

1. Find the directional derivative $D_{\mathbf{u}}f(3, 2, 1)$ of the function $f(x, y, z) = xy^2z^3$ in the direction of $\mathbf{v} = \langle 2, 2, 1 \rangle$. Find the direction that leads to the maximum value of the directional derivative at the point $(3, 2, 1)$ and the maximum value of the directional derivative at the point $(3, 2, 1)$.

Solution. $\nabla f(x, y, z) = \langle y^2z^3, 2xyz^3, 3xy^2z^2 \rangle$ and therefore $\nabla f(3, 2, 1) = \langle 4, 12, 36 \rangle$.

$$\mathbf{u} = \frac{1}{\|\mathbf{v}\|} \mathbf{v} = \left\langle \frac{2}{3}, \frac{2}{3}, \frac{1}{3} \right\rangle$$

Thus,

$$D_{\mathbf{u}}f(3, 2, 1) = \nabla f(3, 2, 1) \cdot \mathbf{u} = 4 \times \frac{2}{3} + 12 \times \frac{2}{3} + 36 \times \frac{1}{3} = \frac{68}{3}$$

The direction leading to the maximal rate is $\nabla f(3, 2, 1) = \langle 4, 12, 36 \rangle$. The maximal rate is

$$\|\nabla f(3, 2, 1)\| = \sqrt{4^2 + 12^2 + 36^2} = 4\sqrt{1 + 3^2 + 9^2} = 4\sqrt{91}$$

2. Find the equation of the tangent plane to the surface $x^2 + y^2 + z^2 = 1$ at the point $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$.

Solution. Set $F(x, y, z) = x^2 + y^2 + z^2$. Then $\nabla F(x, y, z) = \langle 2x, 2y, 2z \rangle$ and the normal vector $\mathbf{n} = \nabla F(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0) = \langle \sqrt{2}, \sqrt{2}, 0 \rangle$. The equation of the tangent plane is

$$\sqrt{2}(x - \frac{1}{\sqrt{2}}) + \sqrt{2}(y - \frac{1}{\sqrt{2}}) = 0 \quad \text{or} \quad x + y = \sqrt{2}$$