On the Law of the Iterated Logarithm for Local Times of Recurrent Random Walks

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ABSTRACT We consider the law of the iterated logarithm (LIL) for the local time of one-dimensional recurrent random walks. First we show that the constants in the LIL for the local time and for its supremum (with respect to the space variable) are equal under a very general condition given in Jain and Pruitt (1984). Second we evaluate the common value of the constants, as the random walk is in the domain of attraction of a not necessarily symmetric stable law. The first problem relies on a special maximal inequality established in this paper and the second on the LIL for Markovian additive functionals given in the author's recent work.

1 Introduction and main results

Consider a random walk $\{S_n\}_{n\geq 0}$ on the lattice $\mathbf{Z}$

(1.1)
$$S_0 = 0 \quad \text{and} \quad S_n = \sum_{k=1}^{n} X_k \qquad n = 1, 2, \cdots$$

generated by an i.i.d. sequence $\{X, X_n\}_{n\geq 1}$ with integer values. We assume for convenience that the law of X is not supported on a proper subgroup of $\mathbf{Z}$. The local time $L_n(x)$ and its maxima L_n^* are defined as

(1.2)
$$L_n(x) = \sum_{k=0}^{n} I_{\{S_k = x\}} \quad \text{and} \quad L_n^* = \sup_{y\in\mathbf{Z}} L_n(y) \qquad x \in \mathbf{Z},\ n = 1, 2, \cdots .$$

To make things interesting, we always consider the case when $\{S_n\}_{n\geq 0}$ is recurrent. This requires that $EX = 0$ whenever $E|X| < +\infty$. So the condition $EX = 0$ is assumed throughout without further mention.

Asymptotic properties for the local times have been extensively studied since the pioneering work by Chung and Hunt (1948), who established the first LIL for the local times of a symmetric simple random walk. Refer to Révész (1990) and the references therein for the historical development of this subject. Jain and Pruitt (1984) obtained some interesting LIL results for local times of the random walk under a very general condition, which we shall now describe.

Let $F(x)$ be the distribution function of X and write

$$G(x) = 1 - F(x) + F(-x) \quad \text{and} \quad K(x) = x^{-2} \int_{-x}^{x} y^2 dF(y)\,, \tag{1.3}$$

$$Q(x) = G(x) + K(x) = x^{-2} E\big(|X| \wedge x\big)^2 \tag{1.4}$$

for $x > 0$. The function Q is continuous and strictly decreasing for x large enough. Thus we can define the function $a(y)$ by

$$Q\big(a(y)\big) = \frac{1}{y} \tag{1.5}$$

for sufficiently large y, and $a(y) \uparrow \infty$. The basic assumption in Jain and Pruitt (1984) is

$$\limsup_{x\to\infty} \frac{G(x)}{K(x)} < 1 \tag{A}$$

which implies, in particular, that $E|X| < +\infty$. Under (A), the random walk $\{S_n\}_{n\geq 0}$ is recurrent, in which case Jain and Pruitt (1984) proved that there exist $0 < \theta_1, \theta_2 < \infty$ such that

$$\limsup_{n\to\infty} n^{-1} a\Big(\frac{n}{\log\log n}\Big) L_n(0) = \theta_1 \quad a.s. \quad , \tag{1.6}$$

$$\limsup_{n\to\infty} n^{-1} a\Big(\frac{n}{\log\log n}\Big) L_n^* = \theta_2 \quad a.s. \tag{1.7}$$

The quantity on the left of (A) was introduced by Feller (1966) to describe the compactness and convergence of the normalized random walks. If X is in the domain of attraction of a stable law of index α, for example, then

$$\lim_{x\to\infty} \frac{G(x)}{K(x)} = \frac{2-\alpha}{\alpha} \tag{1.8}$$

so that Jain and Pruitt's results include all cases when X is in the domain of attraction of a stable law of index $\alpha > 1$ and of zero mean. As pointed out by Jain and Pruitt, the class of the distributions described by (A) is much larger than this. Jain and Pruitt also pointed out that condition (A) excludes the case when the local time has a slowly varying increasing rate.

Jain and Pruitt (1984) then asked whether or not $\theta_1 = \theta_2$ under their condition. To this author's best knowledge, the first work related to this question is by Kesten (1965) who proved equality when the random walk is replaced by Brownian motion. By approximation, Kesten's result can be used to show that the equality holds for a symmetric simple random walk (Révész (1990)). As an application of large deviation theory, Donsker and Varadhan (1977) extended Kesten's observation to the case of symmetric stable processes with index $\alpha > 1$. In all mentioned cases, the common value of θ_1 and θ_2 is obtained.

Our first goal is to answer this question under condition (A). Indeed, we have

Theorem 1. *Under condition (A),*

$$\theta_1 = \theta_2 \tag{1.9}$$

where θ_1 and θ_2 are constants given by (1.6) and (1.7), respectively.

We are not able to determine the common value in (1.9) explicitly under condition (A). However, we have

Theorem 2. *Assume that*

$$\lim_{x\to\infty} \frac{G(x)}{K(x)} = \frac{2-\alpha}{\alpha} \quad \text{and} \quad p \equiv \lim_{x\to\infty} \frac{1-F(x)}{G(x)} \tag{B}$$

exist for some $1 < \alpha < 2$. Then

$$\limsup_{n\to\infty} n^{-1} a\Big(\frac{n}{\log\log n}\Big) L_n(0) = \limsup_{n\to\infty} n^{-1} a\Big(\frac{n}{\log\log n}\Big) L_n^* = \Lambda(\alpha) \quad a.s. \tag{1.10}$$

where

$$\Lambda(\alpha) = \frac{\Gamma(1-1/\alpha)\Gamma(1/\alpha)}{\pi}\left(\frac{2\cos\big((\rho-1/2)\alpha\pi\big)}{\Gamma(3-\alpha)\sin\frac{(\alpha-1)\pi}{2}}\right)^{1/\alpha}(\alpha-1)^{\frac{2-\alpha}{\alpha}}\sin(\rho\pi), \tag{1.11}$$

$$\rho = \frac{1}{2} + \frac{1}{\alpha\pi}\arctan\Big((2p-1)\tan\big(\frac{\alpha\pi}{2}\big)\Big). \tag{1.12}$$

Theorem 3. *Assume that*

$$\lim_{x\to\infty} \frac{G(x)}{K(x)} = 0. \tag{C}$$

Then

$$\limsup_{n\to\infty} n^{-1} a\Big(\frac{n}{\log\log n}\Big) L_n(0) = \limsup_{n\to\infty} n^{-1} a\Big(\frac{n}{\log\log n}\Big) L_n^* = \sqrt{2} \quad a.s. \tag{1.13}$$

Remark. According to Chap. IX, Theorem 8.1, p.303 in Feller (1966), (B) and (C) are the necessary and sufficient conditions for F being in the domain of attraction of a stable law with $1 < \alpha < 2$ and $\alpha = 2$, respectively. Under (B) (or (C) with $\alpha = 2$),

$$G\big(a(n)\big) \sim \frac{2-\alpha}{2n} \quad \text{and} \quad K\big(a(n)\big) \sim \frac{\alpha}{2n}. \tag{1.14}$$

By Chap. XI, Theorem 8.2 and the comment on p. 305 in Feller (1966) there is a stable law $\mathfrak{L}$ with index α such that

$$S_n/a(n) \longrightarrow \mathfrak{L} \quad \text{in distribution.} \tag{1.15}$$

It turns out that when $1 < \alpha < 2$, the characteristic exponent of $\mathfrak{L}$ is of the form:

$$\begin{aligned} \Psi(\lambda) = \; & \frac{\Gamma(3-\alpha)}{2(\alpha-1)} \sin\frac{(\alpha-1)\pi}{2} \\ & \times|\lambda|^{\alpha}\Big(1 - i(2p-1)\mathrm{sgn}(\lambda)\tan\big(\frac{\alpha\pi}{2}\big)\Big) \end{aligned} \tag{1.16}$$

where $\lambda \in \mathbf{R}$. Hence (see, e.g., Section 2.6 in Zolotarev (1986))

$$\rho = 1 - \mathfrak{L}(0) = \lim_{n\to\infty} P\{S_n \geq 0\}. \tag{1.17}$$

Moreover, one can easily see that $1-1/\alpha \leq \rho \leq 1/\alpha$. The extreme points $\rho = 1/\alpha$ and $\rho = 1 - 1/\alpha$ correspond to the cases when $p = 0$ and $p = 1$, respectively. In particular $\cos\big((\rho-1/2)\alpha\pi\big) > 0$ and therefore the constant we obtain in Theorem 2 remains positive for all $0 \leq p \leq 1$. It is also interesting to see that the maximal limit value in Theorem 2 is achieved when $p = 1/2$, especially when X is symmetrically distributed — in which case Theorem 2 essentially belongs to (in the light of Theorem 1) Marcus and Rosen (1994). Bertoin (1995) obtains a law of the iterated logarithm for the local times of Lévy processes which are not necessarily symmetric. In Bertoin's result (Corollary 2), the constant of the LIL is given under a condition essentially the same as (B) (The author thanks the referee of this paper for pointing out Bertoin's contribution).

When X has a finite second moment σ^2, i.e.,

$$\sigma^2 \equiv E|X|^2 < +\infty\,, \tag{1.18}$$

one can easily see that $a(n) \sim \sqrt{n}\sigma$ as $n \to \infty$. Hence by Theorem 3 we have

$$\limsup_{n\to\infty} \frac{L_n(0)}{\sqrt{2n\log\log n}} = \limsup_{n\to\infty} \frac{L_n^*}{\sqrt{2n\log\log n}} = \frac{1}{\sigma} \qquad a.s. \tag{1.19}$$

2 Proof of Theorem 1

Clearly we have $\theta_1 \le \theta_2$. Hence we need to show

$$\theta_1 \ge \theta_2 \,. \tag{2.1}$$

Our approach is to compare the tail probabilities of $L_n(0)$ and L_n^* and we rely on the following inequalities.

Lemma 1. *For any integers $a, b > 0$ and $n \ge 1$,*

$$P\big\{L_n(x) \ge a\big\} \le P\big\{L_n(0) \ge a\big\} \qquad x \in \mathbf{Z}, \tag{2.2}$$

$$P\big\{L_n^* \ge a+b\big\} \le P\big\{L_n^* \ge a\big\} P\big\{L_n^* \ge b\big\}, \tag{2.3}$$

$$P\big\{L_n^* \ge a\big\} \le \Big(1 - P\big\{\max_{k\le n} |S_k| \ge b\big\}\Big)^{-1} P\big\{\sup_{|x|\le b} L_n(x) \ge a\big\}. \tag{2.4}$$

Proof. Define the stopping times as

$$\tau_1 = \inf\{k \ge 1; \quad S_k = x\}, \tag{2.5}$$

$$\tau_2 = \inf\{k \ge 1; \quad L_k^* \ge b\}, \tag{2.6}$$

$$\tau_3 = \inf\{k \ge 1; \quad |S_k| \ge b\}. \tag{2.7}$$

To prove (2.2), we may assume that $x \ne 0$. Hence

$$\begin{aligned} P\big\{L_n(x) \ge a\big\} &= P\Big\{\tau_1 \le n, \ \sum_{k=\tau_1}^{n} I_{\{S_k = x\}} \ge a\Big\} \\ &= \sum_{j=1}^{n} P\Big\{\tau_1 = j, \ \sum_{k=j}^{n} I_{\{S_k - S_j = 0\}} \ge a\Big\} \\ &= \sum_{j=1}^{n} P\big\{\tau_1 = j\big\} P\big\{L_{n-j}(0) \ge a\big\} \\ &\le P\big\{L_n(0) \ge a\big\}. \end{aligned}$$

The proof of (2.3) and (2.4) relies on the fact that for each $1 \le j \le n$, the random variable

$$\sup_{x \in \mathbf{Z}} \sum_{k=j}^{n} I_{\{S_k = x\}} \tag{2.8}$$

is independent of $\{X_1, \cdots, X_j\}$. Notice that $L^*_{\tau_2} = b$.

$$\begin{aligned} P\big\{L^*_n \geq a+b\big\} &= P\big\{\tau_2 \leq n,\ L^*_n - L^*_{\tau_2} \geq a\big\} \\ &\leq P\Big\{\tau_2 \leq n,\ \sup_{x\in\mathbf{Z}} \sum_{k=\tau_2}^{n} I_{\{S_k=x\}} \geq a\Big\} \\ &\leq P\big\{L^*_n \geq a\big\}P\big\{L^*_n \geq b\big\}. \end{aligned}$$

We shall now prove (2.4). Notice that

$$\begin{aligned} P\big\{L^*_n \geq a\big\} &\leq P\big\{\sup_{|x|\leq b} L_n(x) \geq a\big\} + P\big\{\sup_{|x|>b} L_n(x) \geq a\big\} \\ &= (I) + (II) \qquad \text{(say)} \end{aligned}$$

and that

(2.9)
$$(II) \leq P\Big\{\tau_3 \leq n,\ \sup_x \sum_{k=\tau_3}^{n} I_{\{S_k=x\}} \geq a\Big\} \leq P\big\{\max_{k\leq n}|S_k| \geq b\big\}P\big\{L^*_n \geq a\big\}.$$

Therefore

$$\Big(1 - P\big\{\max_{k\leq n}|S_k| \geq b\big\}\Big)P\big\{L^*_n \geq a\big\} \leq P\big\{\sup_{|x|\leq b} L_n(x) \geq a\big\} \tag{2.10}$$

which gives (2.4). Q.E.D.

To prove (2.1), we let

$$b_n = n\Big[a\Big(\frac{n}{\log\log n}\Big)\Big]^{-1} \quad \text{and} \quad c_n = a\Big(\frac{n}{\log\log n}\Big). \tag{2.11}$$

We let $\epsilon > 0$ be fixed but arbitrary. Applying (2.3) gives

$$P\big\{L^*_n \geq (\theta_1 + 4\epsilon)b_n\big\} \leq P\big\{L^*_n \geq 2\epsilon b_n\big\}P\big\{L^*_n \geq (\theta_1 + 2\epsilon)b_n\big\}. \tag{2.12}$$

Note (Lemma 3, Jain and Pruitt (1984)) that there is an $M > 0$ such that

$$P\big\{\max_{k\leq n}|S_k| \geq Ma(n)\big\} \leq \frac{1}{2}. \tag{2.13}$$

Thus by (2.4),

$$P\big\{L^*_n \geq 2\epsilon b_n\big\} \leq 2P\big\{\sup_{|x|\leq Ma(n)} L_n(x) \geq 2\epsilon b_n\big\}, \tag{2.14}$$

$$P\big\{L^*_n \geq (\theta_1 + 2\epsilon)b_n\big\} \leq 2P\big\{\sup_{|x|\leq Ma(n)} L_n(x) \geq (\theta_1 + 2\epsilon)b_n\big\}. \tag{2.15}$$

According to Lemma 11 in Jain and Pruitt (1984), there is a $\delta > 0$ and a $C > 0$ such that

$$P\Big\{ \sup_{|x-y|\le \delta c_n} \big|L_n(x) - L_n(y)\big| \ge \epsilon b_n \Big\} \le \frac{C}{\log^2 n} \tag{2.16}$$

for all $n \ge 1$. On the other hand, notice that the interval $[-Ma(n), Ma(n)]$ can be covered by no more than $(K \log\log n)^\lambda$ intervals with diameters less than δc_n, where $\lambda > 0$ and $K > 0$ are constants. Hence

$$\begin{aligned} &P\Big\{ \sup_{|x|\le Ma(n)} L_n(x) \ge 2\epsilon b_n \Big\} \\ &\le (K \log\log n)^\lambda \sup_{x\in\mathbf{Z}} P\big\{ L_n(x) \ge \epsilon b_n \big\} + \frac{C}{\log^2 n}, \end{aligned}$$

$$\begin{aligned} &P\Big\{ \sup_{|x|\le Ma(n)} L_n(x) \ge (\theta_1 + 2\epsilon) b_n \Big\} \\ &\le (K \log\log n)^\lambda \sup_{x\in\mathbf{Z}} P\big\{ L_n(x) \ge (\theta_1 + \epsilon) b_n \big\} + \frac{C}{\log^2 n}. \end{aligned}$$

In view of (2.2) and (2.12),

$$\begin{aligned} &P\big\{ L_n^* \ge (\theta_1 + 4\epsilon) b_n \big\} \\ &\le 4(K \log\log n)^{2\lambda} P\big\{ L_n(0) \ge \epsilon b_n \big\} P\big\{ L_n(0) \ge (\theta_1 + \epsilon) b_n \big\} \\ &+ C \frac{(\log\log n)^\lambda}{\log^2 n} \end{aligned}$$

where the constant $C > 0$ may differ from before. By examining the proof of Lemma 8 in Jain and Pruitt (1984) one can see that

$$P\big\{ L_n(0) \ge \epsilon b_n \big\} \le \frac{1}{(\log n)^\delta} \tag{2.17}$$

eventually holds for some small constant $\delta > 0$. Hence,

$$\begin{aligned} &P\big\{ L_n^* \ge (\theta_1 + 4\epsilon) b_n \big\} \\ &\le P\big\{ L_n(0) \ge (\theta_1 + \epsilon) b_n \big\} + C \frac{(\log\log n)^\lambda}{\log^2 n} \end{aligned} \tag{2.18}$$

for large n.

On the other hand, given a $\gamma > 1$ we take a subsequence $\{n_k\}$ such that

$$b_{n_k} \sim \gamma^k \qquad (k \to \infty). \tag{2.19}$$

By Lemma 3.3 in Chen (1999),

$$\sum_k P\big\{ L_{n_k}(0) \ge (\theta_1 + \epsilon) b_{n_k} \big\} < +\infty. \tag{2.20}$$

From (2.18),

$$\sum_k P\{L^*_{n_k} \geq (\theta_1 + 4\epsilon) b_{n_k}\} < +\infty. \tag{2.21}$$

Since $\gamma > 1$ and $\epsilon > 0$ are arbitrary, a standard argument via the Borel-Cantelli lemma gives

$$\limsup_{n\to\infty} L^*_n / b_n \leq \theta_1 \quad a.s. \tag{2.22}$$

Hence, (2.1) is proved. Q.E.D.

3 Proof of Theorem 2 and Theorem 3

In view of Theorem 1, we only need to show

$$\limsup_{n\to\infty} n^{-1} a\Big(\frac{n}{\log\log n}\Big) L_n(0) = \Lambda(\alpha) \quad a.s. \tag{3.1}$$

for $1 < \alpha < 2$, and

$$\limsup_{n\to\infty} n^{-1} a\Big(\frac{n}{\log\log n}\Big) L_n(0) = \sqrt{2} \quad a.s. \tag{3.2}$$

for $\alpha = 2$.

The proof we present here appears as an application of the LIL (Chen (1999)) for additive functionals of Harris recurrent Markov chains. Let the stable law $\mathfrak{L}$ be given by (1.15). From (2.j) in Le Gall-Rosen (1991), as $n \to \infty$,

$$g(n) \equiv \sum_{k=1}^n P\{S_k = 0\} \sim p(0) \sum_{k=1}^n \frac{1}{a(k)}, \tag{3.3}$$

where $p(\cdot)$ is the density of $\mathfrak{L}$. Note that $a(y)$ is non-decreasing and varies regularly at infinity with index α^{-1}. As a routine exercise one can show that, as $n \to \infty$,

$$g(n) \sim p(0) \frac{n}{a(n)} \int_0^1 \frac{1}{x^{1/\alpha}} dx = p(0) \frac{\alpha}{\alpha - 1} \frac{n}{a(n)}. \tag{3.4}$$

In particular, $g(n)$ varies regularly at infinity with index $1 - 1/\alpha$. Viewing $\{S_n\}_{n\geq 0}$ as a recurrent Markov chain with the counting measure on $\mathbf{Z}$ as its invariant measure, we obtain from Theorem 2.4 in Chen (1999) that

(3.5)

$$\limsup_{n\to\infty} \Big(g\Big(\frac{n}{\log\log n}\Big) \log\log n\Big)^{-1} L_n(0) = (\alpha - 1)^{1/\alpha} \Gamma(1 - 1/\alpha) \quad a.s.$$

In view of (3.4),

$$\limsup_{n\to\infty} n^{-1}a\Big(\frac{n}{\log\log n}\Big)L_n(0) = \frac{\alpha\Gamma(1-1/\alpha)}{(\alpha-1)^{1-1/\alpha}}p(0) \quad a.s. \tag{3.6}$$

When $\alpha = 2$, we have from (1.14) that

$$nP\{|X| > a(n)\} \longrightarrow 0 \quad \text{and} \quad E|X|^2 I_{\{|X|\le a(n)\}} \sim \frac{a(n)^2}{n} \quad (n\to\infty). \tag{3.7}$$

By performing the truncation at the level $a(n)$ and by the classical method for the central limit theorem we can obtain that $\mathfrak{L} = N(0,1)$. In particular $p(0) = (2\pi)^{-1/2}$. Hence (3.2) follows from (3.5). We now assume $1 < \alpha < 2$.

To evaluate $p(0)$, we first verify that the characteristic exponent $\Psi(\lambda)$ of $\mathfrak{L}$ is given by (1.16) (This may be known, but we fail to find a complete statement of this result in the literature). By Theorem C.1 and (I.11) in Zolotarev (1986)

$$P\{X > a_n\} \sim \frac{C_1}{n} \quad \text{and} \quad P\{X \le -a_n\} \sim \frac{C_2}{n} \tag{3.8}$$

where $C_1, C_2 \ge 0$ are two parameters of the spectral function of $\mathfrak{L}$ given in that theorem. From the first part in (1.14) and the second part in (B) we have

$$C_1 + C_2 = \frac{2-\alpha}{2} > 0 \quad \text{and} \quad p = \frac{2C_1}{2-\alpha}. \tag{3.9}$$

Hence (1.16) follows from the construction given in the proof of Theorem C.2 in Zolotarev (1986) and the fact that the law $\mathfrak{L}$ has zero mean.

We finally evaluate $p(0)$ in the case when $0 < \alpha < 2$. Write

$$c = \frac{\Gamma(3-\alpha)}{2(\alpha-1)}\sin\frac{(\alpha-1)\pi}{2}. \tag{3.10}$$

Taking the inverse Fourier transformation gives

$$\begin{aligned} p(0) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\exp\Big\{-c|\lambda|^\alpha\Big(1 - i(2p-1)\mathrm{sgn}(\lambda)\tan\big(\frac{\alpha\pi}{2}\big)\Big)\Big\}d\lambda \\ &= \frac{1}{\alpha\pi}\int_0^{+\infty} x^{1/\alpha-1}e^{-cx}\cos\Big[\big(c(2p-1)\tan\big(\frac{\alpha\pi}{2}\big)\big)x\Big]dx \\ &= \frac{\cos\Big(\alpha^{-1}\arctan\big((2p-1)\tan\big(\frac{\alpha\pi}{2}\big)\big)\Big)}{\alpha\pi c^{1/\alpha}\Big[1+(2p-1)^2\tan^2\big(\frac{\alpha\pi}{2}\big)\Big]^{1/2\alpha}}\Gamma(1/\alpha) \qquad (3.11)\\ &= (\alpha\pi)^{-1}\Big[c^{-1}\cos\big((\rho-1/2)\alpha\pi\big)\Big]^{1/\alpha}\sin(\rho\pi)\Gamma(1/\alpha) \\ &= (\alpha\pi)^{-1}\Big[\frac{2(\alpha-1)\cos\big((\rho-1/2)\alpha\pi\big)}{\Gamma(3-\alpha)\sin\frac{(\alpha-1)\pi}{2}}\Big]^{1/\alpha}\sin(\rho\pi)\Gamma(1/\alpha) \end{aligned}$$

where the third step follows from the fact that

$$\int_0^\infty x^{t-1}e^{-ax}\cos(bx)dx = \frac{\sin\left(t\arctan\left(\frac{b}{a}\right)\right)}{(a^2+b^2)^{t/2}}\Gamma(t) \tag{3.12}$$

for all real numbers $t > -1$, $a > 0$ and b.

Bringing our computation of $p(0)$ back to (3.5) yields (3.1). Q.E.D.

4 Further remark

Since the LIL in a general Markovian context applied to Theorem 2 and Theorem 3 holds also in the case of non-discrete state space, we can achieve some results for non-lattice valued random walks, which take a form similar to (3.1) or (3.2). Here we consider a random walk $\{S_n\}_{n\geq 0}$ on $\mathbf{R}$. In the terminology of Revuz (1975), $\{S_n\}_{n\geq 0}$ is called *spread out*, if there is an integer $k \geq 1$ such that the kth convolution F^{*k} is not singular to the Lebesgue measure on $\mathbf{R}$, where F is the distribution of its i.i.d. increment. Revuz (1975) points out that this class is much larger than the class of absolutely continuous F. It is known (see, e.g., Sections 4 and 5, Chapter 3 in Revuz (1975)) that when viewed as a Markov chain with Lebesgue measure as its invariant measure, $\{S_n\}_{n\geq 0}$ is Harris recurrent if it is spread out.

We adopt all notations introduced in Section 1. By an argument almost identical to the one carried out in Section 3, we can prove

Theorem 4. *Assume that $\{S_n\}_{n\geq 0}$ is spread out and satisfies condition (B) for some $1 < \alpha < 2$. Then for any non-negative Lebesgue integrable function f on $\boldsymbol{R}$,*

$$\limsup_{n\to\infty} n^{-1}a\Big(\frac{n}{\log\log n}\Big)\sum_{k=1}^n f(S_k) = \Lambda(\alpha)\int_{-\infty}^\infty f(x)dx \quad a.s. \tag{4.1}$$

where the constant $\Lambda(\alpha) > 0$ is given as in Theorem 2.

Theorem 5. *Assume that $\{S_n\}_{n\geq 0}$ is spread out and satisfies condition (C). Then for any non-negative Lebesgue integrable function f on $\boldsymbol{R}$,*

$$\limsup_{n\to\infty} n^{-1}a\Big(\frac{n}{\log\log n}\Big)\sum_{k=1}^n f(S_k) = \sqrt{2}\int_{-\infty}^\infty f(x)dx \quad a.s. \tag{4.2}$$

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