cdots+l_p=2n}\mathbb{E}\prod_{j=1}^p S_{l_j}\big(g_{l_j}(\cdot,t_j,0)\big)$$
$$\geq\log 2\sup_{\|\varphi\|_2=1}\int_{\mathbb{R}^d}\mu(d\xi)\Big[\int_{\mathbb{R}^d}d\eta\frac{\varphi(\eta)\varphi(\eta+\xi)}{\sqrt{(1+|\eta|^2)(1+|\xi+\eta|^2)}}\Big]^2.$$

Finally, the proof is completed by Theorem 1.5, [7] (with $p=\beta=2$, $\sigma=\alpha$ and $|\cdot|^{-\alpha}$ being replaced by $\gamma(\cdot)$) that states

$$\sup_{\|\varphi\|_2=1}\int_{\mathbb{R}^d}\mu(d\xi)\bigg[\int_{\mathbb{R}^d}d\eta\frac{\varphi(\eta)\varphi(\eta+\xi)}{\sqrt{(1+|\eta|^2)(1+|\xi+\eta|^2)}}\bigg]^2=\mathcal{M}^{\frac{4-\alpha}{2}}. \tag{5.20}$$

This completes the proof (5.16). □

Combining (5.14) with (5.16) yields

$$\lim_{n\to\infty}\frac{1}{n}\log\frac{1}{n!}\int_{(\mathbb{R}^+)^p}dt_1\cdots dt_p\exp\Big\{-\sum_{j=1}^p t_j\Big\}\bigg(\sum_{l_1+\cdots+l_p=2n}\mathbb{E}\prod_{j=1}^p S_{l_j}\big(g_{l_j}(\cdot,t_j,0)\big)\bigg)$$

$$=\log 2\mathcal{M}^{\frac{4-\alpha}{2}}. \tag{5.21}$$

We are not able to establish the lower bound correspondent to (5.6) as $\lambda_1,\cdots,\lambda_p$ are not equal, nor is (5.6) with different $\lambda_1,\cdots,\lambda_p$ needed for the upper bound (5.15). The only reason to keep possibly different $\lambda_1,\cdots,\lambda_p$ in (5.6) is for the installation of the following lower bound that corresponds to the upper bound (5.15).

LEMMA 5.3. *Under the condition in Theorem 1.2, we have*

$$\liminf_{n\to\infty}\frac{1}{n}\log(n!)^{3-\alpha}\bigg(\sum_{l_1+\cdots+l_p=2n}\mathbb{E}\prod_{j=1}^p S_{l_j}\big(g_{l_j}(\cdot,1,0)\big)\bigg)\ge\log\Big(\frac{1}{2}\Big)^{3-\alpha}p^{4-\alpha}\Big(\frac{2\mathcal{M}^{1/2}}{4-\alpha}\Big)^{4-\alpha}. \tag{5.22}$$

PROOF. We adopt some idea from the proof for the lower bound of Gärtner-Ellis large deviations (Theorem 2.3.6, p.44, [17]). The crucial observation made here is the concentration behavior $t_1,\cdots,t_p\approx\frac{(4-\alpha)n}{p}$ (as $n\to\infty$) in a dynamics that creates (5.21). To show it, we define the probability measures on $(\mathbb{R}^+)^p$ as follows

$$\mu_n(A)=\frac{\displaystyle\int_A dt_1\cdots dt_p\exp\big\{-(t_1+\cdots+t_p)\big\}\sum_{l_1+\cdots+l_p=2n}\mathbb{E}\prod_{j=1}^p S_{l_j}\big(g_{l_j}(\cdot,t_j,0)\big)}{\displaystyle\int_{(\mathbb{R}^+)^p}dt_1\cdots dt_p\exp\big\{-(t_1+\cdots+t_p)\big\}\sum_{l_1+\cdots+l_p=2n}\mathbb{E}\prod_{j=1}^p S_{l_j}\big(g_{l_j}(\cdot,t_j,0)\big)}$$

for $n=1,2,\cdots$. Notice that for any $\theta_1,\cdots,\theta_p<1$,

$$\int_{(\mathbb{R}^+)^p}\exp\big\{\theta_1t_1+\cdots+\theta_pt_p)\big\}\mu_n(dt_1\cdots dt_p)$$

$$=\frac{\displaystyle\int_{(\mathbb{R}^+)^p}dt_1\cdots dt_p\exp\Big\{-\sum_{j=1}^p(1-\theta_j)t_j\Big\}\bigg(\sum_{l_1+\cdots+l_p=2n}\mathbb{E}\prod_{j=1}^p S_{l_j}\big(g_{l_j}(\cdot,t_j,0)\big)\bigg)}{\displaystyle\int_{(\mathbb{R}^+)^p}dt_1\cdots dt_p\exp\Big\{-\sum_{j=1}^p t_j\Big\}\bigg(\sum_{l_1+\cdots+l_p=2n}\mathbb{E}\prod_{j=1}^p S_{l_j}\big(g_{l_j}(\cdot,t_j,0)\big)\bigg)}$$

and the right hand side blows up as long as $\theta_j\ge1$ for any $1\le j\le p$.

By (5.6) and (5.21), we see

$$\limsup_{n\to\infty}\frac{1}{n}\log\int_{(\mathbb{R}^+)^p}dt_1\cdots dt_p\exp\big\{\theta_1t_1+\cdots+\theta_pt_p)\big\}\mu_n(dt_1\cdots dt_p)\le\Lambda(\theta_1,\cdots,\theta_p)\tag{5.23}$$

for any $(\theta_1,\cdots,\theta_p)\in\mathbb{R}^p$, where

$$\Lambda(\theta_1,\cdots,\theta_p):=\begin{cases}\dfrac{4-\alpha}{2}\displaystyle\sum_{j=1}^p\frac{(1-\theta_j)^{-2}\log(1-\theta_j)^{-2}}{(1-\theta_1)^{-2}+\cdots+(1-\theta_p)^{-2}} & \text{if }\theta_1,\cdots,\theta_p<1\\ \infty & \text{otherwise}.\end{cases}$$

By the upper bound of Gärtner-Ellis theorem (Theorem 2.3.6 (a), p.44, [17])

$$\limsup_{n\to\infty}\frac{1}{n}\log\mu_n(nF)\le-\inf_{(t_1,\cdots,t_p)\in F}\Lambda^*(t_1,\cdots,t_p)\tag{5.24}$$

for any close $F\subset(\mathbb{R}^+)^p$, where

$$\Lambda^*(t_1,\cdots,t_p)=\sup_{\theta_1,\cdots,\theta_p<1}\Big\{\sum_{j=1}^p\theta_jt_j-\Lambda(\theta_1,\cdots,\theta_p)\Big\},\qquad t_1,\cdots,t_p\ge0.$$

In fact, the statement of Theorem 2.3.6 (a), p.44, [17] requires the equality in (5.23). However, a careful reading of its proof finds that (5.23) is sufficient for (5.24).

Finding the close form of $\Lambda^*(\theta_1,\cdots,\theta_p)$ might not be easy. On the other hand, some properties of $\Lambda^*(\theta_1,\cdots,\theta_p)$ as a rate function exists even in the general context. For example, $\Lambda^*(\theta_1,\cdots,\theta_p)$ is non-negative, lower semi-continuous and has compact level sets (goodness). What important to our purpose is that

$$\Lambda^*(t_1,\cdots,t_p)>0,\qquad\forall(t_1,\cdots,t_p)\ne\Big(\frac{4-\alpha}{p},\cdots,\frac{4-\alpha}{p}\Big).\tag{5.25}$$

Indeed, assume that $\Lambda^*(t_1,\cdots,t_p)=0$ for some $(t_1,\cdots,t_p)$. Then we have that

$$\sum_{j=1}^p\theta_jt_j\le\frac{4-\alpha}{2}\sum_{j=1}^p\frac{(1-\theta_j)^{-2}\log(1-\theta_j)^{-2}}{(1-\theta_1)^{-2}+\cdots+(1-\theta_p)^{-2}}$$

for any $\theta_1,\cdots,\theta_p<1$. For fixed $1\le j\le p$, taking $\theta_k=0$ for all $k\ne j$, the above inequality gives

$$\theta_jt_j\le\frac{4-\alpha}{2}\frac{(1-\theta_j)^{-2}}{p-1+(1-\theta_j)^{-2}}\log(1-\theta_j)^{-2}=(4-\alpha)\frac{(1-\theta_j)^{-2}}{p-1+(1-\theta_j)^{-2}}\log(1-\theta_j)^{-1}$$

for any $\theta_j<1$. So we have that

$$t_j\le(4-\alpha)\frac{(1-\theta_j)^{-2}}{p-1+(1-\theta_j)^{-2}}\frac{1}{\theta_j}\log(1-\theta_j)^{-1}\qquad\text{as }\theta_j>0$$

and

$$t_j\ge(4-\alpha)\frac{(1-\theta_j)^{-2}}{p-1+(1-\theta_j)^{-2}}\frac{1}{\theta_j}\log(1-\theta_j)^{-1}\qquad\text{as }\theta_j<0.$$

By the fact that

$$\lim_{\theta\to0}\frac{1}{\theta}\log(1-\theta)^{-1}=1$$

we have $t_j = \frac{4-\alpha}{p}$ $(j = 1, \cdots, p)$. This shows the claim (5.25).

By (5.25), the lower semi-continuity and goodness we have

$$\inf_{(t_1,\cdots,t_p)\notin G} \Lambda^*(t_1, \cdots, t_p) > 0$$

for any open neighborhood G of $(\frac{4-\alpha}{p}, \cdots, \frac{4-\alpha}{p})$. For any given small $\delta > 0$ taking

$$G_\delta = \Big(\frac{4-\alpha-\delta}{p}, \quad \frac{4-\alpha+\delta}{p}\Big)^p$$

and $F = G_\delta^c$ in (5.24) yields

$$\limsup_{n\to\infty} \frac{1}{n} \log \mu_n(nG_\delta^c) < 0\,.$$

Consequently,

$$\int_{(\frac{4-\alpha-\delta}{p}n,\ \frac{4-\alpha+\delta}{p}n)^p} dt_1 \cdots dt_p \exp\Big\{-\sum_{j=1}^p t_j\Big\}\Big(\sum_{l_1+\cdots+l_p=2n} \mathbb{E}\prod_{j=1}^p S_{l_j}\big(g_{l_j}(\cdot, t_j, 0)\big)\Big)$$

$$\sim \int_{(\mathbb{R}^+)^p} dt_1 \cdots dt_p \exp\Big\{-\sum_{j=1}^p t_j\Big\}\Big(\sum_{l_1+\cdots+l_p=2n} \mathbb{E}\prod_{j=1}^p S_{l_j}\big(g_{l_j}(\cdot, t_j, 0)\big)\Big) \tag{5.26}$$

as $n \to \infty$.

When $(t_1, \cdots, t_p) \in \big(\frac{4-\alpha-\delta}{p}n,\ \frac{4-\alpha+\delta}{p}n\big)^p$, it is easy to see that

$$t_j \le \frac{4-\alpha+\delta}{p} n \le \frac{4-\alpha+\delta}{4-\alpha-\delta} \min_{1\le k\le p} t_k\,, \qquad j = 1, \cdots, p$$

and by the scaling property (5.1) we have

$$\sum_{l_1+\cdots+l_p=2n} \mathbb{E}\prod_{j=1}^p S_{l_j}\big(g_{l_j}(\cdot, t_j, 0)\big) \le \sum_{l_1+\cdots+l_p=2n} \mathbb{E}\prod_{j=1}^p S_{l_j}\Big(g_{l_j}\Big(\cdot,\ \frac{4-\alpha+\delta}{4-\alpha-\delta} \min_{1\le k\le p} t_k,\ 0\Big)\Big)$$

$$= \Big(\frac{4-\alpha+\delta}{4-\alpha-\delta} \min_{1\le k\le p} t_k\Big)^{(4-\alpha)n} \sum_{l_1+\cdots+l_p=2n} \mathbb{E}\prod_{j=1}^p S_{l_j}\big(g_{l_j}(\cdot, 1, 0)\big)\,.$$

Therefore,

$$\int_{(n\frac{4-\alpha-\delta}{p},\ n\frac{4-\alpha+\delta}{p})^p} dt_1 \cdots dt_p \exp\Big\{-\sum_{j=1}^p t_j\Big\} \sum_{l_1+\cdots+l_p=2n} \mathbb{E}\prod_{j=1}^p S_{lj}\big(g_n(\cdot, t_j, 0)\big)$$

$$\le \Big\{\sum_{l_1+\cdots+l_p=2n} \mathbb{E}\prod_{j=1}^p S_{l_j}\big(g_{l_j}(\cdot, 1, 0)\big)\Big\}\Big(\frac{4-\alpha+\delta}{4-\alpha-\delta}\Big)^{(4-\alpha)n}$$

$$\times \int_{(\mathbb{R}^+)^p} dt_1 \cdots dt_p \exp\Big\{-\sum_{j=1}^p t_j\Big\}\Big(\min_{1\le j\le p} t_j\Big)^{(4-\alpha)n}$$

$$= \Big\{\sum_{l_1+\cdots+l_p=2n} \mathbb{E}\prod_{j=1}^p S_{l_j}\big(g_{l_j}(\cdot, 1, 0)\big)\Big\}\Big(\frac{4-\alpha+\delta}{4-\alpha-\delta}\Big)^{(4-\alpha)n}\Big(\frac{1}{p}\Big)^{(4-\alpha)n}\Gamma\Big(1 + (4-\alpha)n\Big)\,,$$

where the last step follows from (5.12). Finally, (5.22) follows from the above inequality together with (5.21), (5.26) and the Stirling formula. □

5.3. *Lower bounds for (1.10).* In this subsection we prove the lower bound part of (1.10):

$$\liminf_{p\to\infty} p^{-\frac{4-\alpha}{3-\alpha}} \log \mathbb{E}|u(t,0)|^p \geq \frac{3-\alpha}{2} t^{\frac{4-\alpha}{3-\alpha}} \left(\frac{2\sqrt{\mathcal{M}}}{4-\alpha}\right)^{\frac{4-\alpha}{3-\alpha}}. \tag{5.27}$$

It should be pointed out that the Gärtner-Ellis type argument used for the proof of Lemma 5.3 is good only for fixed p. Different from the approaches used so far, the treatment below is independent of the Stratonovich moment representation developed in Section 3.

Let $\mathcal{H}$ be the Hilbert space given as the closure of the space

$$\left\{f : \mathbb{R}^d \to \mathbb{R};\quad \int_{\mathbb{R}^d\times\mathbb{R}^d} \gamma(x-y) f(x) f(y) dx dy < \infty\right\}$$

under the inner product

$$\langle f, g\rangle_{\mathcal{H}} = \int_{\mathbb{R}^d\times\mathbb{R}^d} \gamma(x-y) f(x) g(y) dx dy\,.$$

The space $\mathcal{H}$ may contain generalized functions (distributions). For each integer $n \geq 1$, we write $\mathcal{H}^{\otimes n}$ for the n-th product with inner product

$$\langle f, g\rangle_{\mathcal{H}^{\otimes n}} = \int_{(\mathbb{R}^d)^{2n}} d\mathbf{x} d\mathbf{y} \left(\prod_{k=1}^n \gamma(x_k - y_k)\right) f(x_1, \cdots, x_n) g(y_1, \cdots .y_n)\,. \tag{5.28}$$

LEMMA 5.4. *Given any real number $p > 1$,*

$$\|u(t,0)\|_p \geq \exp\Big\{-\frac{1}{2(p-1)}\|f\|_{\mathcal{H}}^2\Big\} \sum_{n=0}^\infty \langle f^{\otimes n}, g_n(\cdot, t, 0)\rangle_{\mathcal{H}^{\otimes n}} \tag{5.29}$$

for any $t > 0$ and $f \in \mathcal{H}$ with $f(\cdot) \geq 0$.

PROOF. Let $q > 1$ be the conjugate of p. By Hölder's inequality

$$\|u(t,x)\|_p \geq \mathbb{E}\left[u(t,x) X\right]$$

for any random variable X with $\|X\|_q = 1$. Take

$$\begin{aligned} X &= \left\|\exp\Big\{\int_{\mathbb{R}^d} f(x) W(dx)\Big\}\right\|_q^{-1} \exp\Big\{\int_{\mathbb{R}^d} f(x) W(dx)\Big\} \\ &= \exp\Big\{-\frac{q}{2}\|f\|_{\mathcal{H}}^2\Big\} \exp\Big\{\int_{\mathbb{R}^d} f(x) W(dx)\Big\}. \end{aligned}$$

Then for any $f \in \mathcal{H}$,

(5.30)

$$\begin{aligned} \|u(t,0)\|_p &\geq \exp\Big\{-\frac{q}{2}\|f\|_{\mathcal{H}}^2\Big\} \mathbb{E} u(t,0) \exp\Big\{\int_{\mathbb{R}^d} f(x) W(dx)\Big\} \\ &= \exp\Big\{-\frac{q}{2}\|f\|_{\mathcal{H}}^2\Big\} \sum_{n=0}^\infty \Big\{\sum_{l=0}^n \frac{1}{l!} \mathbb{E}\Big(\int_{\mathbb{R}^d} f(x) W(dx)\Big)^l S_{n-l}\big(g_{n-l}(\cdot, t, 0)\big)\Big\} \\ &= \exp\Big\{-\frac{q}{2}\|f\|_{\mathcal{H}}^2\Big\} \sum_{n=0}^\infty \Big\{\sum_{l=0}^{2n} \frac{1}{l!} \mathbb{E}\Big(\int_{\mathbb{R}^d} f(x) W(dx)\Big)^l S_{2n-l}\big(g_{2n-l}(\cdot, t, 0)\big)\Big\} \end{aligned}$$

$$\geq \exp\Big\{-\frac{q}{2}\|f\|_{\mathcal{H}}^2\Big\}\sum_{n=0}^{\infty}\Big\{\sum_{l=n}^{2n}\frac{1}{l!}\mathbb{E}\Big(\int_{\mathbb{R}^d}f(x)W(dx)\Big)^l S_{2n-l}\big(g_{2n-l}(\cdot,t,0)\big)\Big\}$$
$$=\exp\Big\{-\frac{q}{2}\|f\|_{\mathcal{H}}^2\Big\}\sum_{n=0}^{\infty}\Big\{\sum_{l=0}^{n}\frac{1}{(n+l)!}\mathbb{E}\Big(\int_{\mathbb{R}^d}f(x)W(dx)\Big)^{n+l} S_{n-l}\big(g_{n-l}(\cdot,t,0)\big)\Big\},$$

where the second equality follows from (2.20), and the second inequality follows from the fact that all terms are non-negative.

For each $0\leq l\leq n$, by (2.17) and (2.19)

$$\mathbb{E}\Big(\int_{\mathbb{R}^d}f(x)W(dx)\Big)^{n+l}S_{n-l}\big(g_{n-l}(\cdot,t,0)\big)$$
$$=\mathbb{E}\int_{(\mathbb{R}^d)^{2n}}g_{n-l}(x_1,\cdots,x_{n-l},t,0)\Big(\prod_{k=n-l+1}^{2n}f(x_k)\Big)W(dx_1)\cdots W(dx_{2n})$$
$$=\sum_{\mathcal{D}\in\Pi_n}\int_{(\mathbb{R}^d)^{2n}}d\mathbf{x}\Big(\prod_{(j,k)\in\mathcal{D}}\gamma(x_j-x_k)\Big)g_{n-l}(x_1,\cdots,x_{n-l},t,0)\Big(\prod_{k=n-l+1}^{2n}f(x_k)\Big).$$

We now count how many pair partitions $\mathcal{D}\in\Pi_n$ that make

(5.31)
$$\int_{(\mathbb{R}^d)^{2n}}d\mathbf{x}\Big(\prod_{(j,k)\in\mathcal{D}}\gamma(x_j-x_k)\Big)g_{n-l}(x_1,\cdots,x_{n-l},t,0)\Big(\prod_{k=n-l+1}^{2n}f(x_k)\Big)$$
$$=\|f\|_{\mathcal{H}}^{2l}\Big\{\int_{(\mathbb{R}^d)^{2(n-l)}}d\mathbf{x}d\mathbf{y}\Big(\prod_{k=1}^{n-l}\gamma(x_k-y_k)\Big)g_{n-l}(x_1,\cdots,x_{n-l},t,0)\Big(\prod_{k=1}^{n-l}f(y_k)\Big)\Big\}.$$

To produce such $\mathcal{D}$, we first partition $\{n-l+1,\cdots,2n\}$ into two disjoint sets A_1 and A_2 such that $\#(A_1)=n-l$ and $\#(A_2)=2l$. The number of ways to carry out this step is

$$\binom{n+l}{2l}.$$

Then we use the elements in A_1 to make $n-l$ pairs with the numbers $1,\cdots,n-l$, there are $(n-l)!$ ways to do this step. Finally, we pick a pair partition $\mathcal{D}_0$ on A_2 together with the earlier $n-l$ pairs to form a pair partition $\mathcal{D}\in\Pi_n$ —there are $\frac{(2l)!}{2^l l!}$ ways to finish this step. By the Fubini theorem, one can see that the pair partitions $\mathcal{D}$ produced in such way satisfy (5.31). By multiplication principle, there are at least

$$\binom{n+l}{2l}(n-l)!\frac{(2l)!}{2^l l!}=\frac{(n+l)!}{2^l l!}$$

pair partitions that make Equation (5.31) happen.

Write

$$\int_{(\mathbb{R}^d)^{2(n-l)}}d\mathbf{x}d\mathbf{y}\Big(\prod_{k=1}^{n-l}\gamma(x_k-y_k)\Big)g_{n-l}(x_1,\cdots,x_{n-l},t,0)\Big(\prod_{k=1}^{n-l}f(y_k)\Big)$$
$$=\langle f^{\otimes(n-l)},g_{n-l}(\cdot,t,0)\rangle_{\mathcal{H}^{\otimes(n-l)}}.$$

In summary,

$$\mathbb{E}\left[\left(\int_{\mathbb{R}^d} f(x)W(dx)\right)^{n+l} S_{n-l}\big(g_{n-l}(\cdot,t,0)\big)\right] \ge \frac{(n+l)!}{2^l l!}\|f\|_{\mathcal{H}}^{2l}\langle f^{\otimes(n-l)}, g_{n-l}(\cdot,t,0)\rangle_{\mathcal{H}^{\otimes(n-l)}}\,.$$

Therefore,

$$\begin{aligned}&\sum_{n=0}^\infty\left\{\sum_{l=0}^n \frac{1}{(n+l)!}\mathbb{E}\left(\int_{\mathbb{R}^d} f(x)W(dx)\right)^{n+l} S_{n-l}\big(g_{n-l}(\cdot,t,0)\big)\right\}\\ &\ge \sum_{n=0}^\infty\left\{\sum_{l=0}^n \frac{1}{l!2^l}\|f\|_{\mathcal{H}}^{2l}\langle f^{\otimes(n-l)}, g_{n-l}(\cdot,t,0)\rangle_{\mathcal{H}^{\otimes(n-l)}}\right\}\\ &=\left\{\sum_{n=0}^\infty \frac{1}{n!2^n}\|f\|_{\mathcal{H}}^{2n}\right\}\left\{\sum_{n=0}^\infty\langle f^{\otimes n}, g_n(\cdot,t,0)\rangle_{\mathcal{H}^{\otimes n}}\right\}\\ &=\exp\left\{\frac{1}{2}\|f\|_{\mathcal{H}}^2\right\}\left\{\sum_{n=0}^\infty\langle f^{\otimes n}, g_n(\cdot,t,0)\rangle_{\mathcal{H}^{\otimes n}}\right\}.\end{aligned}$$

In view of (5.30), we have completed the proof of the lemma. □

PROOF OF (5.27). Replacing $f(x)$ by

$$f_p(x) = \big((p-1)t\big)^{(2-\alpha+d)\frac{1}{3-\alpha}} f\Big(\big((p-1)t\big)^{\frac{1}{3-\alpha}}x\Big)$$

in Lemma 5.4 we get

$$\|u(t,0)\|_p \ge \exp\left\{-\frac{1}{2(p-1)}\|f_p\|_{\mathcal{H}}^2\right\}\sum_{n=0}^\infty\langle f_p^{\otimes n}, g_n(\cdot,t,0)\rangle_{\mathcal{H}^{\otimes n}}\,.$$

Set

$$t_p = (p-1)^{\frac{1}{3-\alpha}} t^{\frac{4-\alpha}{3-\alpha}}\,.$$

First notice that

$$\|f_p\|_{\mathcal{H}}^2 = \big((p-1)t\big)^{\frac{4-\alpha}{3-\alpha}}\|f\|_{\mathcal{H}}^2$$

and by time change and homogeneity of $\gamma(\cdot)$ and $G(t,x)$,

$$\sum_{n=0}^\infty\langle f_p^{\otimes n}, g_n(\cdot,t,0)\rangle_{\mathcal{H}^{\otimes n}} = \sum_{n=0}^\infty\langle f^{\otimes n}, g_n(\cdot,t_p,0)\rangle_{\mathcal{H}^{\otimes n}}\,.$$

Hence,

$$\begin{aligned}\|u(t,0)\|_p &\ge \exp\left\{-\frac{t_p}{2}\|f\|_{\mathcal{H}}^2\right\}\sum_{n=0}^\infty\langle f^{\otimes n}, g_n(\cdot,t_p,0)\rangle_{\mathcal{H}^{\otimes n}}\\ &\ge \exp\left\{-\frac{t_p}{2}\|f\|_{\mathcal{H}}^2\right\}\langle f^{\otimes n}, g_n(\cdot,t_p,0)\rangle_{\mathcal{H}^{\otimes n}}\,,\qquad n=0,1,2,\cdots\end{aligned}$$

Let $a>0$ be fixed but arbitrary. Take supremum over $\|f\|_{\mathcal{H}}=a$. The action can be taken alternatively as f is replaced by af and supremum is over $\|f\|_{\mathcal{H}}=1$:

$$\begin{aligned}\|u(t,0)\|_p &\ge \exp\left\{-\frac{t_p}{2}a^2\right\}a^n\sup_{\|f\|_{\mathcal{H}}=1}\langle f^{\otimes n}, g_n(\cdot,t_p,0)\rangle_{\mathcal{H}^{\otimes n}}\\ &=\exp\left\{-\frac{t_p}{2}a^2\right\}a^n t_p^{\frac{4-\alpha}{2}n}\sup_{\|f\|_{\mathcal{H}}=1}\langle f^{\otimes n}, g_n(\cdot,1,0)\rangle_{\mathcal{H}^{\otimes n}}\,,\end{aligned}\tag{5.32}$$

where the last step follows from the scaling property

$$\sup_{\|f\|_{\mathcal{H}}=1}\langle f^{\otimes n}, g_n(\cdot,t,0)\rangle_{\mathcal{H}^{\otimes n}} = t^{\frac{4-\alpha}{2}n}\sup_{\|f\|_{\mathcal{H}}=1}\langle f^{\otimes n}, g_n(\cdot,1,0)\rangle_{\mathcal{H}^{\otimes n}}\,, \qquad \forall t>0\,. \tag{5.33}$$

Here we should mention that the supremum should be taken over the functions f with $\|f\|_{\mathcal{H}}=1$ and $f\geq 0$ where the constraint "$f\geq 0$" is inherited from Lemma 5.4. We removed "$f\geq 0$" from the above discussion as $g_n(\cdot,1,0)\geq 0$ and therefore

$$\sup_{\substack{\|f\|_{\mathcal{H}}=1\\ f\geq 0}}\langle f^{\otimes n}, g_n(\cdot,t,0)\rangle_{\mathcal{H}^{\otimes n}} = \sup_{\|f\|_{\mathcal{H}}=1}\langle f^{\otimes n}, g_n(\cdot,t,0)\rangle_{\mathcal{H}^{\otimes n}}.$$

Let $0<\theta<1$ be fixed but arbitrary. Multiplying $(1-\theta)\theta^n$ on the both sides of (5.32) and summing up both sides over $n=0,1,2,\cdots$,

$$\|u(t,0)\|_p \geq (1-\theta)\exp\Big\{-\frac{t_p}{2}a^2\Big\}\sum_{n=0}^\infty(\theta a)^n t_p^{\frac{4-\alpha}{2}n}\sup_{\|f\|_{\mathcal{H}}=1}\langle f^{\otimes n}, g_n(\cdot,1,0)\rangle_{\mathcal{H}^{\otimes n}}\,. \tag{5.34}$$

On the other hand,

$$\int_0^\infty dt e^{-t}\sup_{\|f\|_{\mathcal{H}}=1}\langle f^{\otimes n}, g_n(\cdot,t,0)\rangle_{\mathcal{H}^{\otimes n}} \geq \sup_{\|f\|_{\mathcal{H}}=1}\int_0^\infty dt e^{-t}\langle f^{\otimes n}, g_n(\cdot,t,0)\rangle_{\mathcal{H}^{\otimes n}}$$
$$=\sup_{\|f\|_{\mathcal{H}}=1}\int_{(\mathbb{R}^d)^n}\mu^{\otimes n}(d\xi)\Big(\prod_{k=1}^n\mathcal{F}(f)(\xi_k)\Big)\prod_{k=1}^n\Big\{1+\Big|\sum_{j=k}^n\xi_j\Big|^2\Big\}^{-1},$$

where

$$\mathcal{F}(f)(\xi)=\int_{\mathbb{R}^d}e^{i\xi\cdot x}f(x)dx$$

is the Fourier transform of f and the last step follows from a treatment similar to the one conducted in (5.18). In view of the scaling identity (5.33), this inequality can be written as

$$\sup_{\|f\|_{\mathcal{H}}=1}\langle f^{\otimes n}, g_n(\cdot,1,0)\rangle_{\mathcal{H}^{\otimes n}}$$
$$\geq\Big(\int_0^\infty e^{-t}t^{\frac{4-\alpha}{2}n}dt\Big)^{-1}\sup_{\|f\|_{\mathcal{H}}=1}\int_{(\mathbb{R}^d)}\mu^{\otimes n}(d\xi)\Big(\prod_{k=1}^n\mathcal{F}(f)(\xi_k)\Big)\prod_{k=1}^n\Big\{1+\Big|\sum_{j=k}^n\xi_j\Big|^2\Big\}^{-1}$$
$$=\Gamma\Big(1+\frac{4-\alpha}{2}n\Big)^{-1}\sup_{\|f\|_{\mathcal{H}}=1}\int_{(\mathbb{R}^d)}\mu^{\otimes n}(d\xi)\Big(\prod_{k=1}^n\mathcal{F}(f)(\xi_k)\Big)\prod_{k=1}^n\Big\{1+\Big|\sum_{j=k}^n\xi_j\Big|^2\Big\}^{-1}.$$

By (5.19), (5.20) and the Stirling formula

$$\liminf_{n\to\infty}\frac{1}{n}\log(n!)^{\frac{4-\alpha}{2}}\sup_{\|f\|_{\mathcal{H}}=1}\langle f^{\otimes n}, g_n(\cdot,1,0)\rangle_{\mathcal{H}^{\otimes n}}\geq\log\Big(\frac{2\mathcal{M}^{1/2}}{4-\alpha}\Big)^{\frac{4-\alpha}{2}}.$$

Hence,

$$\liminf_{p\to\infty}\frac{1}{t_p}\log\sum_{n=0}^\infty(\theta a)^n t_p^{\frac{4-\alpha}{2}n}\sup_{\|f\|_{\mathcal{H}}=1}\langle f^{\otimes n}, g_n(\cdot,1,0)\rangle_{\mathcal{H}^{\otimes n}}$$
$$\geq\lim_{p\to\infty}\frac{1}{t_p}\log\sum_{n=0}^\infty(n!)^{-\frac{4-\alpha}{2}}\Big((\theta a)\Big(\frac{2\mathcal{M}^{1/2}}{4-\alpha}\Big)^{\frac{4-\alpha}{2}}\Big)^n t_p^{\frac{4-\alpha}{2}n}$$
$$=\frac{4-\alpha}{2}\Big(\theta a\Big(\frac{2\mathcal{M}^{1/2}}{4-\alpha}\Big)^{\frac{4-\alpha}{2}}\Big)^{\frac{2}{4-\alpha}}=(\theta a)^{\frac{2}{4-\alpha}}\mathcal{M}^{1/2},$$

where the second step follows from (5.5) with $\gamma = \dfrac{4-\alpha}{2}$ and $b = t_p^{\frac{4-\alpha}{2}}$.

By (5.34), therefore,

$$\liminf_{p\to\infty} \frac{1}{t_p} \log \|u(t,0)\|_p \ge -\frac{1}{2}a^2 + (\theta a)^{\frac{2}{4-\alpha}} \mathcal{M}^{1/2}.$$

Letting $\theta \to 1^-$ yields

$$\liminf_{p\to\infty} \frac{1}{t_p} \log \|u(t,0)\|_p \ge -\frac{1}{2}a^2 + a^{\frac{2}{4-\alpha}} \mathcal{M}^{1/2}.$$

Taking the supremum over $a > 0$ on the right hand side,

$$\liminf_{p\to\infty} \frac{1}{t_p} \log \|u(t,0)\|_p \ge \frac{3-\alpha}{2}\left(\frac{2\mathcal{M}^{1/2}}{4-\alpha}\right)^{\frac{4-\alpha}{3-\alpha}}. \tag{5.35}$$

By definition of t_p this is (5.27). □

6. Appendix.

6.1. *Moment bounds for Brownian intersection local times.* Let $B(t), B_1(t), B_2(t)$ be independent d-dimensional Brownian motions.

LEMMA 6.1. *Assume Dalang's condition (1.6). There is a constant $C > 0$, independent of n and t, such that*

$$\mathbb{E}_0 \left[\int_0^t\int_0^t \gamma\big(B(s) - B(r)\big) ds dr\right]^n \le C(n!)^2 (t \vee t^2)^n, \qquad n = 1, 2, \cdots \tag{6.1}$$

$$\mathbb{E}_0 \left[\int_0^t\int_0^t \gamma\big(B_1(s) - B_2(r)\big) ds dr\right]^n \le C(n!)^2 (t \vee t^2)^n, \qquad n = 1, 2, \cdots \tag{6.2}$$

PROOF. Write

$$Z_t = \left(\int_0^t\int_0^t \gamma\big(B(s) - B(r)\big) ds dr\right)^{1/2}, \qquad t \ge 0.$$

To prove (6.1) all we need is the bound

$$\mathbb{E}_0 Z_t^n \le n! C^n (\sqrt{t} \vee t)^n, \qquad n = 1, 2, \cdots \tag{6.3}$$

First, Z_t is non-decreasing, almost surely continuous with $Z_0 = 0$. From (A.9), [9] Z_t is sub-additive: For any $t_1, t_2 > 0$, there is a random variable Z'_{t_2} such that $Z'_{t_2} \overset{d}{=} Z_{t_2}$ and Z'_{t_2} is independent of $\{Z_s;\ \ s \le t_1\}$. By (1.3.7), p.21, [8], therefore,

$$\mathbb{P}_0\big\{Z_{t_0} \ge a + b\big\} \le \mathbb{P}_0\big\{Z_{t_0} \ge a\big\}\mathbb{P}_0\big\{Z_{t_0} \ge b\big\}$$

for any $t_0, a, b > 0$. Thus, for any integer $m \ge 1$,

$$\begin{aligned}
\mathbb{E}_0 Z_t^n &= (e\mathbb{E}_0 Z_t)^n \mathbb{E}_0\Big(\frac{Z_t}{e\mathbb{E}_0 Z_t}\Big)^n \\
&= (e\mathbb{E}_0 Z_t)^n n \int_0^\infty b^{n-1} \mathbb{P}_0\{Z_t \ge eb\mathbb{E}_0 Z_t\} db \\
&= (e\mathbb{E}_0 Z_t)^n \left\{ n\int_0^1 b^{n-1} db + n\int_1^\infty \mathbb{P}_0\{Z_t \ge eb\mathbb{E}_0 Z_t\} db \right\} \\
&\le (e\mathbb{E}_0 Z_t)^n \left\{ 1 + n\int_1^\infty b^{n-1} \Big(\mathbb{P}_0\{Z_t \ge e\mathbb{E}_0 Z_t\}\Big)^{b-1} db \right\}.
\end{aligned}$$

The claimed bound (6.3) follows from the following estimation

$$\int_1^\infty b^{n-1}\Big(\mathbb{P}_0\{Z_t \geq e\mathbb{E}_0 Z_t\}\Big)^{b-1} db \leq e\int_0^\infty b^{n-1}e^{-b}db = en!$$

and the bound ((A.6), Appendix, [9])

$$\mathbb{E}_0 Z_t \leq \Big(\mathbb{E}_0 Z_t^2\Big)^{1/2} \leq \Big(C(t\vee t^2)\Big)^{1/2}.$$

We now prove (6.2). Let $\dot{W}(x)$ be a Gaussian noise independent of B, B_1, B_2 and having covariance $\gamma(\cdot)$. Conditioning on the Brownian motions

$$\mathbb{E}\bigg[\int_0^t \dot{W}\big(B_1(s)\big)ds\bigg]\bigg[\int_0^t \dot{W}\big(B_2(s)\big)ds\bigg] = \int_0^t\int_0^t \gamma\big(B_1(s)-B_2(r)\big)dsdr\,.$$

In addition, by the Cauchy-Schwartz inequality

$$\begin{aligned}
&\mathbb{E}\bigg[\int_0^t \dot{W}\big(B_1(s)\big)ds\bigg]\bigg[\int_0^t \dot{W}\big(B_2(s)\big)ds\bigg]\\
&\quad\leq \bigg\{\mathbb{E}\bigg[\int_0^t \dot{W}\big(B_1(s)\big)ds\bigg]^2\bigg\}^{1/2}\bigg\{\mathbb{E}\bigg[\int_0^t \dot{W}\big(B_2(s)\big)ds\bigg]^2\bigg\}^{1/2}\\
&\quad=\bigg\{\int_0^t\int_0^t \gamma\big(B_1(s)-B_1(r)\big)dsdr\bigg\}^{1/2}\bigg\{\int_0^t\int_0^t \gamma\big(B_2(s)-B_2(r)\big)dsdr\bigg\}^{1/2}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\int_0^t\int_0^t \gamma\big(B_1(s)-B_2(r)\big)dsdr \leq &\bigg\{\int_0^t\int_0^t \gamma\big(B_1(s)-B_1(r)\big)dsdr\bigg\}^{1/2}\\
&\times\bigg\{\int_0^t\int_0^t \gamma\big(B_2(s)-B_2(r)\big)dsdr\bigg\}^{1/2}.
\end{aligned}$$

By the independence between B_1 and B_2,

$$\begin{aligned}
&\mathbb{E}_0\bigg[\int_0^t\int_0^t \gamma\big(B_1(s)-B_2(r)\big)dsdr\bigg]^n \leq \bigg\{\mathbb{E}_0\bigg[\int_0^t\int_0^t \gamma\big(B(s)-B(r)\big)dsdr\bigg]^{n/2}\bigg\}^2\\
&\leq \mathbb{E}_0\bigg[\int_0^t\int_0^t \gamma\big(B(s)-B(r)\big)dsdr\bigg]^n.
\end{aligned}$$

Therefore, (6.2) follows from (6.1). □

6.2. *Hu-Meyer formula.* Although Lemma 2.4 gives a way for us to show the existence of a multiple Stratonovich integral we also need to know what kind general conditions to impose on f so that its multiple Stratonovich integral $S_n(f)$ exists, namely the approximation in (2.14) has a limit in $\mathcal{L}^2(\Omega,\mathcal{F},\mathbb{P})$. If the multiple Stratonovich integral $S_n(f)$ exists in $\mathcal{L}^2(\Omega,\mathcal{F},\mathbb{P})$, then according to general Itô-Wiener's chaos expansion theorem it admits a chaos expansion and it is interesting to find this chaos expansion. For this we shall establish a Hu-Meyer formula along the line of [22, 23]. If $f\in\mathcal{H}^{\otimes n}$ is a (generalized) symmetric function of n-variables such that

$$\begin{aligned}
\|f\|^2_{\mathcal{H}^{\otimes n}} :=&\int_{(\mathbb{R}^d)^{2n}} f(x_1,\cdots,x_n)f(y_1,\cdots,y_n)\\
&\times\gamma(x_1-y_1)\cdots\gamma(x_n-y_n)dx_1dy_1\cdots dx_ndy_n<\infty\,,
\end{aligned}\tag{6.4}$$

then its multiple Itô-Skorohod integral exists and is denoted by

$$I_n(f)=\int_{(\mathbb{R}^d)^n} f(x_1,\cdots,x_n)\delta W(x_1)\cdots\delta W(x_n)\,,$$

where δW denotes the Itô-Skorohod stochastic integral. To precisely define $\mathcal{H}^{\otimes n}$, we can complete the set of all symmetric smooth functions with compact supports under the Hilbert norm defined by (6.4). It is well-known that the Hilbert space $\mathcal{H}^{\otimes n}$ contains generalized functions (see e.g. [28]).

Recall our definition (2.7) that $W_\varepsilon(x)=\int_{\mathbb{R}^d} p_\varepsilon(x-y)W(dy)=I_1(p_\varepsilon(x-\cdot))$. From [21, Corollary 5.1, Equation 5.3.15], it follows that the chaos expansion of $\prod_{k=1}^n \dot W_\varepsilon(x_k)$ is

$$\begin{aligned}\prod_{k=1}^n \dot W_\varepsilon(x_k)&=\sum_{k\le n/2}\sum_{i_1<j_1,\cdots,i_k<j_k}\prod_{\ell=1}^k\int_{\mathbb{R}^{2d}} p_\varepsilon(x_{i_\ell}-y)\gamma(y-z)p_\varepsilon(x_{j_\ell}-z)dydz\\
&\qquad I_{n-2k}(\Lambda_{i_1,j_1,\cdots,i_k,j_k}\otimes_{m=1}^n p_\varepsilon(x_m-\cdot))\\
&=\sum_{k\le n/2}\sum_{i_1<j_1,\cdots,i_k<j_k}\prod_{\ell=1}^k\gamma_{2\varepsilon}(x_{i_\ell}-x_{j_\ell})\\
&\qquad I_{n-2k}(\Lambda_{i_1,j_1,\cdots,i_k,j_k}\otimes_{m=1}^n p_\varepsilon(x_m-\cdot))\,,\end{aligned}\tag{6.5}$$

where

(i) The set of distinct elements $i_1<j_1,\cdots,i_k<j_k$ is a subset of $\{1,2,\cdots,n\}$ and the summation $\sum_{i_1<j_1,\cdots,i_k<j_k}$ is over all such distinct pairs;

(ii) The function $\Lambda_{i_1,j_1,\cdots,i_k,j_k}\otimes_{m=1}^n p_\varepsilon(x_m-\cdot)$ is defined as the symmetrization of the function

$$\prod_{m\in[1,n]\setminus\{i_1,j_1,\cdots,i_k,j_k\}} p_\epsilon(x_m-y_m)$$

over the variables $\big(y_m;\ m\in[1,n]\setminus\{i_1,j_1,\cdots,i_k,j_k\}\big)$, i.e.,

$$\Lambda_{i_1,j_1,\cdots,i_k,j_k}\otimes_{m=1}^n p_\varepsilon(x_m-y_m)=\frac{1}{(n-2k)!}\sum_\sigma\prod_{m\in[1,n]\setminus\{i_1,j_1,\cdots,i_k,j_k\}} p_\epsilon(x_m-y_{\sigma(m)})\,,$$

where the summation is over all permutations σ on $[1,n]\setminus\{i_1,j_1,\cdots,i_k,j_k\}$. When $k=0$, in particular, we follow the natural convention that

$$\Lambda_{i_1,j_1,\cdots,i_k,j_k}\otimes_{m=1}^n p_\varepsilon(x_m-y_m)=\frac{1}{n!}\sum_{\sigma\in\Sigma_n}\prod_{m=1}^n p_\varepsilon(x_m-y_{\sigma(m)})\,,$$

where Σ_n is the permutation group on $\{1,\cdots,n\}$.

(iii) $I_{n-2k}(\cdots)$ is the multiple Itô-Wiener (Itô-Skorohod) integral with the integration variables $\{y_m\,;\ m\in[1,n]\setminus\{i_1,j_1,\cdots,i_k,j_k\}\}$.

With the above chaos expansion (6.5) we see that the chaos expansion of the approximated Stratonovich integral is

$$\begin{aligned}S_{n,\varepsilon}(f)&=\int_{(\mathbb{R}^d)^n} f(x_1,\cdots,x_n)\Big(\prod_{k=1}^n \dot W_\varepsilon(x_k)\Big)dx_1\cdots dx_n\\
&=\sum_{k\le n/2}\sum_{i_1<j_1,\cdots,i_k<j_k}\int_{(\mathbb{R}^d)^n} f(x_1,\cdots,x_n)\Big(\prod_{\ell=1}^k\gamma_{2\varepsilon}(x_{i_\ell}-x_{j_\ell})\Big)\end{aligned}$$

$$\times I_{n-2k}\Big(\Lambda_{i_1,j_1,\cdots,i_k,j_k}\otimes_{m=1}^{n}p_\varepsilon(x_m-\cdot)\Big)dx_1\cdots dx_n\,. \tag{6.6}$$

By the symmetry of f on $x_1,\cdots,x_n$ and with a combinatorial analysis as in [23] the above equation can be written

$$\begin{aligned} S_{n,\varepsilon}(f) = \sum_{k\le n/2}\frac{n!}{2^k k!(n-2k)!}I_{n-2k}\Bigg(&\int_{(\mathbb{R}^d)^{2n}} f(x_1,\cdots,x_n)\prod_{\ell=1}^{k}\gamma_{2\varepsilon}(x_{2\ell-1}-x_{2\ell}) \\ &\times\prod_{j=2k+1}^{n}p_\varepsilon(x_j-\cdot))dx_1\cdots dx_n\Bigg)\,. \end{aligned} \tag{6.7}$$

Since the approximated multiple integral can be decomposed to finite sum of multiple Itô-Wiener integrals which are orthogonal, we see that the convergence in $\mathcal{L}^2(\Omega,\mathcal{F},\mathbb{P})$ of (2.14) is equivalent to that each of the multiple Itô-Wiener integrals in (6.7) converges in $\mathcal{L}^2(\Omega,\mathcal{F},\mathbb{P})$. Thus, we have the following theorem which is used to justify (2.19).

THEOREM 6.2. *Let $f\in\mathcal{H}^{\otimes n}$ be deterministic and symmetric. If the trace*

$$\begin{aligned} \mathrm{Tr}^k f(y_{2k+1},\cdots,y_n) := \lim_{\varepsilon\to 0}\int_{(\mathbb{R}^d)^n} & f(x_1,\cdots,x_n)\prod_{\ell=1}^{k}\gamma_{2\varepsilon}(x_{2\ell-1}-x_{2\ell}) \\ &\times\prod_{j=2k+1}^{n}p_\varepsilon(x_j-y_j))dx_1\cdots dx_n \end{aligned} \tag{6.8}$$

exists in $\mathcal{H}^{\otimes(n-2k)}$ for all $k\le n/2$, then the Stratonovich integral $S_n(f)$ exists as an $\mathcal{L}^2(\Omega,\mathcal{F},\mathbb{P})$ limit of $S_{n,\varepsilon}(f)$ as $\varepsilon\to 0$ and we have the following Hu-Meyer formula:

$$S_n(f)=\sum_{k\le n/2}\frac{n!}{2^k k!(n-2k)!}I_{n-2k}(\mathrm{Tr}^k f)\,. \tag{6.9}$$

Conversely, if $S_{n,\varepsilon}(f)$ is a Cauchy sequence in $\mathcal{L}^2(\Omega,\mathcal{F},\mathbb{P})$, then the right hand side of (6.8) is a Cauchy sequence in $\mathcal{H}^{\otimes(n-2k)}$ for all $k\le n/2$, whose limit is denoted by the left hand side of (6.8) and $S_{n,\varepsilon}(f)$ converges to $S_n(f)$ defined by (6.9) in $\mathcal{L}^2(\Omega,\mathcal{F},\mathbb{P})$. Moreover, if $S_{n,\varepsilon}(f)$ converges to $S_n(f)$ in $\mathcal{L}^2(\Omega,\mathcal{F},\mathbb{P})$, then this convergence also takes place in $\mathcal{L}^p(\Omega,\mathcal{F},\mathbb{P})$ for any $p\in[1,\infty)$. This means that $S_n(f)$ is in $\mathcal{L}^p(\Omega,\mathcal{F},\mathbb{P})$ for any $p\in[1,\infty)$.

REMARK 6.3. It is obvious that if f is the symmetrization of $\hat f$, then by the above definition it is easy to verify that $S_n(f)=S_n(\hat f)$.

PROOF OF THE THEOREM. Denote

$$g_{n,k,\varepsilon}(y_{2k+1},\cdots,y_n) := \int_{(\mathbb{R}^d)^{2n}} f(x_1,\cdots,x_n)\prod_{\ell=1}^{k}\gamma_{2\varepsilon}(x_{2\ell-1}-x_{2\ell})\prod_{j=2k+1}^{n}p_\varepsilon(x_j-y_j))dx_1\cdots dx_n\,.$$

Equation (6.8) means $\|g_{n,k,\varepsilon}-\mathrm{Tr}^k f\|_{\mathcal{H}^{\otimes(n-2k)}}\to 0$ as $\varepsilon\to 0$. By the Itô isometry,

$$\begin{aligned} \mathbb{E}|I_{n-2k}(g_{n,k,\varepsilon})-I_{n-2k}(\mathrm{Tr}^k f)|^2 &= \mathbb{E}|I_{n-2k}(g_{n,k,\varepsilon}-\mathrm{Tr}^k f)|^2 \\ &=(n-2k)!\|g_{n,k,\varepsilon}-\mathrm{Tr}^k f\|^2_{\mathcal{H}^{\otimes(n-2k)}}\to 0 \end{aligned} \tag{6.10}$$

by (6.8). Equation (6.7) tells that $S_{n,\varepsilon}(f)$ converges to $S_n(f)$ given by (6.9).

Now we assume that $S_{n,\varepsilon}(f)$ is a Cauchy sequence in $\mathcal{L}^2(\Omega,\mathcal{F},\mathbb{P})$. With our notation $g_{n,k,\varepsilon}$ we can write

$$S_{n,\varepsilon}(f)=\sum_{k\le n/2}\frac{n!}{2^k k!(n-2k)!}I_{n-2k}\left(g_{n,k,\varepsilon}\right).$$

Thus, by the orthogonality of multiple Itô-Wiener integrals,

$$\begin{aligned}\mathbb{E}\left[S_{n,\varepsilon}(f)-S_{n,\varepsilon'}(f)\right]^2&=\sum_{k\le n/2}\left(\frac{n!}{2^k k!(n-2k)!}\right)^2\mathbb{E}\left[I_{n-2k}\left(g_{n,k,\varepsilon}\right)-I_{n-2k}\left(g_{n,k,\varepsilon'}\right)\right]^2\\&=\sum_{k\le n/2}\left(\frac{n!}{2^k k!(n-2k)!}\right)^2(n-2k)!\|g_{n,k,\varepsilon}-g_{n,k,\varepsilon'}\|^2_{\mathcal{H}^{\otimes(n-2k)}}.\end{aligned}$$

This can be used to prove the second part of the theorem easily.

Recall that if $F=\sum_{n=0}^{\infty}I_n(f_n)$ is the chaos expansion of F, then the second quantization operator (e.g. [21]) of a number $\alpha\in[-1,1]$ is defined as

$$\Gamma(\alpha)F=\sum_{n=0}^{\infty}\alpha^n I_n(f_n).$$

Now for any $p>2$, let $\alpha=\sqrt{\frac{1}{p-1}}$ and let

$$F_{t,n,\varepsilon}=\sum_{k\le n/2}(1/\alpha)^{n-2k}\frac{n!}{2^k k!(n-2k)!}\left[I_{n-2k}\left(g_{n,k,\varepsilon}\right)-I_{n-2k}(\mathrm{Tr}^{\,k}f)\right]. \tag{6.11}$$

Then by the hypercontractivity inequality (e.g. [21, p. 54, Theorem 3.20], we have

$$\begin{aligned}(\mathbb{E}|S_{n,\varepsilon}(f)\ -S_n(f)|^p)^{1/p}&=(\mathbb{E}|\Gamma(\alpha)F_{t,n,\varepsilon}|^p)^{1/p}\le\left(\mathbb{E}|F_{t,n,\varepsilon}|^2\right)^{1/2}\\&=\left(\mathbb{E}|\sum_{k\le n/2}(1/\alpha)^{n-2k}\frac{n!}{2^k k!(n-2k)!}\left[I_{n-2k}\left(g_{n,k,\varepsilon}\right)-I_{n-2k}(\mathrm{Tr}^{\,k}f)\right]|^2\right)^{1/2}\\&\le\left(\sum_{k\le n/2}(1/\alpha)^{2n-4k}\frac{(n!)^2}{2^{2k}(k!)^2((n-2k)!)^2}\mathbb{E}\left[I_{n-2k}\left(g_{n,k,\varepsilon}\right)-I_{n-2k}(\mathrm{Tr}^{\,k}f)\right]^2\right)^{1/2},\end{aligned}$$

which converges to 0 by (6.10). This proves the theorem. □

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