Math 231 Quiz #7 Solution Spring 2026

1. In free undamped motion, a mass weighing 1 pounds stretches a spring 1/2 ft. The mass is initially released from rest from 1 ft. below the equilibrium position. Find the equation of motion. (Lets agree that the Newton's constant $g = 32$).

Solution. $m = W/g = 1/32 = 1/32$. The stiffness k of the spring satisfies $W = (1/2)k$ or $1 = (1/2)k$. Therefore $k = 2$. We have the equation

$$\begin{cases} \frac{1}{32}\frac{d^2x}{dt^2} = -2x \\ x(0) = 1 \quad \text{and} \quad x'(0) = 0 \end{cases} \quad \text{or} \quad \begin{cases} \frac{d^2x}{dt^2} + 64x = 0 \\ x(0) = 1 \quad \text{and} \quad x'(0) = 0 \end{cases}$$

So we have $x(t) = C_1 \cos 8t + C_2 \sin 8t$ and therefore $x'(t) = -8C_1 \sin 8t + 8C_2 \cos 8t$. Let $t = 0$: $C_1 = 2$ and $C_2 = 0$. Therefore $x(t) = \cos 8t$.

2. If an external force $f(t) = \sin \omega t$ is added to the above motion, find the external frequency so a resonance occurs (i.e., the oscillation gets wilder and wilder). Find the equation of motion in this case.

Solution. A resonance occurs when $\omega = 8$. In this case $x_p = t(A \cos 8t + B \sin 8t)$, $x_p'(t) = (A \cos 8t + B \sin 8t) + t(-8A \sin 8t + 8B \cos 8t)$ and

$$x_p''(t) = 2(-8A \sin 8t + 8B \cos 8t) + t(-64 \cos 8t - 64 \sin 8t)$$

Therefore,

$$\frac{1}{32}x_p'' + 2x_p = -\frac{1}{2}A \sin 8t + \frac{1}{2}B \cos 8t$$

Thus, $B = 0$ and $\frac{-1}{2}A = 1$ or $A = -2$. So $x_p = -2t \cos 8t$. The general solution s

$$x(t) = x_c(t) + x_p(t) = C_1 \cos 8t + C_2 \sin 8t - 2t \cos 8t$$

$$x'(t) = -8C_1 \sin 8t + 8C_2 \cos 8t - 2 \cos 8t + 16 \sin 8t = (16 - 8C_1) \sin 8t + (8C_2 - 2t) \cos 8t$$

Let $t = 0$: $1 = C_1$; $0 = 8C_2$ so $C_2 = 0$. The equation of motion is

$$x(t) = \cos 8t - 2t \cos 8t$$