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A functional LIL for symmetric stable processes

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Summary: A functional law of the iterated logarithm is obtained for symmetric stable processes with stationary independent increments. This extends the classical liminf results of Chung(1948) for Brownian motion, and of Taylor(1967) for the remaining such processes. It also extends an earlier result of Wichura(1973) on Brownian motion. Proofs depend on small ball probability estimates, and yield the small ball probabilities of the weighted sup-norm for these processes.

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1. Introduction. Let $\{X(t) : t \geq 0\}$ be a symmetric stable process of index $\alpha \in (0, 2]$ with stationary independent increments. Furthermore, assume the process is taken to have sample paths in $D[0, \infty)$, and $X(0) = 0$ with probability one. For $t \geq 0$, $n \geq 1$, define

$$M(t) = \sup_{0 \leq s \leq t} |X(s)|,$$

and

$$\eta_n(t) = M(nt)/(c_\alpha n/LLn)^{1/\alpha},$$

where the constant $0 < c_\alpha < \infty$ is given by

$$c_\alpha = - \lim_{\varepsilon \to 0+} \varepsilon^\alpha \log P\big(\sup_{0 \leq s \leq 1} |X(s)| \leq \varepsilon\big) \tag{1.1}$$

and $LLn = \max(1, \log(\log n))$. The existence of the limit defining c_α in (1.1) can be found in Mogul'skii (1974). The earlier paper Taylor (1967) obtained strictly positive, finite bounds for the liminf and limsup of the right hand side of (1.1), and there is also a variational representation of c_α to be found in Donsker and Varadhan (1977). When $\alpha = 2$ the process is Brownian motion, and it is well known that $c_2 = \pi^2/8$ provided $\{X(t) : t \geq 0\}$ is normalized to have $E(X^2(1)) = 1$. If $\alpha \in (0, 2)$, the constant c_α is also clearly X-dependent, but due to the scaling property of $\{X(t) : t \geq 0\}$ it only affects c_α in multiplicative fashion.

If $\alpha = 2$, then it was shown by Chung (1948) that

$$\varliminf_{n} \eta_n(1) = 1 \quad \text{a.s.}, \tag{1.2}$$

and for general $\alpha \in (0, 2]$, Taylor (1967) showed that

$$\varliminf_{n} M(n)/(n/LLn)^{1/\alpha} = \beta_\alpha \quad \text{a.s.} \tag{1.3}$$

where $0 < \beta_\alpha < \infty$. Of course, once one knows (1.1) holds with $c_\alpha \in (0, \infty)$ then $\beta_\alpha = c_\alpha^{1/\alpha}$. This follows from (1.6) below. The equality in (1.3) is also derived in Donsker and Varadhan (1977) as an application of their functional law, and β_α is defined in terms of the rate function for large deviations of the Markov process $\{X(t) : t \geq 0\}$. Of course, if $\alpha = 2$, and in the definition of $\eta_n(t)$, $M(\cdot)$ is replaced by $X(\cdot)$, then the rates of convergence in the functional LIL of Strassen initiated by Csáki (1980) and de Acosta (1983) generalize (1.2) considerably, and involve the entire function $\eta_n(t)$, $0 \leq t \leq 1$ (see Kuelbs, Li and Talagrand (1994) for further details and references). Another possibly extension of (1.2), or (1.3), is to examine the functional cluster set $C(\{\eta_n(\cdot)\})$ in a weak topology. This was done when $\alpha = 2$ by Wichura (1973) in an unpublished paper. The proof in Wichura (1973) obtains a related cluster set for the first passage time process via properties of Bessel diffusions. Then the cluster set for the maximal process $\{M(t) : t \geq 0\}$ is obtained from the fact that the first passage time process is the inverse of $\{M(t) : t \geq 0\}$ and various continuity considerations.

Our main result studies the cluster set $C(\{\eta_n\})$ for all $\alpha \in (0, 2]$, and recovers the related fact in Wichura (1973) when $\alpha = 2$. Our proof is quite different, and we study the maximal process $\{M(t) : t \geq 0\}$ directly. Of course, our results then apply to the first passage time process by reversing the steps in Wichura (1973). See the remark following (1.6).

To describe these results, denote by $\mathcal{M}$ the space of functions $f : [0, \infty) \to [0, \infty]$ such that $f(0) = 0$, f is right continuous on $(0, \infty)$, non-decreasing, and $\lim_{t\to+\infty} f(t) = \infty$. Let

$$K_\alpha = \{f \in \mathcal{M} : \int_0^\infty f^{-\alpha}(t)dt \leq 1\},$$

and endow $\mathcal{M}$ with the topology of weak convergence, i.e. pointwise convergence at all continuity points of the limit function.

The topology of weak convergence on $\mathcal{M}$ is metrizable and separable. This can be seen as follows. Let $\mathcal{N}$ denote the functions $g : (-\infty, \infty) \to [0, 1]$ with $g(t) = 0$ for $t \leq 0$, right continuous on $(0, \infty)$, non-decreasing, and such that $\lim_{t\to\infty} g(t) = 1$. Let $\lambda(s) = s/(1+s)$ for $s \in [0, \infty]$, with ∞/∞ understood to be one, and for $f \in \mathcal{M}$ define

$$\Psi(f)(t) = f^*(t) = \begin{cases} 0 & \text{for } t \leq 0 \\ \lambda(f(t)) & \text{for } t > 0. \end{cases}$$

Then the map $\Psi : f \to f^*$ is one-to-one from $\mathcal{M}$ onto $\mathcal{N}$, and we define a metric d on $\mathcal{M}$ by setting

$$d(f,g) = L(f^*, g^*),$$

where L is Levy's metric on $\mathcal{N}$, i.e.

$$L(f^*, g^*) = \inf\{\varepsilon > 0 : f^*(t-\varepsilon) - \varepsilon \le g^*(t) \le f^*(t+\varepsilon) + \varepsilon \text{ for } -\infty < t < \infty\}.$$

Now $\lim_n d(f_n, f) = 0$ for $f_n, f \in \mathcal{M}$ iff $\lim_n L(f_n^*, f^*) = 0$, and this holds iff

$$\lim_n f_n^*(t) = f^*(t) \tag{1.4}$$

for all t in the continuity set of f^*. Taking the usual topology on $[0,\infty]$, and the definition of the map $\Psi : f \to f^*$, we see that (1.4) holds for all $t \in C(f^*)$, the continuity set of f^*, if and only if $\lim_n f_n(t) = f(t)$ for all $t \in C(f)$, the continuity set of f. Since Levy's metric makes $\mathcal{N}$ a complete separable metric space, we have $(\mathcal{M}, d)$ a complete separable metric space, with d-convergence equivalent to weak convergence on $\mathcal{M}$.

If $\{f_n\}$ is a sequence of points in $\mathcal{M}$, then $C(\{f_n\})$ denotes the cluster set of $\{f_n\}$, i.e. all possible subsequential limits of $\{f_n\}$ in the weak topology. If $A \subseteq \mathcal{M}$, we write $\{f_n\} \twoheadrightarrow A$ if $\{f_n\}$ is conditionally compact and $C(\{f_n\}) = A$ in the weak topology. Then the following hold.

Theorem 1.1. *Let $\{X(t) : t \ge 0\}$ be a stationary independent increment symmetric stable process of index $\alpha \in (0,2]$ with sample paths in $D[0,\infty)$ and such that $X(0) = 0$. Then*

$$P(\{\eta_n\} \twoheadrightarrow K_\alpha) = 1. \tag{1.5}$$

Corollary 1.1. Let $\{\eta_n\}$ be as in (1.5). Then

$$P(\underline{\lim}_n \eta_n(1) = 1) = 1. \tag{1.6}$$

Remark. Let $D_0^+[0,\infty)$ denote the non-decreasing functions which vanish at zero, are right continuous on $(0,\infty)$, and have left limits on $(0,\infty)$. If $f \in D_0^+[0,\infty)$, we define

$$\mathcal{F}f(y) = \begin{cases} 0 & \text{if } y = 0 \\ \inf\{t : f(t) > y\} & \text{if } y > 0 \end{cases}$$

where $\inf \phi = \infty$. Then $\mathcal{F}$ maps $D_0^+[0,\infty)$ into $D_0^+[0,\infty)$ and $\mathcal{F}f$ is a right continuous inverse of f in the sense that $\mathcal{F}(\mathcal{F}f) = f$. Furthermore, looking at the Levy metric, and considering compact subintervals of $[0,\infty)$, we see $\{f_n\}$ converging weakly to f in $\mathcal{M}$ implies $\{\mathcal{F}f_n\}$ converges weakly to $\mathcal{F}f$ in $D_0^+[0,\infty)$. Of course, the weak topology on $D_0^+[0,\infty)$ can be described as for $\mathcal{M}$ with $\mathcal{N}$ expanded to include functions g with $\lim_{t\to\infty} g(t) \le 1$. We also have

$$\mathcal{F}(K_\alpha) = \{\mathcal{F}f : f \in K_\alpha\} = \{g \in D_0^+[0,\infty) : \int_0^\infty u^{-\alpha} dg(u) \le 1\},$$

where $dg(u)$ denotes integration with respect to the measure on $[0,\infty)$ given by the non-decreasing function g. Hence (1.5) implies

$$P(\{\mathcal{F}(\eta_n)\} \twoheadrightarrow \mathcal{F}(K_\alpha)) = 1.$$

Now

$$\begin{aligned}(\mathcal{F}\eta_n)(s) &= \inf\{t : \eta_n(t) > s\} \\ &= \inf\{t : M(nt) > s(c_\alpha n/LLn)^{1/\alpha}\} \\ &= \frac{1}{n}\mathcal{F}M(s(c_\alpha n/LLn)^{1/\alpha})\end{aligned}$$

Letting $m = m_n = (c_\alpha n/LLn)^{1/\alpha}$ we get $n \sim c_\alpha^{-1} m^\alpha LLm$, and hence as $n \to \infty$, $\mathcal{F}\eta_n(\cdot) \sim \mathcal{F}M(m(\cdot))/(c_\alpha^{-1}m^\alpha LLm)$. Since $\mathcal{F}M$ is increasing with the values of $\{m_n : n \ge 1\}$ within distance one of any large integer, we may replace $m = m_n$ by the greatest integer less than or equal to m_n when we investigate the asymptotic behavior of $\{\mathcal{F}M(m(\cdot))/(c_\alpha^{-1}m^\alpha LLm)\}$. Thus the following corollary holds.

Corollary 1.2. *Let $N(0) = 0$ and $N(s) = \inf\{t : M(t) > s\}$ for $s > 0$ denote the first passage time process for $\{X(t) : t \ge 0\}$. Then $\{N(s) : s \ge 0\} = \{\mathcal{F}M(s) : s \ge 0\}$, and* *with probability one*

$$\{N(m(\cdot))/(c_\alpha^{-1}m^\alpha LLm)\}_{m=1}^{\infty} \twoheadrightarrow \{g \in D_0^+[0,\infty) : \int_0^\infty u^{-\alpha} dg(u) \le 1\}$$

in the weak topology.

There are various applications of the functional LIL given in Theorem 1.1, very much in the same spirit as for Strassen's LIL. For example, we know from Corollary 1.1 that with probability one $\underline{\lim}_n \eta_n(1) = 1$, but how fast does $\eta_n(\cdot)$ get away from the zero function, say over the interval $[0,1]$, or how many samples $\eta_n(1)$, $n \le t$ fall in the interval $[0,c]$, $c \ge 1$? One measure of these quantities is the weighted occupation measure

$$\Psi_c(t) = t^{-1} \int_0^t I_{[0,c]}(\eta_s(1)\theta(s/t))ds, \tag{1.7}$$

where $c \ge 1$, $\theta(\cdot)$ maps $(0,1]$ into $(0,\infty)$ with $\theta(1) = 1$, $\eta_s(u) = M(su)/(c_\alpha s/LLs)^{1/\alpha}$ for $s > 0$, $u \ge 0$, and $\eta_0(u) = 0$ for all $u \ge 0$. As the continuous parameter s converges to infinity, the family of functions $\{\eta_s(\cdot)\}$ satisfies (3.1), (3.2), and (3.3). The analogue of (3.3) follows immediately from the case $n \to \infty$ through the integers, as there can only be more cluster points when s converges to infinity continuously. Furthermore, both (3.1) and (3.2) follow in the continuous parameter case from the proofs in Propositions 3.2 and 3.1, respectively.

Beyond the properties already mentioned for θ, we will also assume θ satisfies:

$$s \mapsto s^{1/\alpha}/\theta(s) \text{ is increasing on } (0,1], \tag{1.8}$$

$$\int_0^1 \theta^\alpha(s)/s \; ds = \infty, \tag{1.9}$$

and the function

$$h(s) = \theta^\alpha(s) + \int_s^1 \theta^\alpha(u)/u \; du \tag{1.10}$$

maps $(0,1]$ onto $[1,\infty)$ in continuous and one-to-one fashion. For example, suppose (1.8) and (1.9) hold, and θ is continuous and decreasing on $(0,1]$ with $\theta(1) = 1$. Then $h(s)$

is strictly decreasing and continuous on $(0,1]$ with range $[1,\infty)$. The functions $\theta(s)=1$ and $\theta(s)=(\log(e/s))^{1/\alpha}$ are such examples, but $\theta(s)=1/(\log(e/s))^{1/\alpha}$ also satisfies the conditions formulated in (1.8), (1.9), and (1.10). With this notation we now can state the following proposition. Its proof is in Section 4.

Theorem 1.2. *Let $\theta:(0,1]\to(0,\infty)$ satisfy $\theta(1)=1$, (1.8), (1.9), and that $h(s)$ as defined in (1.10) is continuous and one-to-one on $(0,1]$ into $[1,\infty)$. Let $\{X(t):\geq 0\}$ be a symmetric stable process with homogeneous independent increments, sample paths in $D[0,\infty)$, $X(0)=0$ w.p. 1, and parameter $\alpha\in(0,2]$. Then, with probability one,*

$$\overline{\lim_{t\to\infty}}\ \Psi_c(t)=1-s_c, \tag{1.11}$$

where $s=s_c$ is the (unique) solution to $h(s)=c^\alpha$, $c\geq 1$.

Examples. If $\theta(s)=1$ on $[0,1]$, then $h(s)=1-\log s$ and $h(s)=c^\alpha$ has solution $s_c=e^{-(c^\alpha-1)}$ for $c\geq 1$. Thus (1.11) implies that with probability one

$$\overline{\lim_{t\to\infty}}\ t^{-1}\int_0^t I_{[0,c]}(\eta_s(1))ds=1-e^{-(c^\alpha-1)}$$

for each $c\geq 1$.

If $\theta(s)=(\log(e/s))^{1/\alpha}$ on $(0,1]$, then for $0<s\leq 1$, $h(s)=1-2\log s+(\log s)^2/2$. Solving $h(s)=c^\alpha$, $0<s\leq 1$ and $c\geq 1$, we get $s_c=\exp(2-2\sqrt{1+(c^\alpha-1)/2})$, and hence with probability one

$$\overline{\lim_{t\to\infty}}\ t^{-1}\int_0^t I_{[0,c]}(\eta_s(1)(\log(et/s))^{1/\alpha})ds=1-\exp(2-2\sqrt{1+(c^\alpha-1)/2})$$

for $c\geq 1$.

If $\theta(s)=(\log(e/s))^{-1/\alpha}$ on $(0,1]$, then for $0<s\leq 1$, $h(s)=(1-\log s)^{-1}+\log(1-\log s)$, and $h(s)$ is continuous and strictly decreasing on $(0,1]$ with $h(1)=1$. Thus $h(s)$ has a unique continuous solution s_c and Theorem 1.2 applies. However, an explicit formula for the value of s_c is not immediate in this case.

Another gauge of the rate of escape is the quantity $t^{-1}\int_0^t I_{[0,t]}(\eta_t(s/t))ds$, which is similar to $\Psi_c(t)$ (as $t\to\infty$), provided $\theta(s)=s^{1/\alpha}$. With this choice of θ, (1.8) applies, but (1.9) fails and $h(s) = 1$ for all $s \in (0,1]$. Thus Theorem 1.2 is not applicable, but the techniques for its proof imply

$$\overline{\lim_{t\to\infty}}\, t^{-1} \int_0^t I_{[0,c]}(\eta_t(s/t))ds = \begin{cases} 1 & \text{if } c \geq 1 \\ c^\alpha & \text{if } 0 \leq c < 1. \end{cases} \tag{1.12}$$

The rate of escape with respect to the L^p norms is given by the following theorem, whose proof is in Section 4.

Theorem 1.3. *Let $\{X(t) : t \geq 0\}$ be as above and suppose $0 < p < \infty$. Then, with probability one*

$$\lim_{t\to\infty} \int_0^1 |\eta_t(u)|^p du = \inf_{f\in K_\alpha} \int_0^1 |f(u)|^p du = 1. \tag{1.13}$$

Remark. Since $\eta_t(\cdot)$ is increasing, the analogue of (1.13) for the sup-norm on $[0,1]$ follows immediately from (1.6).

2. Probability Estimates. The proof of Theorem 1.1 depends on the probability estimates obtained in this section. The first result is an Anderson type inequality for symmetric α-stable measures. It is a known fact, but we give a proof for completeness.

Lemma 2.1. Let $\{X(t) : t \in T\}$ be a symmetric stable process of index $\alpha \in (0,2]$ such that T is a countable set and $P(\sup_{t\in T} |X(t)| < \infty) = 1$. Then for all $\lambda > 0$, and all real numbers x,

$$P\big(\sup_{t\in T} |X(t) + x| \leq \lambda\big) \leq P\big(\sup_{t\in T} |X(t)| \leq \lambda\big). \tag{2.1}$$

Proof. The proof of (2.1) follows from Anderson's inequality if $\alpha = 2$. If $\alpha \in (0,2)$, then by Lemma 1.6 of Marcus and Pisier (1984), we can find probability spaces $(\Omega, \mathcal{F}, P)$ and $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{P})$ and a real-valued stochastic process $\{Y(t) : t \in T\}$ on $(\Omega \times \widetilde{\Omega}, \mathcal{F} \times \widetilde{\mathcal{F}}, P \times \widetilde{P})$ such that the processes $\{Y(t) : t \in T\}$ and $\{X(t) : t \in T\}$ have the same distribution and for each fixed $w \in \Omega$, the stochastic process $\{Y(t,\omega,\cdot) : t \in T\}$ is a symmetric Gaussian process. Hence for $\lambda > 0$ and all x real, the $\alpha = 2$ case implies

$$\widetilde{P}\big(\sup_{t\in T} |Y(t,\omega,\cdot) + x| < \lambda\big) \le \widetilde{P}\big(\sup |Y(t,\omega,\cdot)| < \lambda\big). \tag{2.2}$$

Since (2.2) holds for all $\omega \in \Omega$, Fubini's theorem and (2.2) combine to give (2.1).

Proposition 2.2. *Let $\{X(t) : t \ge 0\}$ be a symmetric stable process with homogeneous independent increments, sample paths in $D[0,\infty)$, and parameter $\alpha \in (0,2]$. Fix sequences $\{t_i\}_{i=0}^m$, $\{a_i\}_{i=0}^m$, and $\{b_i\}_{i=0}^m$ such that $0 = t_0 < t_1 < \cdots < t_m$ and $a_1 < b_1 \le a_2 < b_2 \le \cdots \le a_m < b_m$. Then*

$$\overline{\lim_{\varepsilon\to 0^+}} \varepsilon^\alpha \log P(a_i\varepsilon \le M(t_i) \le b_i\varepsilon, 1 \le i \le m) \le -c_\alpha \sum_{i=1}^m (t_i - t_{i-1})/b_i^\alpha.$$

Proof. Let $A_i = \big\{\sup_{t_{i-1}\le s<t_i} |X(s)| \le b_i\varepsilon\big\}$ for $i = 1,\dots,m$. Then it is easy to see

$$P(a_i\varepsilon \le M(t_i) \le b_i\varepsilon,\ 1 \le i \le m) \le P\big(\bigcap_{i=1}^m A_i\big). \tag{2.3}$$

Furthermore, we have

$$\begin{aligned}
&P\big(\bigcap_{i=1}^m A_i\big)\\
&= \int_{\mathbb{R}} P\big(\bigcap_{i=1}^{m-1} A_i, \sup_{t_{m-1}\le s<t_m} |X(s) - X(t_{m-1}) + x| \le b_m\varepsilon \mid X(t_{m-1}) = x\big) dP_{X(t_{m-1})}(x)\\
&= \int_{\mathbb{R}} P\big(\sup_{t_{m-1}\le s<t_m} |X(s) - X(t_{m-1}) + x| \le b_m\varepsilon\big) P\big(\bigcap_{i=1}^{m-1} A_i \mid X(t_{m-1}) = x\big) dP_{X(t_{m-1})}(x),
\end{aligned}$$

since $\sup_{t_{m-1}\le s<t_m} |X(s) - X(t_{m-1}) + x|$ is independent of $X(t_{m-1})$ and $\bigcap_{i=1}^{m-1} A_i$ by the independent increments property of $X(t)$.

Now Lemma 2.1, and that the sample paths are in $D[0,\infty)$, together imply

$$\begin{aligned}
&P\big(\sup_{t_{m-1}\le s<t_m} |X(s) - X(t_{m-1}) + x| \le b_m\varepsilon\big)\\
&\le P\big(\sup_{t_{m-1}\le s<t_m} |X(s) - X(t_{m-1})| \le b_m\varepsilon\big)\\
&= P\big(\sup_{0\le s\le 1} |X(s)| \le b_m\varepsilon/(t_m - t_{m-1})^{1/\alpha}\big),
\end{aligned}$$

where the equality follows from the scaling property of $\{X(t) : t \geq 0\}$ and the homogeneity of the increments. Thus

$$P\Big(\bigcap_{i=1}^{m} A_i\Big) \leq P\Big(\bigcap_{i=1}^{m-1} A_i\Big) \cdot P\Big(\sup_{0\leq s\leq 1} |X(s)| \leq b_m\varepsilon/(t_m - t_{m-1})^{1/\alpha}\Big),$$

and iterating the above estimate, along with (2.3), implies

$$\begin{aligned} &\overline{\lim_{\varepsilon\to 0^+}} \varepsilon^\alpha \log P(a_i\varepsilon \leq M(t_i) \leq b_i\varepsilon,\ 1 \leq i \leq m) \\ &\leq \sum_{i=1}^{m} \overline{\lim_{\varepsilon\to 0^+}} \varepsilon^\alpha \log P\Big(\sup_{0\leq s\leq 1} |X(s)| \leq \frac{b_i\varepsilon}{(t_i - t_{i-1})^{1/\alpha}}\Big) \\ &= -c_\alpha \sum_{i=1}^{m} (t_i - t_{i-1})/b_i^\alpha, \end{aligned}$$

where the equality follows from (1.1).

Thus Proposition 2.2 is proven. To obtain a reverse estimate, we need the following lemma.

Lemma 2.3. *Given $\delta > 0$,*

(2.4) $$\lim_{\varepsilon\to 0^+} \varepsilon^\alpha \log P(M(1) \leq \varepsilon,\ |X(1)| \leq \varepsilon\delta) = -c_\alpha.$$

Remark. From (2.4) one can see that for given positive numbers $a < b$ and $\delta > 0$,

$$\lim_{\varepsilon\to 0^+} \varepsilon^\alpha \log P(a\varepsilon \leq M(1) \leq b\varepsilon,\ |X(1)| \leq \varepsilon\delta) = -c_\alpha/b^\alpha.$$

Proof of lemma 2.3. If $\delta \geq 1$, then (2.4) follows immediately from (1.1). Hence assume $\delta \in (0, 1)$, and suppose $T = \{t_j\}$ is a countable dense subset of $(0, 1)$. Let $\{Y(t) : t \in T\}$ be a sochastic process on $(\Omega \times \widetilde{\Omega}, \mathcal{F} \times \widetilde{\mathcal{F}}, P \times \widetilde{P})$ as in the proof of Lemma 2.1. Then

$$\begin{aligned} &P(M(1) \leq \varepsilon, |X(1)| \leq \varepsilon\delta) \\ &= \lim_n P\Big(\sup_{1\leq j\leq n} |X(t_j)| \leq \varepsilon, |X(1)| \leq \varepsilon\delta\Big) \\ &= \lim_n E_\omega\Big(P_{\omega'}\Big(\sup_{1\leq j\leq n} |Y(t_j, \omega, \omega')| \leq \varepsilon,\ |Y(1, \omega, \omega')| \leq \varepsilon\delta\Big)\Big) \\ &\geq \lim_n E_\omega\Big(P_{\omega'}\Big(\sup_{1\leq j\leq n} |Y(t_j, \omega, \omega')| \leq \varepsilon,\ |Y(1, \omega, \omega') + \theta| \leq \varepsilon\delta\Big)\Big) \end{aligned}$$

for all $\theta \in \mathbb{R}$, where the inequality is due to Anderson's inequality applied conditionally to the Gaussian probability in $\mathbb{R}^{n+1}$, i.e. we are translating only the $(n+1)^{\text{st}}$ coordinate. Continuing with (2.11) we have for $\theta \in \mathbb{R}$ that

$$\begin{aligned}
&P(M(1) \le \varepsilon, |X(1)| \le \varepsilon\delta) \\
&\quad \ge (P \times P')\big(\sup_T |Y(t,\omega,\omega')| \le \varepsilon,\ |Y(1,\omega,\omega') + \theta| \le \varepsilon\delta\big) \\
&\quad = P\big(\sup_T |X(t)| \le \varepsilon,\ |X(1) + \theta| \le \varepsilon\delta\big) \\
&\quad = P(M(1) \le \varepsilon,\ |X(1) + \theta| \le \varepsilon\delta).
\end{aligned}$$

Thus

$$\begin{aligned}
P(M(1) \le \varepsilon) &\le \sum_{j=-[1/\delta]}^{[1/\delta]} P(M(1) \le \varepsilon,\ |X(1) + j\varepsilon\delta| \le \varepsilon\delta) \\
&\le \left(2\left[1/\delta\right] + 1\right) P(M(1) < \varepsilon,\ |X(1)| \le \varepsilon\delta).
\end{aligned}$$

Hence the above estimate implies (2.4).

Proposition 2.4. *Let $\{X(t) : t \ge 0\}$ be a symmetric stable process with homogeneous independent increments, sample paths in $D[0,\infty)$, and parameter $\alpha \in (0,2]$. Fix sequences $\{t_i\}_{i=0}^m$, $\{a_i\}_{i=0}^m$, $\{b_i\}_{i=0}^m$ such that $0 = t_0 < t_1 < \cdots < t_m$ and $a_1 < b_1 \le a_2 < b_2 \le \cdots \le a_m < b_m$. Then, for every $\gamma > 0$,*

$$\varliminf_{\varepsilon \to 0} \varepsilon^\alpha \log P(a_i\varepsilon \le M(t_i) \le b_i\varepsilon, 1 \le i \le m, |X(t_m)| \le b_m\gamma\varepsilon) \ge -c_\alpha \sum_{i=1}^m \frac{t_i - t_{i-1}}{b_i^\alpha}. \tag{2.5}$$

Proof. Take a small $\delta > 0$ such that $\delta < \gamma$ and $a_i(1+\delta) < b_i(1-\delta)$ for all $1 \le i \le m$. Define

$$B_i = \left\{ a_i\varepsilon \le \sup_{t_{i-1} \le s \le t_i} |X(s)| \le b_i\varepsilon,\ |X(t_i)| \le b_i\delta\varepsilon \right\}$$

for $i = 1, \ldots, m$. Then

$$\{a_i\varepsilon \le M(t_i) \le b_i\varepsilon,\ 1 \le i \le m,\ |X(t_m)| \le b_m\gamma\varepsilon\} \supseteq \bigcap_{i=1}^m B_i.$$

On the other hand, if for $i = 1, \ldots, m$.

$$A_i = \Big\{ a_i(1+\delta)\varepsilon \le \sup_{t_{i-1} \le s \le t_i} |X(s) - X(t_{i-1})| \le b_i(1-\delta)\varepsilon,$$
$$|X(t_i) - X(t_{i-1})| \le (b_i - b_{i-1})\delta\varepsilon \Big\}$$

then

$$P(A_i) = P\Big(\frac{a_i(1+\delta)\varepsilon}{(t_i - t_{i-1})^{1/\alpha}} \le M(1) \le \frac{b_i(1-\delta)\varepsilon}{(t_i - t_{i-1})^{1/\alpha}},\ |X(1)| \le \frac{(b_i - b_{i-1})\delta\varepsilon}{(t_i - t_{i-1})^{1/\alpha}} \Big)$$

and

$$P\Big(\bigcap_{i=1}^{m} B_i\Big) \ge P\Big(\bigcap_{i=1}^{m-1} B_i \cap A_m\Big) = P\Big(\bigcap_{i=1}^{m-1} B_i\Big) \cdot P(A_m) \ge \prod_{i=1}^{m} P(A_i). \tag{2.6}$$

By the remark after Lemma 2.3, (2.5) follows from (2.6), and the proposition is proven.

As a direct consequence of our Proposition 2.2 and Proposition 2.4, we have the following small ball estimates for $X(t)$ under weighted norms. The case $\alpha = 2$ was given in Mogul'skii (1982) and its connection with Gaussian Markov processes was studied in Li (1998).

Proposition 2.5. *Let $\{X(t) : t \ge 0\}$ be a symmetric stable process with homogeneous independent increments, sample paths in $D[0,\infty)$, and parameter $\alpha \in (0,2]$. Let $\rho : [0,1] \to [0,\infty)$ be a bounded function such that $\rho(t)^\alpha$ is Riemann integrable on $[0,1]$. Then*

$$\lim_{\varepsilon \to 0} \varepsilon^2 \log P(\sup_{0 \le t \le 1} |\rho(t) X(t)| \le \varepsilon) = -c_\alpha \int_0^1 \rho(t)^\alpha dt.$$

3. Proof of Theorem 1.1 and Corollary 1.1.

The proof of Theorem 1.1 follows immediately from the following three facts.

$$P(C(\{\eta_n\}) \subset K_\alpha) = 1, \tag{3.1}$$

$$P(\{\eta_n\} \quad \text{is conditionally compact in} \quad \mathcal{M}) = 1, \tag{3.2}$$

and

$$P(K_\alpha \subset C(\{\eta_n\})) = 1. \tag{3.3}$$

Of course, the topology on $\mathcal{M}$ is that of weak convergence, which is separable and metric.

In order to prove (3.2) we first observe that a subset F of $\mathcal{M}$ is conditionally compact if for every $\Gamma > 0$ there exists $t_0 = t_0(\Gamma)$ such that $t \geq t_0$ implies $\inf_{f \in F} f(t) \geq \Gamma$. This characterization of conditional compactness in $\mathcal{M}$ is immediate from the homeomorphism of $\mathcal{N}$ and $\mathcal{M}$.

Proposition 3.1. $P(\{\eta_n\} \quad \textit{is conditionally compact in} \quad \mathcal{M}) = 1.$

Proof. Let $n_k = 2^k$, and observe that for $n_{k-1} \leq n \leq n_k$, and all k sufficiently large,

$$\eta_n(t) = \eta_{n_k}(nt/n_k)\,(n_k LLn/(nLLn_k))^{1/\alpha} \geq \eta_{n_k}(t/2). \tag{3.4}$$

Hence for $\Gamma > 0$, (3.4) implies

$$P(\eta_n(t) > \Gamma \ \text{ eventually in n}) \geq P(\eta_{n_k}(t/2) > \Gamma \ \text{ eventually in n}). \tag{3.5}$$

Rescaling, and applying (1.1), we have for all k sufficiently large that

$$P(\eta_{n_k}(t/2) \leq \Gamma) = P(M(1) \leq \Gamma(2c_\alpha/(tLLn_k))^{1/\alpha}) \leq \exp\{-(tLLn_k)/(4\Gamma^\alpha)\}.$$

Hence if $t \geq 8\Gamma^\alpha$, we have

$$\sum_{k \geq 1} P(\eta_{n_k}(t/2) \leq \Gamma) < \infty,$$

and the Borel-Cantelli lemma implies $P(\eta_{n_k}(t/2) \leq \Gamma \quad \text{i.o.}) = 0$. Thus (3.5) implies $P(\eta_n(t) > \Gamma \quad \text{eventually in n}) = 1$ for $t \geq 8\Gamma^\alpha$. Letting $\Gamma \nearrow \infty$ through a countable set implies (3.2), and the proposition is proven.

Proposition 3.2 $P(C(\{\eta_n\}) \subset K_\alpha) = 1.$

Proof. Fix $f \in \mathcal{M} \cap K_\alpha^c$, and hence

$$\int_0^\infty (f(t))^{-\alpha} dt > 1. \tag{3.6}$$

Let $t_f^* = \sup\{t : f(t) < \infty\}$. Then $t_f^* = 0$ negates (3.6), so $t_f^* = \infty$ or $0 < t_f^* < \infty$.

Suppose (3.6) holds. Since $f(t) = \infty$ for $t = t_f^*$ we have

$$\int_0^{t_f^*} (f(t))^{-\alpha} dt = \int_0^{\infty} (f(t))^{-\alpha} dt > 1. \tag{3.7}$$

Furthermore, since f is increasing and non-negative, the integrals in (3.7) exist as improper Riemann integrals. Hence there exist points $0 = t_0 < t_1 < \cdots < t_r < t_f^*$ and $\delta > 0$ such that $0 < f(t_1) < \cdots < f(t_r)$ and

$$\sum_{j=1}^{r} (f(t_j) + \delta)^{-\alpha}(t_j - t_{j-1}) > 1. \tag{3.8}$$

Furthermore, we may assume the t_j's are continuity points of f. That is, if t_j is not a continuity point, then we choose a point t_j^* such that $t_j < t_j^*, t_j^*$ is a continuity point of f, and for t_j^* sufficiently close to t_j we have

$$\begin{aligned}(f(t_j^*) + \delta)^{-\alpha}(t_j^* - t_{j-1}) + (f(t_{j+1}) + \delta)^{-\alpha}(t_{j+1} - t_j^*)\\ > (f(t_j) + \delta)^{-\alpha}(t_j - t_{j-1}) + (f(t_{j+1}) + \delta)^{-\alpha}(t_{j+1} - t_j) - \beta/(2r),\end{aligned} \tag{3.9}$$

where, by (3.8),

$$\beta = -1 + \sum_{j=1}^{r} (f(t_j) + \delta)^{-\alpha}(t_j - t_{j-1}) > 0.$$

The inequality in (3.9) holds since f is right continuous on $(0, \infty)$ and continuous everywhere except possibly a countable set. Modifying each t_j in this way (starting with t_1, then t_2, etc. whenever necessary), we see the t_j's can be taken to be continuity points of f and (3.8) holds.

With $\delta > 0$ as in (3.8) we define

$$N_f = \{g \in \mathcal{M} : f(t_j) - \delta < g(t_j) < f(t_j) + \delta, 1 \leq j \leq r\}.$$

Then for $g \in N_f$,

$$\sum_{j=1}^{n} (g(t_j))^{-\alpha}(t_j - t_{j-1}) \geq \sum_{j=1}^{r} (f(t_j) + \delta)^{-\alpha}(t_j - t_{j-1}) > 1, \tag{3.10}$$

and since $\int_0^\infty (f(t))^{-\alpha} dt$ exists as an improper Riemann integral, with refinements of a partition leading to a increase of the partial sums in (3.10) (they are lower sums), we have $N_f \cap K_\alpha = \phi$. Rescaling, applying Proposition 2.2, and taking $\gamma > 0$ such that

$$c_\alpha \sum_{j=1}^{r} (f(t_j) + \delta)^{-\alpha} (t_j - t_{j-1}) - \gamma > (1+\gamma) c_\alpha,$$

we have for n sufficiently large that

$$\begin{aligned} P(\eta_n \in N_f) &= P(\cap_{j=1}^{r} \{M(nt_j)/n^{1/\alpha} \in (c_\alpha / LLn)^{1/\alpha} (f(t_j) - \delta, f(t_j) + \delta)\}) \\ &\leq \exp\{-\,(LLn/c_\alpha)\,(c_\alpha \sum_{j=1}^{r} \frac{t_j - t_{j-1}}{(f(t_j)+\delta)^\alpha} - \gamma)\} \\ &\leq \exp\{-(1+\gamma) LLn\}. \end{aligned} \tag{3.11}$$

Thus if $n_k = \exp\{k/Lk\}$, (3.11) and the Borel-Cantelli Lemma implies

$$P(\eta_{n_k} \in N_f \ \text{ i.o.}) = 0. \tag{3.12}$$

The above argument shows K_α^c is open, and since $\mathcal{M}$ is separable there are $\{f_j\}$ dense in K_α^c such that $K_\alpha^c \subset \bigcup_{j=1}^\infty N_{f_j}$. Hence

$$\{C(\{\eta_{n_k}\}) \cap K_\alpha^c \neq \phi\} \subset \bigcup_{j=1}^{\infty} \{C(\{\eta_{n_k}\}) \cap N_{f_j} \neq \phi\},$$

and (3.12) implies

$$P(C(\{\eta_{n_k}) \subset K_\alpha) = 1. \tag{3.13}$$

Now $\eta_n(t) = \eta_{n_k}(nt/n_k)\,(n_k LLn/(nLLn_k))^{1/\alpha}$ and $f \in C(\{\eta_n\})$ implies $f \in C(\{\eta_{n_k}\})$ since $\lim n_k/n_{k-1} = 1$ and $n_{k-1} \leq n \leq n_k$. Thus (3.13) implies (3.1) and the proposition is proven.

Proposition 3.3. $P(K_\alpha \subset C(\{\eta_n\})) = 1$.

Proof. Let $\Lambda(f) = \int_0^\infty (f(t))^{-\alpha} dt$. Suppose $\Lambda(f) \leq 1$ and N is an arbitrary weak neighborhood of f. Since $\mathcal{M}$ is metrizable in the weak topology, there is a countable neighborhood base at each point of $\mathcal{M}$, and hence $f \in C(\{\eta_n\})$ with probability one provided

$$P(\eta_n \in N_f \ \text{ i.o.}) = 1. \tag{3.14}$$

Since K_α has a countable dense set, we then have every point of K_α in $C(\{\eta_n\})$ with probability one provided (3.14) holds for $f \in K_\alpha$.

To establish (3.14) for each $f \in K_\alpha$ our first step is to show we may actually assume $\Lambda(f)$ is strictly less than one. To do this we define $t_f^* = \sup\{t : f(t) < \infty\}$ as before, and consider the two possibilities $t_f^* = \infty$ and $0 < t_f^* < \infty$.

If $t_f^* = \infty$, then a typical neighborhood of f is of the form $N = \bigcap_{j=1}^r \Gamma_j$ where $0 < t_1 < \cdots < t_r$,

$$\Gamma_j = \{g : f(t_j) - \gamma < g(t_j) < f(t_j) + \gamma\}, \tag{3.15}$$

and $\gamma > 0$. Hence if we define

$$\tilde{f}(t) = \begin{cases} 0 & t = 0 \\ f(t) + \gamma/4 & 0 < t < \infty, \end{cases}$$

then $\tilde{f} \geq f$, $\tilde{f} \in N$, and $\Lambda(\tilde{f}) < 1$. Defining $\tilde{N}_f = \bigcap_{j=1}^n \tilde{\Gamma}_j$, where

$$\tilde{\Gamma}_j = \{g : \tilde{f}(t_j) - \gamma/2 < g(t_j) < \tilde{f}(t_j) + \gamma/2\}$$

we see $\tilde{N} \subset N_f$, and (3.14) will hold provided $P(\eta_n \in \tilde{N} \ \text{ i.o.}) = 1$.

The other case is $0 < t_f^* < \infty$. Then a typical neighborhood of f is of the form

$$N_f = (\bigcap_{j=1}^r \Gamma_j) \cap (\bigcap_{k=1}^s R_{r+k})$$

where $0 = t_0 < t_1 < \cdots < t_r < t_f^* \leq t_{r+1} < \cdots < t_{r+s}$, Γ_j is defined as in (3.15) and $R_{r+k} = \{g : g(t_{r+k}) > m_k\}$. Now we can define

$$\tilde{f}(t) = \begin{cases} 0 & t = 0 \\ f(t) + \gamma/4 & 0 < t < (t_r + t_f^*)/2 \\ \ell + 1/2 & (t_r + t_f^*)/2 \leq t < t_{r+s} + 1 \\ \infty & t \geq t_{r+s} + 1 \ , \end{cases}$$

and set

$$\tilde{N} = (\bigcap_{j=1}^{r} \tilde{\Gamma}_j) \cap (\bigcap_{k=1}^{s} \tilde{R}_{r+k}),$$

where

$$\tilde{\Gamma}_j = \{g : \tilde{f}(t_j) - \gamma/2 < g(t_j) < \tilde{f}(t_j) + \gamma/2\},$$

$$\tilde{R}_{r+k} = \{g : \ell < g(t_{r+k}) < \ell + 1\}$$

and $\ell > f((t_r + t_f^*)/2) + \gamma/4$ is sufficiently large so that

$$\Lambda(\tilde{f}) \leq \int_0^{(t_r+t_f^*)/2} (f(t) + \gamma/4)^{-\alpha} dt + (t_{r+s} + 1 - (t_r + t_f^*)/2)/\ell^\alpha < 1.$$

Then $\tilde{f} \in \tilde{N} \subset N_f$, $\Lambda(\tilde{f}) < 1$. Hence in both cases it suffices to verify (3.14) with $f \in N_f$ and $\Lambda(f) < 1$.

Assuming $\Lambda(f) < 1$, we consider only the case $t_t^* = \infty$ (the other case is much the same). Then $N_f = \cap_{j=1}^r \Gamma_j$, where Γ_j is given in (3.15). To verify (3.14) we take $n_k = \exp\{k^{1+\delta}\}$ with $\delta > 0$ to be specified later as a function of $\beta = 1 - \Lambda(f) > 0$. Now we observe

$$P(\eta_{n_k} \in N_f \text{ i.o.}) \geq P(A_k \cap B_k \text{ i.o.}), \tag{3.16}$$

where

$$A_k = \{f(t_j) - \gamma/2 < \tilde{\eta}_{n_k}(t_j) < f(t_j) + \gamma/2,\ 1 \leq j \leq r\}$$

$$B_k = \{\sup_{0 \leq s \leq n_{k-1} t_r / n_k} |X(n_k s)| \leq (\gamma/4)(c_\alpha n_k / LLn_k)^{1/\alpha}\},$$

and

$$\tilde{\eta}_{n_k}(t) = \sup_{n_{k-1} t_r / n_k \leq s \leq t} |X(n_k s) - X(n_{k-1} t_r)| / (c_\alpha n_k / LLn_k)^{1/\alpha}.$$

Levy's inequality and rescaling implies

$$\begin{aligned} P(B_k^c) &\leq 2P(|X(n_{k-1}t_r)| > (\gamma/4)(c_\alpha n_k / LLn_k)^{1/\alpha}) \\ &\leq 2E(|X(n_{k-1}t_r)|^{\alpha-\theta})(4/\gamma)^{\alpha-\theta} \left(LLn_k/(c_\alpha n_k)\right)^{1-\theta/\alpha} \\ &= 2(4/\gamma)^{\alpha-\theta} E(|X(t_r)|^{\alpha-\theta}) \left(n_{k-1} LLn_k/(c_\alpha n_k)\right)^{1-\theta/\alpha}, \end{aligned}$$

provided $0 < \theta < \alpha$. Since $n_k = \exp\{k^{1+\delta}\}$, we see $\sum_{k\geq 1} P(B_k^c) < \infty$, and hence $P(B_k^c \text{ i.o.}) = 0$. Thus $P(B_k \text{ eventually}) = 1$ and (3.14) will follow from (3.16) provided $P(A_k \text{ i.o.}) = 1$.

The time homogeneous, independent increments of $\{X(t) : t \geq 0\}$ imply the A_k's are independent provided $n_{k-1}t_r/n_k < t_1$,i.e. for all k sufficiently large, and, furthermore, that

$$P(A_k) = P(\cap_{j=1}^r \{M(n_k(t_j - n_{k-1}t_r/n_k))/(c_\alpha n_k/LLn_k)^{1/\alpha} \in \Gamma_j\}).$$

From Proposition 2.4, and rescaling, we thus have for all $\rho > 0$ that for k sufficiently large

$$\begin{aligned} P(A_k) &= P(\bigcap_{j=1}^r M(t_j - n_{k-1}t_r/n_k) \in (c_\alpha/LLn_k)^{1/\alpha}\Gamma_j) \\ &\geq \exp\{-(LLn_k)(1+\rho)\big(\frac{t_1 - n_{k-1}t_r/n_k}{(f(t_1)+\gamma/2)^\alpha} + \sum_{j=2}^r \frac{(t_j - t_{j-1})}{(f(t_j)+\gamma/2)^\alpha}\big)\} \\ &\geq k^{-(1+\delta)(1+\rho)^2(1-\beta)}, \end{aligned}$$

where $\beta = 1 - \Lambda(f) > 0$. In particular, taking $\rho = \delta$ and $(1+\delta)^3 < (1-\beta)^{-1}$ we have $\sum_{k\geq 1} P(A_k) = \infty$. Independence and the Borel Cantelli lemma now imply $P(A_k \text{ i.o.}) = 1$. Thus (3.16) implies (3.14). Hence we have shown (3.1)-(3.3), and Theorem 1.1 follows immediately.

Proof of Corollary 1.1. Applying the zero-one law we may assume with probability one that $\underline{\lim}_n \eta_n(1) = d$. If $d < 1$, then for every $f \in K_\alpha$ with $t = 1$ a continuity point of f, there is a subsequence (random) such that

$$\lim_{n_k} \eta_{n_k}(1) = f(1) = d < 1 \qquad \text{a.s.}$$

Thus $\int_0^\infty f^{-\alpha}(t)dt \geq \int_0^1 d^{-\alpha}dt > 1$, which contradicts $f \in K_\alpha$. Hence $d \geq 1$.

If $d > 1$ we define

$$f_0(t) = \begin{cases} 0 & t = 0 \\ d & 0 < t < 1+\delta \\ +\infty & t \geq 1+\delta. \end{cases}$$

Then $f_0 \in \mathcal{M}$ and for $\delta > 0$ sufficiently small $f_0 \in K_\alpha$. Furthermore, since 1 is a continuity point of f_0,

$$P(\underline{\lim}_n \eta_n(1) \leq f_0(1) = d) = 1.$$

Since $d > 1$ is arbitrary, this proves Corollary 1.1, and (1.3) holds with $\beta_\alpha = c_\alpha^{1/\alpha}$.

4. Proof of Theorems 1.2 and 1.3. We first establish several lemmas which allow us to identify the left-hand terms in (1.11), (1.12), and (1.13).

Lemma 4.1. *Let* $F_c(f) = \int_0^1 I_{[0,c]}(f(u)r(u))du$, *and*

$$G_c(t) = \int_0^1 I_{[0,c]}\Big(\eta_t(u)r(u)\Big(\frac{LLtu}{LLt}\Big)^{1/\alpha}\Big)du,$$

where $r : (0,1] \to [0,\infty)$ *is measurable. Then for each* $c > 0$*, with probability one*

$$\overline{\lim_{t\to\infty}}\, G_c(t) \leq \sup_{f\in K_\alpha} F_c(f). \tag{4.1}$$

Furthermore, we have equality in (4.1) whenever $\sup_{f\in K_\alpha} F_c(f)$ *is left continuous at* c.

Proof. First we prove $\overline{\lim}_{t\to\infty} G_c(t) \leq \sup_{f\in K_\alpha} F_c(f)$. Suppose the contrary, so there is a set $E \subseteq \Omega$ (our probability space for $\{X(t) : t \geq 0\}$) with $P(E) > 0$, and for $w \in E$

$$\overline{\lim_{t\to\infty}}\, G_c(t) > \sup_{f\in K_\alpha} F_c(f).$$

Let $\Omega_0 \subseteq \Omega$ with $P(\Omega_0) = 1$ and for $w \in \Omega_0$

$$\begin{array}{rl} \text{(i)} & C(\{\eta_t\}, t\to\infty) \subseteq K_\alpha, \\ \text{(ii)} & \{\eta_t\} \text{ is conditionally compact in } \mathcal{M} \text{ as } t\to\infty, \text{ and} \\ \text{(iii)} & K_\alpha \subseteq C(\{\eta_t\}, t\to\infty). \end{array} \tag{4.2}$$

Then for $w \in E\cap\Omega_0$, there exists a possibly random subsequence $\{t_j(w)\} = \{t_j\}$ such that $t_j \to \infty$, $\lim_{j\to\infty} G_c(t_j) > \sup_{f\in K_\alpha} F_c(f)$, and $\eta_{t_j}(\cdot) \overset{\text{weakly}}{\longrightarrow} f_0 \in K_\alpha$. Hence $\lim_{j\to\infty} \eta_{t_j}(u) = f_0(u)$ except possibly for countably many values of u, and therefore

$$\overline{\lim_j}\, I_{[0,c]}\Big(\eta_{t_j}(u)r(u)\Big(\frac{LLt_ju}{LLt_j}\Big)^{1/\alpha}\Big) \leq I_{[0,c]}(f_0(u)r(u))$$

for almost all $u \in [0,1]$ (Lebesgue measure), since the characteristic function of a closed set is upper semi-continuous. Thus the reverse Fatou lemma implies

$$\begin{aligned} \overline{\lim_j}\, G_c(t_j) &\leq \int_0^1 \overline{\lim_j}\, I_{[0,c]}\Big(\eta_{t_j}(u)r(u)\Big(\frac{LLt_ju}{LLt_j}\Big)^{1/\alpha}\Big)du \\ &\leq \int_0^1 I_{0,c]}(f_0(u)r(u))du \\ &= F_c(f_0) \leq \sup_{f\in K_\alpha} F_c(f), \end{aligned}$$

which contradicts that E exists with $P(E) > 0$. Thus $\overline{\lim_t} G_c(t) \le \sup_{f \in K_\alpha} F_c(f)$.

To prove the reverse inequality take $f_0 \in K_\alpha$. Then for all $w \in \Omega_0$ there exists a possibly random subsequence $\{t_j(w)\} = \{t_j\}$, such that

$$\eta_{t_j}(\cdot) \overset{\text{weakly}}{\longrightarrow} f_0 \in K_\alpha.$$

Then for $\delta > 0$, $c - \delta > 0$,

$$\overline{\lim_{t\to\infty}} G_c(t) \ge \overline{\lim_{t\to\infty}} F_c(\eta_t) \ge \overline{\lim_{t_j\to\infty}} F_c(\eta_{t_j}) \ge \underline{\lim}_{t_j\to\infty} F_c(\eta_{t_j}).$$

Hence by Fatou's lemma

$$\begin{aligned}\underline{\lim}_{t_j\to\infty} F_c(\eta_{t_j}) &\ge \int_0^1 \underline{\lim}_{t_j\to\infty} I_{[0,c]}(\eta_{t_j}(u)r(u))du \\ &\ge \int_0^1 I_{[0,c-(\delta/2))}\Big(\underline{\lim}_{t_j\to\infty} \eta_{t_j}(u)r(u)\Big)du \\ &= \int_0^1 [0, c-(\delta/2))(f_0(u)r(u))du,\end{aligned}$$

where the last inequality holds because $[0, c-(\delta/2))$ is open in $[0,\infty]$ and therefore $I_{[0,c-(\delta/2))}$ is lower semi-continuous here. Thus $\overline{\lim_{t\to\infty}} G_c(t) \ge F_{c-\delta}(f_0)$, and since $f_0 \in K_\alpha$ is arbitrary we have $\overline{\lim_{t\to\infty}} G_c(t) \ge \sup_{f\in K_\alpha} F_{c-\delta}(f)$. The left-continuity of $\sup_{f\in K_\alpha} F_\beta(f)$ at c thus implies equality in (4.1), and the lemma is proven.

Lemma 4.2. *If $0 < \alpha \le 2$ and $0 < p \le \infty$, then*

$$\inf_{f\in K_\alpha} \|f\|_p = 1,$$

where $\|f\|_p = \left(\int_0^1 |f(u)|^p du\right)^{1/p}$, $0 < p < \infty$, and $\|f\|_\infty$ is the essential supremum of f on $[0,1]$ with respect to Lebesgue measure.

Proof. If $p = \infty$, then $f \nearrow$ implies

$$\inf_{f\in K_\alpha} \|f\|_\infty \le \inf_{f\in K_\alpha} f(1) \le 1.$$

On the other hand, if $\|f\|_\infty < 1$, then $\int_0^1 f^{-\alpha}(t)dt > 1$ and $f \notin K_\alpha$. Thus $\inf_{f\in K_\alpha} \|f\|_\infty = 1$.

If $0 < p < \infty$ take $r \in (1,\infty)$ such that $\alpha/(r-1) = p$. Then $\int_0^1 |f(u)|^p du < \infty$ and $f \in K_\alpha$ imply both f and f^{-1} are finite and non-negative a.s. on $[0,1]$. Hence with Lebesgue measure one

$$1 = f^{\alpha/r}(u)\, f^{-\alpha/r}(u),$$

and therefore by Holder's inequality with $r > 1$, $q^{-1} = 1 - r^{-1} = (r-1)/r$,

$$\begin{aligned} 1 &= \int_0^1 (f(u))^{\alpha/r}(f(u))^{-\alpha/r} du \\ &\le \Big(\int_0^1 (f(u))^{\alpha/(r-1)} du\Big)^{(r-1)/r} \Big(\int_0^1 (f(u))^{-\alpha} du\Big)^{1/r} \\ &\le \Big(\int_0^1 (f(u))^p du\Big)^{1/p} \cdot 1 \end{aligned}$$

since $f \in K_\alpha$. Thus $\lim_{f\in K_\alpha} \|f\|_p \ge 1$, and it's trivially less than or equal to one by the $p = \infty$ case.

Proof of (1.13). Fix $0 < \alpha \le 2$ and $0 < p < \infty$. Then Lemma 4.2 implies that $\inf_{f\in K_\alpha} \int_0^1 |f(u)|^p du = 1$, so it remains to verify the first equality. Hence assume $\underline{\lim}_{t\to\infty} \int_0^1 |\eta_t(u)|^p du < 1$ on a set $E \subseteq \Omega$ with $P(E) > 0$ and assume $\Omega_0 \subseteq \Omega$ is as in Lemma 4.1. In particular, $w \in \Omega_0$ implies (4.2) holds, and for $w \in E \cap \Omega_0$, there exists a possibly random sequence $\{t_j(w)\} = \{t_j\}$ such that $\lim_{j\to\infty} \int_0^1 |\eta_{t_j}(u)|^p du < 1$ and $\eta_{t_j} \overset{\text{weakly}}{\longrightarrow} f_0$, for some $f_0 \in K_\alpha$. Then $\lim_j \eta_{t_j}(u) = f_0(u)$ except for possibly countably many u and hence Fatou's lemma implies

$$\underline{\lim}_{j} \int_0^1 |\eta_{t_j}(u)|^p \ge \int_0^1 |f_0(u)|^p du \ge \inf_{f\in K_\alpha} \int_0^1 |f(u)|^p du = 1.$$

This contradicts $P(E) > 0$, so we have with probability one that $\underline{\lim}_{t\to\infty} \int_0^1 |\eta_t(u)|^p du \ge 1$. On the other hand, $\underline{\lim}_{t\to\infty} \int_0^1 |\eta_t(u)|^p du \le \underline{\lim}_{t\to\infty} |\eta_t(1)|^p = 1$ by Corollary 1.1 and that $\eta_t(\cdot)$ is increasing on $[0,1]$. Hence (1.13) holds and Theorem 1.3 is proven.

Proof of (1.12). Fix $0 < \alpha \leq 2$ and set $u = s/t$ in (1.12). Then

$$\overline{\lim_{t\to\infty}}\, t^{-1} \int_0^t I_{[0,c]}(\eta_t(s/t))ds = \overline{\lim_{t\to\infty}} \int_0^1 I_{[0,c]}(\eta_t(u))du \tag{4.3}$$

with probability one. Let $r(u) = 1$ and define $F_c(f)$ as in Lemma 4.1. Then, for $0 < c < \infty$, consider

$$\sup_{f\in K_\alpha} F_c(f) = \sup_{f\in K_\alpha} \int_0^1 I_{[0,c]}(f(u))du.$$

If $c \geq 1$, then setting

$$f_c(u) = \begin{cases} 0 & \text{if } u = 0 \\ c & \text{if } 0 < u < 1 \\ +\infty & \text{if } u \geq 1, \end{cases}$$

we see $\int_0^1 I_{0,c]}(f_c(u))du = 1$ and since $c \geq 1$ we also have $f_c \in K_\alpha$. Since $\sup_{f\in K_\alpha} \int_0^1 I_{[0,c]}(f(u))du \leq 1$, we have

$$\sup_{f\in K_\alpha} F_c(f) = 1$$

for $c \geq 1$. If $0 < c < 1$, define

$$f_c(u) = \begin{cases} 0 & \text{if } u = 0 \\ c & \text{if } 0 < u < c^\alpha \\ +\infty & \text{if } u \geq c^\alpha. \end{cases}$$

Then $f_c \in K_\alpha$, and since $f \in K_\alpha$ is increasing with $f(0) = 0$, it is easy to see that

$$\sup_{f\in K_\alpha} \int_0^1 I_{[0,c]}(f(u))du = \int_0^1 I_{[0,c]}(f_c(u))du = c^\alpha.$$

Thus $\sup_{f\in K_\alpha} F_c(f)$ is continuous for $0 \leq c < \infty$ and hence the method of proof of Lemma 4.1 implies with probability one

$$\overline{\lim_{t\to\infty}} \int_0^1 I_{[0,c]}(\eta_t(u))du = \sup_{f\in K_\alpha} F_c(f) = \begin{cases} 1 & \text{if } c \geq 1 \\ c^\alpha & \text{if } 0 \leq c < 1. \end{cases} \tag{4.4}$$

Combining (4.3) and (4.4) yields (1.12).

Proof of (1.11). Since $\eta_s(1) = \eta_t(s/t)\left(\frac{tLLs}{sLLt}\right)^{1/\alpha}$ for $s, t > 0$, letting $u = s/t$ implies $\Psi_c(t)$ as given in (1.7) satisfies

$$\Psi_c(t) = \int_0^1 I_{[0,c]}\Big(\eta_t(u)u^{-1/\alpha}\theta(u)\Big(\frac{LLtu}{LLt}\Big)^{1/\alpha}\Big)du.$$

Now Lemma 4.1 with $r(u) = u^{-1/\alpha}\theta(u)$ implies

$$\overline{\lim_{t\to\infty}}\ \Psi_c(t) = \sup_{f\in K_\alpha} F_c(f)$$

with probability one, provided $\sup_{f\in K_\alpha} F_c(f)$ is sufficiently continuous at c.

When $c \geq 1$

$$\sup_{f\in K_\alpha} F_c(f) = \sup_{f\in K_\alpha} \int_0^1 I_{[0,c]}(f(u)u^{-1/\alpha}\theta(u))du \tag{4.5}$$

is taken on by the function $f_c(u)$ where

$$f_c(u) = \begin{cases} 0 & \text{if } u = 0, \\ cu_0^{1/\alpha}/\theta(u_0) & \text{if } 0 < u < u_0, \\ cu^{1/\alpha}/\theta(u) & \text{if } u_0 \leq u < 1, \\ +\infty & \text{if } u \geq 1. \end{cases}$$

That is, if $f(u) > cu^{1/\alpha}/\theta(u)$ for $u \in E \subseteq [0,1]$, then since both $cu^{1/\alpha}/\theta(u)$ and $f(u)$ are increasing on $[0,1]$ with (1.9) holding, we minimize the quantity $\int_0^1 f^{-\alpha}(u)du$ by having the set E be an interval starting at zero. Thus the choice of f_c is optimal provided we choose u_0 such that $h(u_0) = c^\alpha$ where $h(\cdot)$ is as in (1.10). Then $u_0 = s_c$, $f_c \in K_\alpha$, and for all $c \geq 1$

$$\sup_{f\in K_\alpha} \int_0^1 I_{[0,c]}(f(u)u^{-1/\alpha}\theta(u))du = 1 - s_c. \tag{4.6}$$

Now $h(\cdot)$ one-to-one and continuous from $[0,1]$ onto $[1,\infty)$ with $h(1) = 1$ implies s_c is continuous for all $c > 1$ and $s_1 = 1$. Thus Lemma 4.1, (4.5), and (4.6) imply (1.11) for $c > 1$. If $c = 1$, then $s_1 = 1$ and the upper bound in (4.1) imply with probability one that

$$\overline{\lim_{t\to\infty}}\ \Psi_c(t) \leq 0.$$

However, $\overline{\lim}_{t\to\infty} \Psi_c(t) \geq 0$ is trivial, so (1.11) holds even when $c = 1$. Hence Theorem 1.2 is proven.

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