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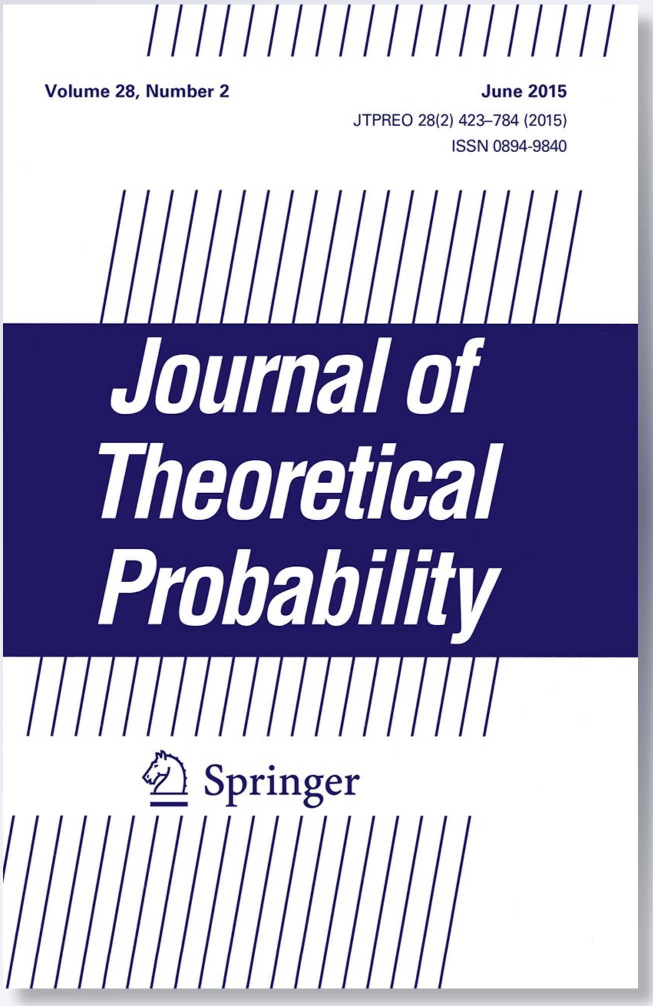

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The Limit Law of the Iterated Logarithm

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Abstract For the partial sum $\{S_n\}$ of an i.i.d. sequence with zero mean and unit variance, it is pointed out that

$$\lim_{n\to\infty}(2\log\log n)^{-1/2}\max_{1\le k\le n}\frac{S_k}{\sqrt{k}}=1\quad \text{a.s.}$$

Keywords The limit law of the iterated logarithm · Brownian motion · Ornstein–Uhlenbeck process

Mathematics Subject Classification (2010) 60F15 · 60G10 · 60G15 · 60G50

1 Theorem

Given a sequence $\{X_k\}_{k\ge 1}$ of i.i.d. random variables with

$$\mathbb{E}X_1=0\quad \text{and}\quad \mathbb{E}X_1^2=1, \tag{1.1}$$

Hartman–Wintner's law of the iterated logarithm states [4] that

$$\limsup_{n\to\infty}(2n\log\log n)^{-1/2}S_n=1\quad \text{a.s.} \tag{1.2}$$

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where $\{S_n\}$ is the process of the partial sum given as

$$S_n = X_1 + \ldots + X_n \quad n = 1, 2, \cdots .$$

We point out the Ref. [3] for an elegant proof of Hartman–Wintner's law of the iterated logarithm.

Hartman–Wintner's law of the iterated logarithm is also regarded as the limsup law of the iterated logarithm in the literature. The liminf law of the iterated logarithm was obtained by Chung [2] who proved that

$$\liminf_{n\to\infty} \sqrt{\frac{\log\log n}{n}} \max_{1\le k\le n} |S_k| = \frac{\pi}{\sqrt{8}} \quad \text{a.s.} \tag{1.3}$$

under the extra assumption of finite third moment. The gap was closed by Jain and Pruitt [5] who point out that the assumption (1.1) is sufficient (and necessary) for Chung's law of the iterated logarithm. We recommend the Ref. [1] for an extensive survey on both limsup and liminf laws of the iterated logarithm.

In this short note we establish the limit law of the iterated logarithm.

Theorem 1.1 *Under the assumption (1.1),*

$$\lim_{n\to\infty} (2\log\log n)^{-1/2} \max_{1\le k\le n} \frac{S_k}{\sqrt{k}} = 1 \quad \text{a.s.}$$

Proof By the almost sure invariance principle [10], one can construct a possibly larger probability space which supports both the i.i.d. sequence $\{X_k\}$ and a linear Brownian motion $B(t)$ such that

$$|S_k - B(k)| = o\left(\sqrt{k \log\log k}\right) \quad \text{a.s.} \quad (k \to \infty). \tag{1.4}$$

On the other hand, it is well known that the time-changed Brownian motion

$$X(t) = \frac{B(e^t)}{\sqrt{e^t}} \quad t \ge 0 \tag{1.5}$$

is an Ornstein–Uhlenbeck process. By Proposition 2.1 below,

$$\lim_{t\to\infty} (2\log t)^{-1/2} \max_{s\le t} \frac{B(e^s)}{\sqrt{e^s}} = 1 \quad \text{a.s.}$$

By variable substitution,

$$\lim_{t\to\infty} (2\log\log t)^{-1/2} \max_{1\le s\le t} \frac{B(s)}{\sqrt{s}} = 1 \quad \text{a.s.}$$

Using the classic fact that

$$\sup_{k\le s\le k+1}|B(s)-B(k)|=O\big(\sqrt{\log k}\big)\quad \text{a.s.}\quad (k\to\infty)$$

we have that

$$\lim_{n\to\infty}(2\log\log n)^{-1/2}\max_{1\le k\le n}\frac{B(k)}{\sqrt{k}}=1\quad \text{a.s.}$$

In view of (1.4), we have completed the proof. □

Remark By a deterministic argument, one can see that Hartman–Wintner's law of the iterated logarithm in (1.2) is a direct corollary of Theorem 1.1. In view of the relation

$$\max_{1\le k\le n}\frac{|S_k|}{\sqrt{k}}=\max_{1\le k\le n}\frac{S_k}{\sqrt{k}}\vee\max_{1\le k\le n}\frac{-S_k}{\sqrt{k}}$$

applying Theorem 1.1 to both $\{X_k\}$ and $\{-X_k\}$ on the right hand side leads to

$$\lim_{n\to\infty}(2\log\log n)^{-1/2}\max_{1\le k\le n}\frac{|S_k|}{\sqrt{k}}=1\quad \text{a.s.} \tag{1.6}$$

2 Law of Logarithm for Ornstein–Uhlenbeck Process

A one dimensional Ornstein–Uhlenbeck process $\{X(t);\ t\ge 0\}$ can be defined as a stationary, centered, and continuous Gaussian process with co-variance function

$$\text{Cov}\,\big(X(0),X(t)\big)=e^{-t/2}\quad t\ge 0. \tag{2.1}$$

By this definition, one can directly verify the representation given in (1.5).

Here we prove the following law of the logarithm for the maximum of Ornstein–Uhlenbeck process.

Proposition 2.1 *The following strong law holds.*

$$\lim_{t\to\infty}(2\log t)^{-1/2}\max_{s\le t}X(s)=1\quad \text{a.s.} \tag{2.2}$$

We mention the Refs. [7] and [8] for some similar results in the context of stationary Gaussian sequence. It is possible that (2.2) can be established by discrete approximation. We provide a direct proof in the following. By a standard argument using Borel–Cantelli lemma, all we need is to show that for any $\epsilon>0$,

$$\sum_k \mathbb{P}\Big\{\max_{s\le 2^k}X(s)\ge(1+\epsilon)\sqrt{2\log 2^k}\Big\}<\infty, \tag{2.3}$$

$$\sum_k \mathbb{P}\Big\{\max_{s\le 2^k}X(s)\le(1-\epsilon)\sqrt{2\log 2^k}\Big\}<\infty. \tag{2.4}$$

By stationarity,

$$\mathbb{P}\Big\{\max_{s\le 2^k} X(s) \ge (1+\epsilon)\sqrt{2\log 2^k}\Big\} \le 2^k \mathbb{P}\Big\{\max_{s\le 1} X(s) \ge (1+\epsilon)\sqrt{2\log 2^k}\Big\}.$$

By the concentration inequality for Gaussian process (see, e.g., (5.152), Theorem 5.4.3, p. 219, [6]), and by stationarity, there is a $u > 0$ such that

$$\mathbb{P}\Big\{\max_{s\le 1} X(s) \ge (1+\epsilon)\sqrt{2\log 2^k}\Big\} \le \exp\Big\{-(1+u)\log 2^k\Big\}.$$

This finishes the proof of (2.3).

To establish (2.4), let $\delta > 0$ be small but fixed. For each t, let $N = N_t = [\delta t]$ and let $0 = \tau_0 < \tau_1 < \cdots < \tau_N = t$ be an uniform partition of $[0, t]$. We have that

$$\mathbb{P}\Big\{\max_{s\le t} X(s) \le (1-\epsilon)\sqrt{2\log t}\Big\} \le \mathbb{P}\Big\{\max_{0\le k\le N} X(\tau_k) \le (1-\epsilon)\sqrt{2\log t}\Big\}. \quad (2.5)$$

On the other hand, let $\eta_0, \eta_1, \cdots, \eta_n$ be an i.i.d. sequence of standard normal random variables and write

$$\zeta_k = \big(1 + 2e^{-t/(2N)}\big)^{-1/2}\Big(\eta_k + \sqrt{2}e^{-t/(4N)}\eta_0\Big) \quad k = 1, \cdots, N.$$

In view of (2.1), it is straightforward to show that when $\delta < 2^{-1}(\log 2)^{-1}$,

$$\begin{aligned}&\mathrm{Var}\,\big(X(\tau_k)\big) = \mathrm{Var}\,(\zeta_k) = 1 \text{ and}\\ &\mathrm{Cov}\,\big(X(\tau_i), X(\tau_j)\big) \le \mathrm{Cov}\,(\zeta_i, \zeta_j) \quad i, j, k = 1, \cdots, n.\end{aligned}$$

By Slepian lemma ([9], see also Lemma 5.5.1, [6]),

$$\mathbb{P}\Big\{\max_{0\le k\le N} X(\tau_k) \le (1-\epsilon)\sqrt{2\log t}\Big\} \le \mathbb{P}\Big\{\max_{0\le k\le N} \zeta_k \le (1-\epsilon)\sqrt{2\log t}\Big\}.$$

Notice that

$$\max_{k\le n} \zeta_k = \big(1 + 2e^{-t/(2N)}\big)^{-1/2}\Big(\sqrt{2}e^{-t/(4N)}\eta_0 + \max_{k\le n}\eta_k\Big).$$

By the triangle inequality

$$\begin{aligned}&\mathbb{P}\Big\{\max_{k\le N} X(\tau_k) \le (1-\epsilon)\sqrt{2\log t}\Big\}\\ &\le \mathbb{P}\Big\{\max_{k\le N}\eta_k \le \sqrt{1+2e^{-t/(2N)}}(1-2^{-1}\epsilon)\sqrt{2\log t}\Big\}\\ &\quad + \mathbb{P}\Big\{\eta_0 \le -\frac{\epsilon}{2}e^{t/(4N)}\sqrt{2\log t}\Big\}\end{aligned}$$

$$= \left(\mathbb{P}\left\{\eta_1 \le \sqrt{1+2e^{-t/(2N)}}(1-2^{-1}\epsilon)\sqrt{2\log t}\right\}\right)^{N+1}$$
$$+\mathbb{P}\left\{\eta_0 \ge \frac{\epsilon}{2}e^{t/(4N)}\sqrt{2\log t}\right\}.$$

Taking $\delta > 0$ sufficiently small, we have

$$\mathbb{P}\left\{\eta_0 \ge \frac{\epsilon}{2}e^{t/(4N)}\sqrt{2\log t}\right\} \le \exp\left\{-\log t\right\},$$
$$\mathbb{P}\left\{\eta_1 \ge \sqrt{1+2e^{-t/(2N)}}(1-2^{-1}\epsilon)\sqrt{2\log t}\right\} \ge \exp\left\{-(1-u)\log t\right\}$$

for large t, where $u > 0$ is a small number depending only on ϵ and δ.

As a consequence of the second inequality,

$$\left(\mathbb{P}\left\{\eta_1 \le \sqrt{1+2e^{-t/(2N)}}(1-2^{-1}\epsilon)\sqrt{2\log t}\right\}\right)^{N+1}$$
$$\le \left(1-\frac{1}{t^{1-u}}\right)^{\delta t} \le \exp\left\{-ct^u\right\}$$

for large t, where $c > 0$ is independent of t.

In summary of the steps since (2.5), we have completed the proof of (2.4). □

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References

1. Bingham, N.H.: Variants on the law of the iterated logarithm. Bull. Lond. Math. Soc. **18**, 433–467 (1986)
2. Chung, K.L.: On the maximum partial sums of sequences of independent random variables. Trans. Am. Math. Soc. **64**, 205–233 (1948)
3. de Acosta, A.: A new proof of the Hartman–Wintner law of the iterated logarithm. Ann. Probab. **11**, 270–276 (1983)
4. Hartman, P., Wintner, A.: On the law of the iterated logarithm. Am. J. Math. **63**, 169–176 (1941)
5. Jain, N.C., Pruitt, W.E.: The other law of the iterated logarithm. Ann. Probab. **3**, 1046–1049 (1975)
6. Marcus, M.B., Rosen, J.: Markov processes, Gaussian processes, and local times. Cambridge studies in advanced mathematics, p. 100. Cambridge University Press, New York (2006)
7. Mittal, Y.: Limiting behavior of maxima in stationary Gaussian sequences. Ann. Probab. **2**, 231–242 (1974)
8. Pickands, J.: An iterated logarithm law for the maximum of a stationary Gaussian sequence. Z. Wahrsch. Verw. Gebiete **12**, 344–353 (1969)
9. Slepian, D.: The one-sided barrier problem for Gaussian noise. Bell Syst. Tech. J. **41**, 463–501 (1962)
10. Strassen, V.: An invariance principle for the law of the iterated logarithm. Z. Wahrsch. Verw. Gebiete **49**, 23–32 (1964)