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FEYNMAN–KAC REPRESENTATION FOR PARABOLIC ANDERSON EQUATIONS WITH GENERAL GAUSSIAN NOISE

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We provide the Feynman–Kac representation for the parabolic Anderson equations driven by a general Gaussian noise. As a feature of the idea, we can mention the argument of subadditivity used to establish the required exponential integrability.

1. Introduction

Consider a parabolic Anderson equation

$$\begin{aligned} &\frac{\partial u}{\partial t}(t,x) = \frac{1}{2}\Delta u(t,x) + \dot{W}(t,x)u(t,x), \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d, \\ &u(0,x) = u_0(x), \quad x \in \mathbb{R}^d, \end{aligned} \tag{1.1}$$

run by the mean zero and, possibly, a generalized time-space Gaussian noise $\dot{W}(t,x),\ (t,x) \in \mathbb{R} \times \mathbb{R}^d$, with the covariance function

$$\text{Cov}\left(\dot{W}(t,x), \dot{W}(s,y)\right) = |t-s|^{-\alpha_0}\gamma(x-y), \quad x, y \in \mathbb{R}^d, \tag{1.2}$$

where $0 < \alpha_0 < 1$. Throughout the paper, we assume that $\gamma(\cdot) \geq 0$. With maximal generality, the present paper allows that $\gamma(\cdot)$ can be a generalized function defined as a linear functional on $\mathcal{S}(\mathbb{R}^d)$, the set of all rapidly decreasing functions known as the Schwartz space. Since $\gamma(\cdot)$ is nonnegative definite as a covariance function, by Bochner's theorem, there is a unique measure on $\mathbb{R}^d$ known as the spectral measure of $\gamma(\cdot)$ such that

$$\gamma(x) = \int_{\mathbb{R}^d} e^{i\xi\cdot x}\mu(d\xi). \tag{1.3}$$

Further, $\mu(d\xi)$ is tempered in a sense that

$$\int_{\mathbb{R}^d} \left(\frac{1}{1+|\xi|^2}\right)^p \mu(d\xi) < \infty$$

for some $p > 0$. In particular, $\mu(d\xi)$ is locally finite.

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The singularity of the system does not make (1.1) a rigorous definition. Mathematically, a random field $u(t,x)$, $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^d$, is called a weak solution of (1.1) if

$$\int_{\mathbb{R}^d} u(t,x)\varphi(x)\,dx = \int_{\mathbb{R}^d} u_0(x)\varphi(x)\,dx + \frac{1}{2}\int_0^t \int_{\mathbb{R}^d} u(s,x)\Delta\varphi(x)\,dx\,ds$$

$$+ \int_0^t \int_{\mathbb{R}^d} u(s,x)\varphi(x)W(ds\,dx) \quad \text{a.s.} \tag{1.4}$$

for every C^∞-function φ with compact support, where the stochastic integral on the right-hand side is known as the Stratonovich integral, which is defined as follows:

$$\int_0^t \int_{\mathbb{R}^d} v(s,x)W(ds\,dx) \triangleq \lim_{\epsilon\to 0^+} \int_0^t \int_{\mathbb{R}^d} v(s,x)\dot{W}_\epsilon(t,x)\,dx\,ds \quad \text{in probability}$$

(whenever the limit exists) for all random fields $v(t,x),\ (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d$, satisfying the inequality

$$\int_0^t \int_{\mathbb{R}^d} |v(s,x)|\,dx\,ds < \infty \quad \text{a.s.}$$

Here, $\dot{W}_\epsilon(t,x)$ is a smoothed version of $\dot{W}(t,x)$ (see relation (2.1) in what follows).

In the case where

$$\dot{W}(t,x) = \frac{\partial^{d+1} W^H(t,x)}{\partial t \partial x_1 \ldots \partial x_d}, \qquad x = (x_1,\ldots,x_d), \tag{1.5}$$

is the formal derivative of a fractional Brownian sheet $W^H(t,x)$ with the Hurst parameter $(H_0,\ldots,H_d)$, where $H_0 > 1/2$ and $H_1,\ldots,H_d \ge 1/2$, it was proved in [7] that, under the condition

$$2H_0 + \sum_{j=1}^d H_j > d+1, \tag{1.6}$$

the random field

$$u(t,x) \triangleq \mathbb{E}_x \exp\left\{ \int_0^t \dot{W}(t-s,B_s)\,ds \right\} u_0(B_t), \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d, \tag{1.7}$$

provides a weak solution to the parabolic Anderson equation (1.1). Here B_s is a d-dimensional Brownian motion starting at x and independent of $\dot{W}$, $\mathbb{E}_x$ is the expectation with respect to the Brownian motion, and the time-integral on the right-hand side is properly defined by the way of approximation [see relation (2.1) in what follows].

In the literature, formula (1.7) is known as the Feynman–Kac representation. For the first time, it appeared in the setting of deterministic heat equation (see, e.g., Theorem 2.2 in [5, p. 132]) with $\dot{W}(t,x)$ replaced by a deterministic function with sufficient regularity.

Representation (1.7) has been extended (see Section 6 in [4]) to a class of Gaussian noises with spatial covariance of the homogeneity

$$\gamma(cx) = c^{-\alpha}\gamma(x), \quad x \in \mathbb{R}^d, \quad c > 0, \tag{1.8}$$

with $0 < \alpha < 2(1-\alpha_0)$.

In the present paper, we solve the parabolic Anderson equation by establishing representation (1.7) for the Gaussian noise with the general spatial covariance $\gamma(\cdot)$.

Theorem 1.1. *Assume that $u_0(x)$ is a bounded and measurable function on $\mathbb{R}^d$ and*

$$\int_{\mathbb{R}^d} \left(\frac{1}{1+|\xi|^2}\right)^{1-\alpha_0} \mu(d\xi) < \infty. \tag{1.9}$$

The random field $u(t,x)$ given in (1.7) is well defined and is a weak solution of the parabolic Anderson equation (1.1). Further, $u(t,x) \in \mathcal{L}^m(\Omega, \mathcal{A}, \mathbb{P})$ for all $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^d$ with the following representation:

$$\mathbb{E}u^m(t,x) = \mathbb{E}_x \exp\left\{\sum_{j,k=1}^m \int_0^t \int_0^t \frac{\gamma(B_j(s)-B_k(r))}{|s-r|^{\alpha_0}}\, ds\, dr\right\} \prod_{j=1}^m u_0(B_j(t)), \tag{1.10}$$

where $B_1(t), \dots, B_m(t)$ are independent d-dimensional Brownian motions with $B_j(0) = x$, $\mathbb{E}_x$ is the expectation with respect to the Brownian motions, and the time-Hamiltonians on the right-hand side are defined by an appropriate approximation [see (2.6) and (2.7) in what follows].

For the purpose of comparison, we mention a different regime in which the parabolic Anderson equation (1.1) is defined by

$$u(t,x) = (p_t * u_0)(x) + \int_0^t \int_{\mathbb{R}^d} p_{t-s}(y-x)u(s,y)W(ds\, dy), \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d,$$

where $p_t(x)$ is a Brownian semigroup defined as follows:

$$p_t(x) = \frac{1}{(2\pi t)^{d/2}} \exp\left\{-\frac{1}{2t}|x|^2\right\}, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d,$$

and the stochastic integral on the right-hand side is understood as a Skorokhod integral. In the Skorokhod case, it was proved (see Theorem 3.6 in [6]) that equation (1.1) has the following solution under the Dalang condition:

$$\int_{\mathbb{R}^d} \frac{1}{1+|\xi|^2} \mu(d\xi) < \infty. \tag{1.11}$$

Contrary to (1.11), assumption (1.9) shows that the singularity of the time component (quantified by α_0) of the Gaussian noise $\dot{W}(t,x)$ contributes to the system singularity in the setting of weak solution.

Assumption (1.9) is necessary when $u_0(x) = 1$: by (2.5) and (2.8) in what follows, we get,

$$\mathbb{E} \otimes \mathbb{E}_x \left[\int_0^t \dot{W}(t-s, B_s)\, ds \right]^2 = \mathbb{E}_0 \int_0^t \int_0^t \frac{\gamma(B(s) - B(r))}{|s-r|^{\alpha_0}}\, ds\, dr$$

$$= \int_{\mathbb{R}^d} \mu(d\xi) \int_0^t \int_0^t |s-r|^{-\alpha_0} \exp\left\{ -\frac{|\xi|^2}{2} |s-r| \right\}.$$

One can check [see the computation next to (2.5) in what follows] that condition (1.9) is equivalent to

$$\mathbb{E} \otimes \mathbb{E}_x \left[\int_0^t \dot{W}(t-s, B_s)\, ds \right]^2 < \infty \quad \text{for some} \quad t > 0 \quad \text{or, equivalently, for every} \quad t > 0.$$

Hence, condition (1.9) is necessary for the meaningful and integrable expression given by (1.7).

In view of the homogeneity condition (1.8), by virtue of Lemma 3.10 in [3], we get

$$\int_{\mathbb{R}^d} \left(\frac{1}{1 + |\xi|^2} \right)^{1-\alpha_0} \mu(d\xi) = \alpha \mu(B(0,1)) \int_0^\infty \left(\frac{1}{1+\rho^2} \right)^{1-\alpha_0} \rho^{\alpha - 1}\, d\rho.$$

Since $\mu(d\xi)$ is tempered, $\mu(B(0,1)) < \infty$. Therefore, (1.9) holds if and only if $\alpha < 2(1 - \alpha_0)$.

Corollary 1.1. *Under assumption (1.8) with* $0 < \alpha < 2(1 - \alpha_0)$, *all statements in Theorem 1.1 are true.*

As in the special case where $\dot{W}(t, x)$ is the fractional Gaussian noise given by (1.5), the homogeneity condition (1.8) is satisfied with

$$\alpha_0 = 2 - 2H_0 \qquad \text{and} \qquad \alpha = 2d - 2\sum_{j=1}^d H_j.$$

Consequently, (1.6) is equivalent to $0 < \alpha < 2(1 - \alpha_0)$.

The proof of Theorem 1.1 is given in the next section. We especially mention a striking fact that the exponential integrability [given by (2.10) below] of the Brownian Hamiltonian

$$\int_0^t \int_0^t \frac{\gamma(B_s - B_r)}{|s - r|^{\alpha_0}}\, ds\, dr$$

is determined by its local behavior near $t = 0$ (Lemma 2.1), and the efficiency of the subadditivity approach used to prove this fact.

2. Proof of Theorem 1.1

The time-integral in representation (1.7) is defined as follows:

$$\int_0^t \dot{W}(t-s, B_s)\, ds \triangleq \lim_{\epsilon \to 0^+} \int_0^t \dot{W}_\epsilon(t-s, B_s)\, ds \quad \text{in} \quad \mathcal{L}^2(\Omega, \mathcal{A}, \mathbb{P}_x \otimes \mathbb{P}), \tag{2.1}$$

where $\dot{W}_\epsilon$ is a pointwise-defined Gaussian field $\dot{W}_\epsilon(t,x)$ given by

$$\dot{W}_\epsilon(t,x) \triangleq \int_{\mathbb{R}^{d+1}} \dot{W}(u,y)\left[(2\pi\epsilon)^{-\frac{d+1}{2}}\exp\left\{-\frac{(t-u)^2+|x-y|^2}{2\epsilon}\right\}\right]du\,dy, \quad (t,x)\in\mathbb{R}^+\times\mathbb{R}^d.$$

To make it working, it is necessary to show that the limit on the right-hand side exists. To this end, all we need is to show that the limit

$$\lim_{\epsilon,\epsilon'\to 0^+}\mathbb{E}_x\otimes\mathbb{E}\left(\int_0^t \dot{W}_\epsilon(t-s,B_s)\,ds\right)\left(\int_0^t \dot{W}_{\epsilon'}(t-s,B_s)\,ds\right)$$

exists.

Note that

$$\mathrm{Cov}\left(W_\epsilon(s,x),W_\epsilon(r,y)\right)=\gamma_{0,\epsilon+\epsilon'}(s-r)\gamma_{\epsilon+\epsilon'}(x-y),$$

where

$$\gamma_{0,\epsilon}(u)=\int_{\mathbb{R}}\frac{1}{|v|^{\alpha_0}}\left[\frac{1}{\sqrt{2\pi\epsilon}}\exp\left\{-\frac{(u-v)^2}{2\epsilon}\right\}\right]dv, \tag{2.2}$$

$$\gamma_\epsilon(x)=\int_{\mathbb{R}^d}\gamma(y)\left[\frac{1}{(2\pi\epsilon)^{d/2}}\exp\left\{-\frac{|x-y|^2}{2\epsilon}\right\}\right]dy. \tag{2.3}$$

We have

$$\begin{aligned}&\mathbb{E}_x\otimes\mathbb{E}\left(\int_0^t \dot{W}_\epsilon(t-s,B_s)\,ds\right)\left(\int_0^t \dot{W}_{\epsilon'}(t-s,B_s)\,ds\right)\\ &\qquad=\mathbb{E}_0\int_0^t\int_0^t \mathbb{E}\dot{W}_\epsilon(t-s,B_s)\dot{W}_{\epsilon'}(t-r,B_r)\,ds\,dr\\ &\qquad=\mathbb{E}_0\int_0^t\int_0^t \gamma_{0,\epsilon+\epsilon'}(s-r)\gamma_{\epsilon+\epsilon'}(B_s-B_r)\,ds\,dr.\end{aligned}$$

We also note that, for any $\delta>0$, $\gamma_\delta(\cdot)$ has the spectral measure $e^{-\delta|\xi|^2/2}\mu(d\xi)$. Let $\mu_0(d\lambda)$ be the spectral measure of $|\cdot|^{-\alpha_0}$ (one can easily show that $\mu_0(d\lambda)$ is a constant multiple of $|\lambda|^{-(1-\alpha_0)}d\lambda$). Then $\gamma_{0,\delta}(\cdot)$ has the spectral measure $e^{-\delta\lambda^2/2}\mu_0(d\lambda)$. By the Fourier transform, we get

$$\begin{aligned}&\int_0^t\int_0^t \gamma_{0,\epsilon+\epsilon'}(s-r)\gamma_{\epsilon+\epsilon'}(B_s-B_r)\,ds\,dr\\ &\qquad=\int_{\mathbb{R}^{d+1}}\exp\left\{-\frac{\epsilon+\epsilon'}{2}\left(\lambda^2+|\xi|^2\right)\right\}\left|\int_0^t\exp\left\{i\lambda s+i\xi\cdot B_s\right\}ds\right|^2\mu_0(d\lambda)\mu(d\xi).\end{aligned}$$

Therefore, by the dominated convergence theorem,

$$\lim_{\epsilon,\epsilon'\to 0^+} \mathbb{E}_x \otimes \mathbb{E} \left(\int_0^t \dot{W}_\epsilon(t-s, B_s)\, ds \right) \left(\int_0^t \dot{W}_{\epsilon'}(t-s, B_s) ds \right)$$

$$= \int_{\mathbb{R}^{d+1}} \mathbb{E}_0 \left| \int_0^t \exp\{i\lambda s + i\xi \cdot B_s\}\, ds \right|^2 \mu_0(d\lambda)\mu(d\xi)$$

provided that

$$\int_{\mathbb{R}^{d+1}} \mathbb{E}_0 \left| \int_0^t \exp\{i\lambda s + i\xi \cdot B_s\}\, ds \right|^2 \mu_0(d\lambda)\mu(d\xi) < \infty \quad \forall t > 0. \tag{2.4}$$

Here, we have used the fact that the integral in (2.4) is independent of the starting point x of the Brownian motion (and, hence, we take $x = 0$). Indeed,

$$\int_{\mathbb{R}^{d+1}} \mathbb{E}_0 \left| \int_0^t \exp\{i\lambda s + i\xi \cdot B_s\}\, ds \right|^2 \mu_0(d\lambda)\mu(d\xi)$$

$$= \int_{\mathbb{R}^d} \mu(d\xi) \mathbb{E}_0 \int_0^t \int_0^t |s-r|^{-\alpha_0} \exp\{i\xi \cdot (B_s - B_r)\}\, ds\, dr$$

$$= \int_{\mathbb{R}^d} \mu(d\xi) \int_0^t \int_0^t |s-r|^{-\alpha_0} \exp\left\{ -\frac{|\xi|^2}{2} |s-r| \right\} ds\, dr. \tag{2.5}$$

Note that the right-hand side is monotonic in t. To establish (2.4), all we need is to prove that

$$\int_{\mathbb{R}^d} \mu(d\xi) \int_0^\infty dt e^{-t} \int_0^t \int_0^t |s-r|^{-\alpha_0} \exp\left\{ -\frac{|\xi|^2}{2} |s-r| \right\} ds\, dr < \infty.$$

Indeed,

$$\int_{\mathbb{R}^d} \mu(d\xi) \int_0^\infty dt e^{-t} \int_0^t \int_0^t |s-r|^{-\alpha_0} \exp\left\{ -\frac{|\xi|^2}{2} |s-r| \right\} ds\, dr$$

$$= 2 \int_{\mathbb{R}^d} \mu(d\xi) \int_0^\infty dt e^{-t} \int_0^t \int_r^t (s-r)^{-\alpha_0} \exp\left\{ -\frac{|\xi|^2}{2} (s-r) \right\} ds\, dr$$

$$= 2 \int_{\mathbb{R}^d} \mu(d\xi) \int_0^\infty t^{-\alpha_0} \exp\left\{-\frac{|\xi|^2}{2}\, t\right\} e^{-t}\, dt$$

$$= 2 \left(\int_0^\infty t^{-\alpha_0} e^{-t}\, dt\right) \int_{\mathbb{R}^d} \left(\frac{1}{1 + 2^{-1}|\xi|^2}\right)^{1-\alpha_0} \mu(d\xi),$$

where the last step follows from the integration substitution

$$t \mapsto \left(1 + 2^{-1}|\xi|^2\right)^{-1} t.$$

In summary, by using condition (1.9) we have proved (2.4) and, therefore, justified the definition in (2.1).

Further, we clarify the time-Hamiltonians in (1.10) by introducing the following definition:

$$\int_0^t\!\!\int_0^t \frac{\gamma(B_s - B_r)}{|s-r|^{\alpha_0}}\, ds\, dr \triangleq \lim_{\epsilon \to 0^+} \int_0^t\!\!\int_0^t \gamma_{0,\epsilon}(s-r)\gamma_\epsilon(B_s - B_r)\, ds\, dr \quad \text{in} \quad \mathcal{L}(\Omega, \mathcal{A}, \mathbb{P}_x), \tag{2.6}$$

$$\int_0^t\!\!\int_0^t \frac{\gamma\left(B_s - \widetilde{B}_r\right)}{|s-r|^{\alpha_0}}\, ds\, dr \triangleq \lim_{\epsilon \to 0^+} \int_0^t\!\!\int_0^t \gamma_{0,\epsilon}(s-r)\gamma_\epsilon\left(B_s - \widetilde{B}_r\right) ds\, dr \quad \text{in} \quad \mathcal{L}(\Omega, \mathcal{A}, \mathbb{P}_x) \tag{2.7}$$

for two independent Brownian motions B_t and $\widetilde{B}_t$, where $\gamma_{0,\epsilon}(\cdot)$ and $\gamma_\epsilon(\cdot)$ are given by (2.2) and (2.3), respectively.

Once again, we note that the problem is independent of the starting point of Brownian motions, that

$$\int_0^t\!\!\int_0^t \gamma_{0,\epsilon}(s-r)\gamma_\epsilon(B_s - B_r)\, ds\, dr$$

$$= \int_{\mathbb{R}^{d+1}} \left|\int_0^t e^{i\lambda s + i\xi\cdot B_s}\, ds\right|^2 \exp\left\{-\frac{\epsilon}{2}\left(\lambda^2 + |\xi|^2\right)\right\} \mu_0(d\lambda)\mu(d\xi),$$

and that

$$\int_0^t\!\!\int_0^t \gamma_{0,\epsilon}(s-r)\gamma_\epsilon\left(B_s - \widetilde{B}_r\right) ds\, dr$$

$$= \int_{\mathbb{R}^{d+1}} \left[\int_0^t e^{i\lambda s + i\xi\cdot B_s}\, ds\right]\left[\int_0^t e^{-i\lambda s - i\xi\cdot \widetilde{B}_s}\, ds\right] \exp\left\{-\frac{\epsilon}{2}\left(\lambda^2 + |\xi|^2\right)\right\} \mu_0(d\lambda)\mu(d\xi).$$

Hence, we have

$$\mathbb{E}_0\left|\int_0^t\int_0^t \gamma_{0,\epsilon}(s-r)\gamma_\epsilon(B_s-B_r)\,ds\,dr-\int_0^t\int_0^t \gamma_{0,\epsilon'}(s-r)\gamma_{\epsilon'}(B_s-B_r)\,ds\,dr\right|$$
$$\le \int_{\mathbb{R}^{d+1}}\left|\exp\left\{-\frac{\epsilon}{2}\left(\lambda^2+|\xi|^2\right)\right\}-\exp\left\{-\frac{\epsilon'}{2}\left(\lambda^2+|\xi|^2\right)\right\}\right|$$
$$\times\,\mathbb{E}_0\left|\int_0^t e^{i\lambda s+i\xi\cdot B_s}\,ds\right|^2\mu_0(d\lambda)\mu(d\xi)$$

and

$$\mathbb{E}_0\left|\int_0^t\int_0^t \gamma_{0,\epsilon}(s-r)\gamma_\epsilon\left(B_s-\widetilde{B}_r\right)ds\,dr-\int_0^t\int_0^t \gamma_{0,\epsilon'}(s-r)\gamma_{\epsilon'}\left(B_s-\widetilde{B}_r\right)ds\,dr\right|$$
$$\le \int_{\mathbb{R}^{d+1}}\left|\exp\left\{-\frac{\epsilon}{2}\left(\lambda^2+|\xi|^2\right)\right\}-\exp\left\{-\frac{\epsilon'}{2}\left(\lambda^2+|\xi|^2\right)\right\}\right|$$
$$\times\left\{\mathbb{E}_0\left|\int_0^t e^{i\lambda s+i\xi\cdot B_s}\,ds\right|\right\}^2\mu_0(d\lambda)\mu(d\xi).$$

By (2.4) and dominated convergence, the right-hand sides tend to 0 as $\epsilon,\epsilon'\to 0^+$. This is the justification for (2.6) and (2.7). Further, from above argument, we get

$$\int_0^t\int_0^t \frac{\gamma(B_s-B_r)}{|s-r|^{\alpha_0}}\,ds\,dr=\int_{\mathbb{R}^{d+1}}\left|\int_0^t e^{i\lambda s+i\xi\cdot B_s}\,ds\right|^2\mu_0(d\lambda)\mu(d\xi). \tag{2.8}$$

We now show that the random field $u(t,x)$ in (1.7) is well defined by proving that

$$\mathbb{E}|u(t,x)|<\infty\quad\forall(t,x)\in\mathbb{R}^+\times\mathbb{R}^d. \tag{2.9}$$

By assumption, $|u_0(\cdot)|\le C$ for some constant $C>0$. Thus, we obtain

$$\mathbb{E}|u(t,x)|\le C\mathbb{E}\otimes\mathbb{E}_x\exp\left\{\int_0^t \dot W(t-s,B_s)\,ds\right\}$$
$$=C\mathbb{E}\otimes\mathbb{E}_0\exp\left\{\int_0^t \dot W(t-s,B_s)\,ds\right\}.$$

From (2.1) and (2.6), we can see that conditioning on the Brownian motion, the random variable

$$\int_0^t \dot{W}(t-s, B_s)\, ds$$

is a mean-zero normal with the variance

$$\int_0^t \int_0^t \frac{\gamma(B_s - B_r)}{|s-r|^{\alpha_0}}\, ds\, dr.$$

Hence, we get

$$\mathbb{E} \exp\left\{ \int_0^t \dot{W}(t-s, B_s)\, ds \right\} = \exp\left\{ \frac{1}{2} \int_0^t \int_0^t \frac{\gamma(B_s - B_r)}{|s-r|^{\alpha_0}}\, ds\, dr \right\} \quad \text{a.s.}$$

Therefore, to establish the integrability requested for the definition in (1.7), all we need is the exponential integrability

$$\mathbb{E}_0 \exp\left\{ \theta \int_0^t \int_0^t \frac{\gamma(B_s - B_r)}{|s-r|^{\alpha_0}}\, ds\, dr \right\} < \infty \quad \forall \theta, t > 0. \tag{2.10}$$

To this end, we first establish the following lemma:

Lemma 2.1. *Under condition (1.9),*

$$\lim_{t \to 0^+} \frac{1}{t} \mathbb{E}_0 \int_{\mathbb{R}^{d+1}} \left| \int_0^t e^{i\lambda s + i\xi \cdot B_s}\, ds \right|^2 \mu_0(d\lambda)\mu(d\xi) = 0. \tag{2.11}$$

Proof. From (2.5) and variable substitution, we get

$$\begin{aligned}
&\mathbb{E}_0 \int_{\mathbb{R}^{d+1}} \left| \int_0^t e^{i\lambda s + i\xi \cdot B_s}\, ds \right|^2 \mu_0(d\lambda)\mu(d\xi) \\
&\quad = \int_{\mathbb{R}^d} \frac{\mu(d\xi)}{|\xi|^{4-2\alpha_0}} \int_0^{|\xi|^2 t} \int_0^{|\xi|^2 t} \frac{1}{|s-r|^{\alpha_0}} \exp\left\{ -\frac{1}{2}|s-r| \right\} ds\, dr \\
&\quad = \int_{\{|\xi| \le t^{-1/2}\}} \frac{\mu(d\xi)}{|\xi|^{4-2\alpha_0}} \int_0^{|\xi|^2 t} \int_0^{|\xi|^2 t} \frac{1}{|s-r|^{\alpha_0}} \exp\left\{ -\frac{1}{2}|s-r| \right\} ds\, dr \\
&\qquad + \int_{\{|\xi| > t^{-1/2}\}} \frac{\mu(d\xi)}{|\xi|^{4-2\alpha_0}} \int_0^{|\xi|^2 t} \int_0^{|\xi|^2 t} \frac{1}{|s-r|^{\alpha_0}} \exp\left\{ -\frac{1}{2}|s-r| \right\} ds\, dr.
\end{aligned}$$

For the first term, we obtain

$$\int_{\{|\xi|\le t^{-1/2}\}} \frac{\mu(d\xi)}{|\xi|^{4-2\alpha_0}} \int_0^{|\xi|^2 t}\int_0^{|\xi|^2 t} \frac{1}{|s-r|^{\alpha_0}} \exp\left\{-\frac{1}{2}|s-r|\right\} ds\, dr$$

$$\le \int_{\{|\xi|\le t^{-1/2}\}} \frac{\mu(d\xi)}{|\xi|^{4-2\alpha_0}} \int_0^{|\xi|^2 t}\int_0^{|\xi|^2 t} \frac{1}{|s-r|^{\alpha_0}} ds\, dr$$

$$= \frac{2t^{2-\alpha_0}}{(1-\alpha_0)(2-\alpha_0)}\, \mu\left(B\left(0, t^{-1/2}\right)\right).$$

According to the Kronecker lemma, (1.9) implies that

$$\lim_{t\to 0^+} t^{1-\alpha_0}\mu\left(B\left(0,t^{-1/2}\right)\right) = 0.$$

As for the second term in our decomposition, we use a simple bound

$$\int_{\{|\xi|> t^{-1/2}\}} \frac{\mu(d\xi)}{|\xi|^{4-2\alpha_0}} \int_0^{|\xi|^2 t}\int_0^{|\xi|^2 t} \frac{1}{|s-r|^{\alpha_0}} \exp\left\{-\frac{1}{2}|s-r|\right\} ds\, dr$$

$$\le 2 \int_{\{|\xi|> t^{-1/2}\}} \frac{\mu(d\xi)}{|\xi|^{4-2\alpha_0}} \int_0^{|\xi|^2 t}\int_r^{\infty} \frac{1}{(s-r)^{\alpha_0}} \exp\left\{-\frac{1}{2}(s-r)\right\} ds\, dr$$

$$= 2\left(\int_0^\infty \frac{1}{s^{\alpha_0}} \exp\left\{-\frac{1}{2}s\right\} ds\right) t \int_{\{|\xi|> t^{-1/2}\}} \frac{\mu(d\xi)}{|\xi|^{2(1-\alpha_0)}}$$

and the following obvious fact derived from (1.9):

$$\lim_{t\to 0^+} \int_{\{|\xi|> t^{-1/2}\}} \frac{\mu(d\xi)}{|\xi|^{2(1-\alpha_0)}} = 0.$$

The lemma is proved.

To establish (2.10), we use the subadditivity argument. A stochastic process $Z_t,\ t\ge 0$, is said to be subadditive if, for any $t_1,\ t_2>0$, there exists a random variable Z'_{t_2} such that $Z'_{t_2} \overset{d}{=} Z_{t_1},\ Z'_{t_2}$ is independent of $\{Z_s,\ s\le t_1\}$, and $Z_{t_1+t_2}\le Z_{t_1}+Z'_{t_2}$ a.s. An interested reader is referred to Section 1.3 in [1] for the discussion on this topic. Specifically, a nonnegative, nondecreasing, and sample-path continuous subadditive process Z_t with $Z_0 = 0$ has the following property ([1, p. 21], (1.3.7)):

$$\mathbb{P}\{Z_t \geq a+b\} \leq \mathbb{P}\{Z_t \geq a\}\mathbb{P}\{Z_t \geq b\} \quad \forall a, b, t > 0. \tag{2.12}$$

We now examine subadditivity for the process

$$Z_t \triangleq \left(\int_0^t \int_0^t \frac{\gamma(B_s - B_r)}{|s-r|^{\alpha_0}} \, ds \, dr\right)^{1/2}, \quad t \geq 0.$$

Indeed, by (2.8) and the triangle inequality, the subadditivity $Z_{t_1+t_2} \leq Z_{t_1} + Z'_{t_2}$ holds with

$$\begin{aligned} Z'_{t_2} &= \left(\int_{\mathbb{R}^{d+1}} \left| \int_{t_1}^{t_1+t_2} e^{i\lambda s + i\xi\cdot B_s} \, ds \right|^2 \mu_0(d\lambda)\mu(d\xi)\right)^{1/2} \\ &= \left(\int_{\mathbb{R}^{d+1}} \left| \int_0^{t_2} e^{i\lambda s + i\xi\cdot(B_{t_1+s} - B_{t_1})} \, ds \right|^2 \mu_0(d\lambda)\mu(d\xi)\right)^{1/2}. \end{aligned}$$

Clearly, $Z_0 = 0$ and Z_t is nondecreasing. By (2.8), Z_t is sample-path continuous [more precisely, relation (2.8) provides a sample-path continuous modification of Z_t]. Hence, Z_t satisfies (2.12).

For any $\theta > 0$, by using (2.12) repeatedly, we get

$$\mathbb{P}_0\left\{Z_t \geq m\theta^{-1}\sqrt{t}\right\} \leq \left(\mathbb{P}_0\left\{Z_t \geq \theta^{-1}\sqrt{t}\right\}\right)^m, \quad m = 1, 2, \ldots.$$

By Lemma 2.1, (2.8), and Chebyshev's inequality, there exists a (possibly) small $t_0 > 0$ such that

$$\sup_{t \leq t_0} \mathbb{P}_0\left\{Z_t \geq \theta^{-1}\sqrt{t}\right\} \leq e^{-2}.$$

Hence,

$$\begin{aligned} \mathbb{E}_0 \exp\left\{\theta Z_t/\sqrt{t}\right\} &= 1 + \int_0^\infty e^b \, \mathbb{P}_0\left\{Z_t \geq b\theta^{-1}\sqrt{t}\right\} db \\ &\leq 1 + e + \sum_{m=1}^\infty e^{m+1} \, \mathbb{P}_0\left\{Z_t \geq m\theta^{-1}\sqrt{t}\right\} \\ &\leq 1 + e + \sum_{m=0}^\infty e^{m+1} \, e^{-2m} = \frac{2e^2-1}{e-1} < \infty \end{aligned}$$

for all $0 < t \leq t_0$. Unfortunately, (2.13) is not even close to what is requested by (2.10). To improve this, we first note that the above estimation yields the following uniform bound:

$$\mathbb{E}_0 Z_t^n \le \frac{2e^2-1}{e-1}\,\theta^{-n} n! t^{n/2}, \quad 0<t<t_0, \quad n=1,2,\dots. \tag{2.13}$$

By subadditivity, for any $t_1,\ t_2>0$ and integer $n\ge 1$, we have

$$\mathbb{E}_0 Z_{t_1+t_2}^n \le \mathbb{E}\left[Z_{t_1}+Z'_{t_2}\right]^n = \sum_{l=0}^n \binom{n}{l}\left\{\mathbb{E}Z_{t_1}^l\right\}\left\{\mathbb{E}Z_{t_2}^{n-l}\right\}.$$

For any $t>0$ and integer $m\ge 1$, repeating the above inequality, we obtain

$$\mathbb{E}Z_t^n \le \sum_{l_1+\ldots+l_m=n} \frac{n!}{l_1!\ldots l_m!}\prod_{k=1}^m \mathbb{E}Z_{t/m}^{l_k} = \sum_{l_1+\ldots+l_m=n} \frac{n!}{l_1!\ldots l_m!}\prod_{k=1}^m \mathbb{E}Z_{t/m}^{l_k}.$$

Taking $m=n$ and $t\le t_0$, by virtue of (2.13), we get

$$\begin{aligned}\mathbb{E}Z_t^n &\le \sum_{l_1+\ldots+l_n=n} \frac{n!}{l_1!\ldots l_n!}\prod_{k=1}^n \frac{2e^2-1}{e-1}\,\theta^{-l_j} l_j!\left(\frac{t}{n}\right)^{l_j/2} \\ &= \left(\frac{\theta^{-1}\left(2e^2-1\right)}{e-1}\right)^n n! n^{-n/2} t^{n/2} \sum_{l_1+\ldots+l_n=n} 1.\end{aligned}$$

A simple combinatorial argument gives

$$\sum_{l_1+\ldots+l_n=n} 1 = \binom{2n-1}{n} \le 4^n.$$

Thus, we arrive at the following improved version of (2.13):

$$\mathbb{E}_0 Z_t^n \le \left(\frac{4\theta^{-1}\left(2e^2-1\right)}{e-1}\right)^n \sqrt{n!} t^{n/2}, \quad 0<t\le t_0, \quad n=1,2,\dots.$$

Replacing n with $2n$, we get

$$\mathbb{E}_0 Z_t^{2n} \le \left(\frac{4\theta^{-1}\left(2e^2-1\right)}{e-1}\right)^{2n} \sqrt{(2n)!}\, t^n \le \left(\frac{4\sqrt{2}\theta^{-1}\left(2e^2-1\right)}{e-1}\right)^{2n} n!\, t^n$$

for any $0<t\le t_0$ and $n=1,2,\dots$. Consequently, by the Taylor expansion, we have

$$\sup_{0<t\le t_0} \mathbb{E}_0 \exp\left\{\left(\frac{(e-1)\theta}{8\left(2e^2-1\right)}\right)^2 \frac{Z_t^2}{t}\right\} < \infty. \tag{2.14}$$

In addition, we can show that the process

$$S_t \triangleq \frac{Z_t^2}{t} = \frac{1}{t}\int_0^t\int_0^t \frac{\gamma(B_s - B_r)}{|s-r|^{\alpha_0}}\,ds\,dr, \quad t > 0,$$

is subadditive. Indeed, by (2.8) and Jensen's inequality, we can establish the subadditivity $S_{t_1+t_2} \le S_{t_1} + S'_{t_2}$, where

$$S'_{t_2} = \frac{1}{t_2}\int_{\mathbb{R}^{d+1}} \left|\int_{t_1}^{t_1+t_2} e^{i\lambda s + i\xi\cdot B_s}\,ds\right|^2 \mu_0(d\lambda)\mu(d\xi)$$

$$= \frac{1}{t_2}\int_{\mathbb{R}^{d+1}} \left|\int_0^{t_2} e^{i\lambda s + i\xi\cdot(B_{t_1+s} - B_{t_1})}\,ds\right|^2 \mu_0(d\lambda)\mu(d\xi)$$

satisfies all requirements for subadditivity.[1] Therefore,

$$\mathbb{E}_0 \exp\left\{\left(\frac{(e-1)\theta}{8(2e^2-1)}\right)^2 S_{t_1+t_2}\right\} \le \mathbb{E}_0 \exp\left\{\left(\frac{(e-1)\theta}{8\,(2e^2-1)}\right)^2 S_{t_1}\right\}\mathbb{E}_0 \exp\left\{\left(\frac{(e-1)\theta}{8\,(2e^2-1)}\right)^2 S_{t_2}\right\}$$

for any $0 < t_1,\ t_2 < t_0$. By (2.14), the right-hand side is finite. Therefore, (2.14) can be extended to all $t > 0$:

$$\mathbb{E} \exp\left\{\left(\frac{(e-1)\theta}{8\,(2e^2-1)}\right)^2 S_t\right\} < \infty \quad \forall t > 0.$$

In particular, we take $t = 1$ and note that $\theta > 0$ is arbitrary. Thus, we have reached the conclusion

$$\mathbb{E}_0 \exp\left\{\theta\int_0^1\int_0^1 \frac{\gamma(B_s - B_r)}{|s-r|^{\alpha_0}}\,ds\,dr\right\} < \infty \quad \forall \theta > 0.$$

This can be further extended to (2.10) since [by (2.8)], for any $t_1,\ t_2 > 0$,

$$\int_0^{t_1+t_2}\int_0^{t_1+t_2} \frac{\gamma(B_s - B_r)}{|s-r|^{\alpha_0}}\,ds\,dr \le 2\int_0^{t_1}\int_0^{t_1} \frac{\gamma(B_s - B_r)}{|s-r|^{\alpha_0}}\,ds\,dr + 2\int_0^{t_2}\int_0^{t_2} \frac{\gamma\left(\widetilde{B}_s - \widetilde{B}_r\right)}{|s-r|^{\alpha_0}}\,ds\,dr$$

with $\widetilde{B}(s) = B_{t_1+s} - B_{t_1}$ being a Brownian motion independent of $\{B_s,\ s \le t_1\}$.

More than (2.9), we now show that, for any integer $m \ge 1$, $u(t,x) \in \mathcal{L}^m(\Omega, \mathcal{A}, \mathbb{P})$ with representation (2.10). Indeed, conditioning on the Brownian motions, the random variable

$$\sum_{j=1}^m \int_0^t \dot{W}(t-s, B_j(s))\,ds$$

[1] We do not have (2.12) in this case due to the lack of monotonicity and because S_t is not defined for $t = 0$.

is a mean-zero normal random variable with the following variance:

$$\sum_{j,k=1}^{m} \mathbb{E}\left[\int_0^t \dot{W}(t-s, B_j(s))\, ds\right]\left[\int_0^t \dot{W}(t-s, B_k(s))\, ds\right].$$

On the other hand, for any $\epsilon > 0,$ we get

$$\mathbb{E}\left[\int_0^t \dot{W}_\epsilon(t-s, B_j(s))\, ds\right]\left[\int_0^t \dot{W}_\epsilon(t-s, B_k(s)) ds\right]$$
$$= \int_0^t \int_0^t \gamma_{0,2\epsilon}(s-r)\gamma_{2\epsilon}(B_j(s) - B_k(r))\, ds\, dr.$$

Therefore, by (2.1), (2.6), and (2.7),

$$\mathbb{E}\left[\int_0^t \dot{W}(t-s, B_j(s))\, ds\right]\left[\int_0^t \dot{W}\big(t-s, B_k(s)\big) ds\right]$$
$$= \int_0^t \int_0^t \frac{\gamma(B_j(s) - B_k(r))}{|s-r|^{\alpha_0}}\, ds\, dr. \tag{2.15}$$

Hence, we get

$$\mathbb{E}\exp\left\{\sum_{j=1}^{m} \int_0^t \dot{W}(t-s, B_j(s))\, ds\right\} = \exp\left\{\frac{1}{2}\sum_{j,k=1}^{m} \int_0^t \int_0^t \frac{\gamma(B_j(s) - B_k(r))}{|s-r|^{\alpha_0}}\, ds\, dr\right\}.$$

On the other hand, from (1.9), we obtain

$$u^m(t,x) = \mathbb{E}_x \exp\left\{\sum_{j=1}^{m} \int_0^t \dot{W}(t-s, B_j(s)) ds\right\} \prod_{j=1}^{m} u_0(B_j(t)).$$

By the Fubini theorem,

$$\mathbb{E}u^m(t,x) = \mathbb{E}_x\left(\mathbb{E}\exp\left\{\sum_{j=1}^{m} \int_0^t \dot{W}(t-s, B_j(s))\, ds\right\}\right) \prod_{j=1}^{m} u_0(B_j(t))$$
$$= \mathbb{E}_x \exp\left\{\frac{1}{2}\sum_{j,k=1}^{m} \int_0^t \int_0^t \frac{\gamma(B_j(s) - B_k(r))}{|s-r|^{\alpha_0}}\, ds\, dr\right\} \prod_{j=1}^{m} u_0(B_j(t)).$$

This is (1.10). The integrability issue arising from its right-hand side is resolved by the boundedness of $u_0(\cdot)$, the relation [from (2.15)] that

$$\int_0^t\int_0^t \frac{\gamma(B_j(s)-B_k(r))}{|s-r|^{\alpha_0}}\,ds\,dr \le \frac{1}{2}\int_0^t\int_0^t \frac{\gamma(B_j(s)-B_j(r))}{|s-r|^{\alpha_0}}\,ds\,dr + \frac{1}{2}\int_0^t\int_0^t \frac{\gamma(B_k(s)-B_k(r))}{|s-r|^{\alpha_0}}\,ds\,dr, \quad j\ne k,$$

and (2.10).

We finally come to the step of showing that the random field $u(t,x)$ in (1.7) is a weak solution of the parabolic Anderson equation (1.1). This was done by Hu, Nualart, and Song (see Theorem 4.3 in [7]) in the setting of fractional noise. In their proof, system (1.1) is approximated by its smoothed version

$$\begin{aligned} &\frac{\partial u}{\partial t}(t,x) = \frac{1}{2}\Delta u(t,x) + \dot W_\epsilon(t,x)u(t,x), \quad (t,x)\in\mathbb{R}^+\times\mathbb{R}^d, \\ &u(0,x) = u_0(x), \quad x\in\mathbb{R}^d, \end{aligned} \tag{2.16}$$

where $\epsilon > 0$ is small but fixed (at least for a while) and $\dot W_\epsilon(t,x)$ is given in (2.1).

To follow Hu–Nualart–Song's footstep, we set

$$u_\epsilon(t,x) = \mathbb{E}_x \exp\left\{\int_0^t \dot W_\epsilon(t-s,B_s)\,ds\right\} u_0(B_t), \quad (t,x)\in\mathbb{R}^+\times\mathbb{R}^d.$$

The smoothed Gaussian field $\dot W_\epsilon(t,x)$ has a continuous but unbounded path. As pointed out by the referee, the unboundedness of $\dot W_\epsilon(t,x)$ turns the legitimacy of $u_\epsilon(t,x)$ as a solution of (2.16) into a questionable issue. On the other hand, the argument used by Hu, Nualart, and Song (the proof of Theorem 4.3 in [7]) requires $u_\epsilon(t,x)$ to be a weak solution of (2.16). That is,

$$\begin{aligned} \int_{\mathbb{R}^d} u_\epsilon(t,x)\varphi(x)\,dx &= \int_{\mathbb{R}^d} u_0(x)\varphi(x)\,dx + \frac{1}{2}\int_0^t\int_{\mathbb{R}^d} u_\epsilon(s,x)\Delta\varphi(x)\,dx\,ds \\ &\quad + \int_0^t\int_{\mathbb{R}^d} u_\epsilon(s,x)\varphi(x)\dot W_\epsilon(s,y)\,dy\,ds \quad \text{a.s.} \end{aligned} \tag{2.17}$$

for every C^∞-function φ with compact support. By Lemma 3.1 in what follows (conditionally on $\dot W$), relation (2.17) holds if

$$\begin{aligned} &\int_D \mathbb{E}_x \exp\left\{\int_0^t \dot W_\epsilon(t-s,B_s)\,ds\right\} dx < \infty \quad \text{a.s.}, \\ &\int_0^t\int_D \mathbb{E}_x \exp\left\{\int_0^s \dot W_\epsilon(s-r,B_r)\,dr\right\} dx\,ds < \infty \quad \text{a.s.} \end{aligned} \tag{2.18}$$

for any bounded $D\subset\mathbb{R}^d$ and $t > 0$.

The first inequality in (2.18) follows from the fact that

$$\mathbb{E}\int_D \mathbb{E}_x \exp\left\{\int_0^t \dot{W}_\epsilon(t-s,B_s)\,ds\right\}dx$$

$$= \int_D \mathbb{E}_0 \exp\left\{\frac{1}{2}\int_0^t\int_0^t \gamma_{0,\epsilon}(s-r)\gamma_\epsilon(B_s-B_r)\,ds\,dr\right\}dx$$

$$= |D|\mathbb{E}_0 \exp\left\{\frac{1}{2}\int_0^t\int_0^t \gamma_{0,\epsilon}(s-r)\gamma_\epsilon(B_s-B_r)\,ds\,dr\right\} < \infty.$$

Further computations lead to

$$\mathbb{E}\int_0^t\int_D \mathbb{E}_x \exp\left\{\int_0^s \dot{W}_\epsilon(s-r,B_r)\,dr\right\}dx\,ds$$

$$= |D|\int_0^t \mathbb{E}_0 \exp\left\{\frac{1}{2}\int_0^s\int_0^s \gamma_{0,\epsilon}(r_1-r_2)\gamma_\epsilon(B_{r_1}-B_{r_2})\,dr_1\,dr_2\right\}ds$$

$$\le |D|t\mathbb{E}_0 \exp\left\{\frac{1}{2}\int_0^t\int_0^t \gamma_{0,\epsilon}(s-r)\gamma_\epsilon(B_s-B_r)\,ds\,dr\right\} < \infty,$$

where the second step follows from the time-monotonicity of the integrand. Thus, we have proved the second inequality in (2.18).

Based on the exponential integrability (2.10) and its consequence, on the moment integrability of $u(t,x)$ given in (1.7), on equation (2.17), and on the square integrability stated in Lemma 2.2 below, an argument by approximation via the Malliavin calculus given in the proof of Theorem 4.3 in [7] validates the Feynman–Kac representation (1.7) as a weak solution of (1.1).

Theorem 1.1 is proved.

The following lemma is a generalization of Lemma A.4 in [7] and allows us to follow the argument used in step 5 of the proof of Theorem 4.3 in [7] {see (4.15) and (4.16) in [7] for its relevance}.

Lemma 2.2. *Under assumption (1.9),*

$$\mathbb{E}_0\left[\int_0^t \frac{\gamma(B_s)}{s^{\alpha_0}}\,ds\right]^2 < \infty, \quad t>0.$$

Proof. By the monotonicity in time, all we need is to show that

$$\int_0^\infty e^{-t}\mathbb{E}_0\left[\int_0^t \frac{\gamma(B_s)}{s^{\alpha_0}}\,ds\right]^2 dt < \infty. \tag{2.19}$$

We write

$$\left[\int_0^t \frac{\gamma(B_s)}{s^{\alpha_0}}\, ds\right]^2 = 2\int_0^t dr \frac{\gamma(B_r)}{r^{\alpha_0}} \int_r^t \frac{\gamma(B_s)}{s^{\alpha_0}}\, ds \le 2\int_0^t dr \frac{\gamma(B_r)}{r^{\alpha_0}} \int_r^t \frac{\gamma(B_s)}{(s-r)^{\alpha_0}}\, ds.$$

By the Markov property,

$$\mathbb{E}_0 \left[\int_0^t \frac{\gamma(B_s)}{s^{\alpha_0}} ds\right]^2 \le 2\mathbb{E}_0 \int_0^t dr\, \frac{\gamma(B_r)}{r^{\alpha_0}} \int_r^t \frac{\mathbb{E}_{B_r}\gamma(B_{s-r})}{(s-r)^{\alpha_0}}\, ds.$$

By (1.3), for any $x \in \mathbb{R}^d$,

$$\begin{aligned}\mathbb{E}_x\gamma(B_{s-r}) = \mathbb{E}_0\gamma(x + B_{s-r}) &= \mathbb{E}_0 \int_{\mathbb{R}^d} e^{i\xi\cdot(x+B_{s-r})}\mu(d\xi)\\ &= \int_{\mathbb{R}^d} e^{i\xi\cdot x} \exp\left\{-\frac{1}{2}\,|\xi|^2(s-r)\right\}\mu(d\xi)\\ &\le \int_{\mathbb{R}^d} \exp\left\{-\frac{1}{2}|\xi|^2(s-r)\right\}\mu(d\xi).\end{aligned}$$

Hence,

$$\mathbb{E}_0 \left[\int_0^t \frac{\gamma(B_s)}{s^{\alpha_0}}\, ds\right]^2 \le 2\int_0^t dr \frac{\mathbb{E}_0\gamma(B_r)}{r^{\alpha_0}} \int_r^t \frac{ds}{(s-r)^{\alpha_0}} \int_{\mathbb{R}^d} \exp\left\{-\frac{1}{2}\,|\xi|^2(s-r)\right\}\mu(d\xi).$$

Taking the Laplace transform, we get

$$\int_0^\infty e^{-t}\mathbb{E}_0 \left[\int_0^t \frac{\gamma(B_s)}{s^{\alpha_0}}\, ds\right]^2 dt \le 2\left(\int_0^\infty e^{-t}\frac{\mathbb{E}_0\gamma(B_t)}{t^{\alpha_0}}\, dt\right)\left(\int_0^\infty \frac{dt}{t^{\alpha_0}}\, e^{-t}\int_{\mathbb{R}^d} \exp\left\{-\frac{1}{2}\,|\xi|^2 t\right\}\mu(d\xi)\right).$$

By using (1.3) once again, we obtain

$$\mathbb{E}_0\gamma(B_t) = \int_{\mathbb{R}^d} \exp\left\{-\frac{1}{2}|\xi|^2 t\right\}\mu(d\xi).$$

Hence, we get

$$\int_0^\infty e^{-t}\mathbb{E}_0 \left[\int_0^t \frac{\gamma(B_s)}{s^{\alpha_0}}\, ds\right]^2 dt \le 2\left(\int_0^\infty \frac{dt}{t^{\alpha_0}}\, e^{-t}\int_{\mathbb{R}^d} \exp\left\{-\frac{1}{2}\,|\xi|^2 t\right\}\mu(d\xi)\right)^2.$$

Finally, (2.19) is obtained from the following computation:

$$\int_0^\infty \frac{dt}{t^{\alpha_0}}\, e^{-t} \int_{\mathbb{R}^d} \exp\left\{-\frac{1}{2}\,|\xi|^2 t\right\} \mu(d\xi) = \int_{\mathbb{R}^d} \mu(d\xi) \int_0^\infty \frac{1}{t^{\alpha_0}} \exp\left\{-\left(1+\frac{1}{2}\,|\xi|^2\right) t\right\} dt$$

$$= \left(\int_0^\infty t^{-\alpha_0} e^{-t}\, dt\right) \int_{\mathbb{R}^d} \left(1+\frac{1}{2}|\xi|^2\right)^{-(1-\alpha_0)} \mu(d\xi) < \infty.$$

The lemma is proved.

3. Appendix

Let $c(t,x),\ (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d$, be a continuous function. Consider the deterministic heat equation

$$\begin{aligned} &\frac{\partial u}{\partial t}\,(t,x) = \frac{1}{2}\,\Delta u(t,x) + c(t,x)u(t,x), \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d, \\ &u(0,x) = u_0(x), \quad x \in \mathbb{R}^d. \end{aligned} \tag{3.1}$$

As earlier, $u_0(x)$ is bounded and measurable. We now write the corresponding Feynman–Kac representation

$$u(t,x) = \mathbb{E}_x \exp\left\{\int_0^t c(t-s, B_s)ds\right\} u_0(B_t), \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d, \tag{3.2}$$

whenever the right-hand side of this expression makes sense. It is not clear whether or not $u(t,x)$ in (3.2) is a path-wise solution of (3.1) if $c(t,x)$ is unbounded on $\mathbb{R}^+ \times \mathbb{R}^d$. In the following lemma, we claim that it is at least a weak solution of (3.1).

Lemma 3.1. *Assume that the Feynman–Kac representation in (3.2) is well defined on $\mathbb{R}^+ \times \mathbb{R}^d$ and*

$$\begin{aligned} &\int_D \mathbb{E}_x \exp\left\{\int_0^t c(t-s, B_s)\, ds\right\} dx < \infty, \\ &\int_0^t \int_D \mathbb{E}_x \exp\left\{\int_0^s c(s-r, B_r) dr\right\} dx\, ds < \infty \end{aligned} \tag{3.3}$$

for any bounded $D \subset \mathbb{R}^d$ and $t > 0$. Then the Feynman–Kac representation $u(t,x)$ in (3.2) is a weak solution of (3.1) in the sense that

$$\int_{\mathbb{R}^d} u(t,x)\varphi(x)\, dx = \int_{\mathbb{R}^d} u_0(x)\varphi(x)\, dx$$

$$+\frac{1}{2}\int_0^t\int_{\mathbb{R}^d} u(s,x)\Delta\varphi(x)\,dx\,ds\ +\int_0^t\int_{\mathbb{R}^d} u(s,x)\varphi(x)c(s,y)\,dy\,ds \tag{3.4}$$

for any C^∞-function φ with compact support.

Proof. For any $R>0$, we write $D_R=\{x\in\mathbb{R}^d,\ |x|<R\}$. Consider the heat equation with zero boundary condition:

$$\begin{aligned}\frac{\partial u}{\partial t}(t,x)&=\frac{1}{2}\Delta u(t,x)+c(t,x)u(t,x),\quad (t,x)\in\mathbb{R}^+\times D_R,\\ u(0,x)&=u_0^R(x),\quad x\in D_R,\\ u(t,\partial D_R)&=0,\quad t\in\mathbb{R}^+,\end{aligned}\tag{3.5}$$

where $u_0^R(x)$ is a bounded function supported on D_R and such that $\left|u_0^R(x)\right|\le|u_0|(x)$ and $u_0^R(x)\to u_0(x)$ for any $x\in\mathbb{R}^d$ as $R\to\infty$.

We set the Brownian exit time as follows:

$$\tau_R=\inf\{t>0,\ B_t\notin D_R\}.$$

According to Theorem 2.3 in [5, p. 133], the Feynman–Kac representation

$$u^R(t,x)=\mathbb{E}_x\exp\left\{\int_0^t c(t-s,B_s)\,ds\right\}u_0^R(B_t)1_{\{\tau_R\ge t\}},\quad (t,x)\in\mathbb{R}^+\times D_R,$$

is a path-wise solution of (3.5). Given a C^∞-function φ with compact support D, we take sufficiently large R so that $D\subset D_R$. Thus, we get

$$\begin{aligned}\int_{\mathbb{R}^d} u^R(t,x)\varphi(x)\,dx&=\int_{\mathbb{R}^d} u_0^R(x)\varphi(x)\,dx\\ &\quad+\frac{1}{2}\int_0^t\int_{\mathbb{R}^d} u^R(s,x)\Delta\varphi(x)\,dx\,ds+\int_0^t\int_{\mathbb{R}^d} u^R(s,x)\varphi(x)c(s,y)\,dy\,ds.\end{aligned}\tag{3.6}$$

Note that $u^R(t,x)\longrightarrow u(t,x)$ pointwise as $R\to\infty$. In addition,

$$\left|u^R(t,x)\right|\le\|u_0\|_\infty\mathbb{E}_x\exp\left\{\int_0^t c(t-s,B_s)\,ds\right\},\quad (t,x)\in\mathbb{R}^+\times\mathbb{R}^d.$$

Let $R\to\infty$ in (3.6). In view of assumption (3.3), if we properly apply the dominated convergence theorem to each term in (3.6), then we get (3.4).

The lemma is proved.

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