Math 241 **Quiz #3** **Spring 2025** **Name:**

Consider the curve $\mathbf{r}(t) = \langle t, \cos t, \sin t\rangle$

1. Find the length of the the part of the curve from $t = 0$ to $t = 1$.

Solution. $\mathbf{r}'(t) = \langle 1, -\sin t, \cos t\rangle$ and

$$\|\mathbf{r}'(t)\| = \sqrt{1^2 + \sin^2 t + \cos^2 t} = \sqrt{2}$$

$$s = \int_0^1 \|\mathbf{r}'(t)\| dt = \int_0^1 \sqrt{2} dt = \sqrt{2}$$

2. Find the curvature $\kappa(t)$ of the curve.

Solution. We use the formula

$$\kappa(t) = \frac{\|\mathbf{r}'(t) \times \mathbf{r}"(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{\|\mathbf{r}'(t) \times \mathbf{r}"(t)\|}{\sqrt{8}}$$

Continued from the previous problem: $\mathbf{r}"(t) = \langle 0, -\cos t, -\sin t\rangle$.

$$\mathbf{r}'(t) \times \mathbf{r}"(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -\sin t & \cos t \\ 0 & -\cos t & -\sin t \end{vmatrix} - \mathbf{i} + \sin t\mathbf{j} - \cos t\mathbf{k}$$

$$\|\mathbf{r}'(t) \times \mathbf{r}"(t)\| = \sqrt{1 + \sin^2 t + \cos^2 t} = \sqrt{2}$$

Thus

$$\kappa(t) = \frac{\sqrt{2}}{\sqrt{8}} = \frac{1}{2}$$

Alternative solution. We use the formula

$$\kappa(t) = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\|\mathbf{T}'(t)\|}{\sqrt{2}}$$

where

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{1}{\sqrt{2}}\langle 1, -\sin t, \cos t\rangle$$

Hence,

$$\mathbf{T}'(t) = \frac{1}{\sqrt{2}}\langle 0, -\cos t, -\sin t\rangle$$

$$\|\mathbf{T}'(t)\| = \frac{1}{\sqrt{2}}\sqrt{0^2 + \cos^2 t + \sin^2 t} = \frac{1}{\sqrt{2}}$$

Finally,

$$\kappa(t) = \frac{\|\mathbf{T}'(t)\|}{\sqrt{2}} = \frac{1}{2}$$