Moderate deviations and laws of the iterated logarithm for the renormalized self-intersection local times of planar random walks

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Abstract

We study moderate deviations for the renormalized self-intersection local time of planar random walks. We also prove laws of the iterated logarithm for such local times.

1 Introduction

Let $\{S_n\}$ be a symmetric random walk on $\mathbb{Z}^2$ with covariance matrix Γ. Let

$$B_n = \sum_{1\le j<k\le n} \delta(S_j, S_k) \tag{1.1}$$

where

$$\delta(x,y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases} \tag{1.2}$$

is the usual Kronecker delta. We refer to B_n as the self-intersection local time up to time n. We call $B_n - \mathbb{E}\, B_n$ the renormalized self-intersection local time of the random walk up to time n.

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In [5] it was shown that $B_n - \mathbb{E}\, B_n$, appropriately scaled, converges to the renormalized self-intersection local time of planar Brownian motion. Renormalized self-intersection local time for Brownian motion was originally studied by Varadhan [18] for its role in quantum field theory. Renormalized self-intersection local time turns out to be the right tool for the solution of certain "classical" problems such as the asymptotic expansion of the area of the Wiener sausage in the plane and the range of random walks, [4], [14], [13].

One of the applications of self-intersection local time is to polymer growth. If S_n is a planar random walk and $\mathbb{P}$ is its law, one can construct self-repelling and self-attracting random walks by defining

$$d\mathbb{Q}_n/d\mathbb{P} = c_n e^{\zeta B_n/n},$$

where ζ is a parameter and c_n is chosen to make $\mathbb{Q}_n$ a probability measure. When $\zeta < 0$, more weight is given to those paths with a small number of self-intersections, hence $\mathbb{Q}_n$ is a model for a self-repelling random walk. When $\zeta > 0$, more weight is given to paths with a large number of self-intersections, leading to a self-attracting random walk. Since $\mathbb{E}\, B_n$ is deterministic, by modifying c_n, we can write

$$d\mathbb{Q}_n/d\mathbb{P} = c_n e^{\zeta(B_n - \mathbb{E}\, B_n)/n}.$$

It is known that for small positive ζ the self-attracting random walk grows with n while for large ζ it "collapses," and its diameter remains bounded in mean square. It has been an open problem to determine the critical value of ζ at which the phase transition takes place. The work [2] suggested that the critical value ζ_c could be expressed in terms of the best constant of a certain Gagliardo-Nirenberg inequality, but that work was for planar Brownian motion, not for random walks. In the current paper we obtain moderate deviations estimates for $B_n - \mathbb{E}\, B_n$ and these are in terms of the best constant of the Gagliardo-Nirenberg inequality; see Theorem 1.1. However the critical constant ζ_c is different (see Remark 1.4) and it is still an open problem to determine it. See [6] and [7] for details and further information on these models.

In the present paper we study moderate deviations of $B_n - \mathbb{E}\, B_n$. Before stating our main theorem we recall one of the Gagliardo-Nirenberg inequalities:

$$\|f\|_4 \le C \|\nabla f\|_2^{1/2} \|f\|_2^{1/2},$$

which is valid for $f \in C^1$ with compact support, and can then be extended to more general f's. We define $\kappa(2,2)$ to be the infimum of those values of C for which the above inequality holds. In particular, $0 < \kappa(2,2) < \infty$. For further details, see [8].

In this paper we will always assume that the smallest group which supports $\{S_n\}$ is $\mathbb{Z}^2$. For simplicity we assume further that our random walk is strongly aperiodic.

Theorem 1.1 *Let $\{b_n\}$ be a positive sequence satisfying*

$$\lim_{n\to\infty} b_n = \infty \quad \text{and} \quad b_n = o(n). \tag{1.3}$$

For any $\lambda > 0$,

$$\lim_{n\to\infty} \frac{1}{b_n} \log \mathbb{P}\Big\{B_n - \mathbb{E}\, B_n \geq \lambda n b_n\Big\} = -\lambda\sqrt{\det\Gamma}\ \kappa(2,2)^{-4}. \tag{1.4}$$

We call Theorem 1.1 a moderate deviations theorem rather than a large deviations result for two reasons: The first reason is the second restriction in (1.3). Our techniques do not apply when this restriction is not present, and in fact it is not hard to show that the value on the right hand side of (1.4) should be different when $b_n \approx n$; see Remark 1.4. The second reason is that Theorem 1.1 is closely related to the following weak law (see Le Gall [13] and Rosen [16])

$$\frac{1}{n}(B_n - \mathbb{E}\, B_n) \xrightarrow{d} (\det\Gamma)^{-1/2}\gamma_1$$

where γ_t is the renormalized self-intersection local time of a planar Brownian motion W_t; this can be formally written as

$$\gamma_t = \iint_{0\leq r<s\leq t} \delta_0(W_r - W_s)dr\, ds - \mathbb{E} \iint_{0\leq r<s\leq t} \delta_0(W_r - W_s)dr\, ds.$$

Recently, it has been proved by the authors ([2] and [3]) that

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}\{\gamma_1 \geq t\} = -\kappa(2,2)^{-4}$$

$$\lim_{t\to\infty} t^{-2\pi} \log \mathbb{P}\{-\gamma_1 \geq \log t\} = -L$$

where $0 < L < \infty$. Theorem 1.1 could perhaps be regarded as an invariance principle linking the moderate deviations of $B_n - \mathbb{E}\, B_n$ to the large deviations for γ_1.

Moderate deviations linked to the large deviations for $-\gamma_1$ are more subtle. In the next theorem we obtain the correct rate, but not the precise constant.

Theorem 1.2 *Suppose $\mathbb{E}\,|S_1|^{2+\delta} < \infty$ for some $\delta > 0$. There exist $C_1, C_2 > 0$ such that if $b_n \to \infty$ with $b_n = o(n^{1/\theta})$ for some $\theta > 0$, then*

$$\begin{aligned} -C_1 &\leq \liminf_{n\to\infty} b_n^{-\theta} \log \mathbb{P}\Big\{ \mathbb{E}\, B_n - B_n \geq \theta (2\pi)^{-1} \det(\Gamma)^{-1/2} n \log b_n \Big\} \\ &\leq \limsup_{n\to\infty} b_n^{-\theta} \log \mathbb{P}\Big\{ \mathbb{E}\, B_n - B_n \geq \theta (2\pi)^{-1} \det(\Gamma)^{-1/2} n \log b_n \Big\} \\ &\leq -C_2. \end{aligned} \tag{1.5}$$

Here are the corresponding laws of the iterated logarithm for $B_n - \mathbb{E}\, B_n$.

Theorem 1.3

$$\limsup_{n\to\infty} \frac{B_n - \mathbb{E}\, B_n}{n \log\log n} = \det(\Gamma)^{-1/2} \kappa(2,2)^4 \quad a.s. \tag{1.6}$$

and if $\mathbb{E}\,|S_1|^{2+\delta} < \infty$ for some $\delta > 0$,

$$\liminf_{n\to\infty} \frac{B_n - \mathbb{E}\, B_n}{n \log\log\log n} = -(2\pi)^{-1} \det(\Gamma)^{-1/2} \quad a.s. \tag{1.7}$$

In this paper we deal exclusively with the case where the dimension d is 2. We note that in dimension 1 no renormalization is needed, which makes the results much simpler. See [15, 9]. When $d \geq 3$, the renormalized intersection local time is in the domain of attraction of a centered normal random variable. Consequently the tails of the weak limit are expected to be of Gaussian type, and in particular, the tails are symmetric; see [13].

Theorems 1.1-1.3 are the analogues of the theorems proved in [2] for the renormalized self-intersection local time of planar Brownian motion. Although the proofs for the random walk case have some elements in common with those for Brownian motion, the random walk case is considerably more difficult. The major difficulty is the fact that we do not have Gaussian random variables. Consequently, the argument for the lower bound of Theorem

1.1 needs to be very different from the one given in [2, Lemma 3.4]. This requires several new tools, such as Theorem 4.1, which we expect will have applications beyond the specific needs of this paper.

Remark 1.4 We remark that without the restriction that $b_n = o(n)$, Theorem 1.1 is not true. To see this, let N be an arbitrarily large integer, let $\varepsilon = 2/N^2$, and let X_i be be an i.i.d. sequence of random vectors in $\mathbb{Z}^2$ that take the values $(N,0), (-N,0), (0,N)$, and $(0,-N)$ with probability $\varepsilon/4$ and $\mathbb{P}(X_1 = (0,0)) = 1-\varepsilon$. The covariance matrix of the X_i will be the identity. Let $b_n = (1-\varepsilon)n$. Then the event that $S_i = S_0$ for all $i \le n$ will have probability at least $(1-\varepsilon)^n$, and on this event $B_n = n(n-1)/2$. This shows that

$$\log \mathbb{P}(B_n - \mathbb{E}\, B_n > nb_n/2) \ge n \log(1-\varepsilon),$$

which would contradict (1.4).

The same example shows that the critical constant in the polymer model is different than the one in [2]. Then

$$\mathbb{E}\, \exp\Big\{C\frac{B_n - \mathbb{E}\, B_n}{n}\Big\} \ge \exp\Big\{-C\frac{\mathbb{E}\, B_n}{n}\Big\}(1-\varepsilon)^n \exp\Big\{C\frac{n-1}{2}\Big\}.$$

This shows that the critical constant is no more than $2\log\frac{1}{1-\varepsilon}$.

2 Integrability

Let $\{S'_n\}$ be an independent copy of the random walk $\{S_n\}$. Let

$$I_{m,n} = \sum_{j=1}^{m}\sum_{k=1}^{n} \delta(S_j, S'_k) \tag{2.1}$$

and set $I_n = I_{n,n}$. Thus

$$I_n = \#\{(j,k) \in [1,n]^2;\ \ S_j = S'_k\}. \tag{2.2}$$

Lemma 2.1

$$\mathbb{E}\, I_{m,n} \le c\left((m+n)\log(m+n) - m\log m - n\log n\right). \tag{2.3}$$

In particular

$$\mathbb{E}\,(I_n) \le cn. \tag{2.4}$$

We also have

$$\mathbb{E}\, I_{m,n} \le c\sqrt{mn}. \tag{2.5}$$

Proof Using symmetry and independence

$$\text{(2.6)} \qquad \begin{aligned} \mathbb{E}\, I_{m,n} &= \sum_{j=1}^{m}\sum_{k=1}^{n} \mathbb{E}\,\delta(S_j, S'_k) \\ &= \sum_{j=1}^{m}\sum_{k=1}^{n} \mathbb{E}\,\delta(S_j - S'_k, 0) \\ &= \sum_{j=1}^{m}\sum_{k=1}^{n} \mathbb{E}\,\delta(S_{j+k}, 0) = \sum_{j=1}^{m}\sum_{k=1}^{n} p_{j+k}(0) \end{aligned}$$

where $p_n(a) = P(S_n = a)$. By [17, p. 75],

$$\text{(2.7)} \qquad p_m(0) = \frac{1}{2\pi\sqrt{\det \Gamma}}\,\frac{1}{m} \,+\, o\left(\frac{1}{m}\right)$$

so that

$$\text{(2.8)} \qquad \mathbb{E}\, I_{m,n} \le c \sum_{j=1}^{m}\sum_{k=1}^{n} \frac{1}{j+k} \le c \int_{r=0}^{m}\int_{s=0}^{n} \frac{1}{r+s}\, dr\, ds$$

and (2.3) follows. (2.4) is then immediate. (2.5) follows from (2.8) and the bound $(r+s)^{-1} \le (\sqrt{rs})^{-1}$. □

It follows from the proof of [8, Lemma 5.2] that for any integer $k \ge 1$

$$\text{(2.9)} \qquad \mathbb{E}\,(I_n^k) \le (k!)^2 2^k (1 + \mathbb{E}\,(I_n))^k.$$

Furthermore, by [13, (5.k)] we have that I_n/n converges in distribution to a random variable with finite moments. Hence for any integer $k \ge 1$

$$\text{(2.10)} \qquad \lim_{n\to\infty} \frac{\mathbb{E}\,(I_n^k)}{n^k} = c_k < \infty.$$

Lemma 2.2 *There is a constant $c > 0$ such that*

$$\text{(2.11)} \qquad \sup_n \mathbb{E}\,\exp\left\{\frac{c}{n} I_n\right\} < \infty.$$

Proof. We need only to show that there is a $C > 0$ such that

$$\mathbb{E}\, I_n^m \le C^m m! n^m \quad m, n \ge 1.$$

We first consider the case $m \le n$ and write $l(m,n) = [n/m] + 1$. Using [8, Theorem 5.1] with $p = 2$ and $a = m$, and then (2.4), (2.9) and (2.10), we obtain

$$
\begin{aligned}
(\mathbb{E}\, I_n^m)^{1/2} &\le \sum_{\substack{k_1+\cdots+k_m=m\\ k_1,\cdots,k_m\ge 0}} \frac{m!}{k_1!\cdots k_m!} (\mathbb{E}\, I_{l(m,n)}^{k_1})^{1/2} \cdots (\mathbb{E}\, I_{l(m,n)}^{k_m})^{1/2} \\
&\le \sum_{\substack{k_1+\cdots+k_m=m\\ k_1,\cdots,k_m\ge 0}} \frac{C^m m!}{k_1!\cdots k_m!} k_1!\cdots k_m! (\mathbb{E}\, I_{l(m,n)})^{k_1/2} \cdots (\mathbb{E}\, I_{l(m,n)})^{k_m/2} \qquad (2.12)\\
&\le \binom{2m-1}{m} m! C^m \Big(\frac{n}{m}\Big)^{m/2} \le \binom{2m}{m} m! C^m \Big(\frac{n}{m}\Big)^{m/2}
\end{aligned}
$$

where $C > 0$ can be chosen independently of m and n. Hence

$$
\mathbb{E}\, I_n^m \le \binom{2m}{m}^2 C^m (m!)^2 \Big(\frac{n}{m}\Big)^m \le \binom{2m}{m}^2 C^m m! n^m. \tag{2.13}
$$

Notice that

$$
\binom{2m}{m} \le 4^m. \tag{2.14}
$$

For the case $m > n$, notice that $I_n \le n^2$. Trivially,

$$
\mathbb{E}\, I_n^m \le n^{2m} \le m^m n^m \le C^m m! n^m,
$$

where the last step follows from Stirling's formula. $\square$

For any random variable X we define

$$
\overline{X} =: X - \mathbb{E}\, X.
$$

We write

$$
(m,n]_<^2 = \{(j,k) \in (m,n]^2;\ \ j < k\} \tag{2.15}
$$

For any $A \subset \{(j,k) \in (\mathbb{Z}^+)^2;\ \ j < k\}$, write

$$
B(A) = \sum_{(j,k)\in A} \delta(S_j, S_k). \tag{2.16}
$$

In our proofs we will use several decompositions of B_n. If $J_1, \ldots, J_\ell$ are consecutive disjoint blocks of integers whose union is $\{1, \ldots, n\}$, we have

$$
B_n = \sum_i B((J_i \times J_i) \cap (0,n]_<^2) + \sum_{i<j} B(J_i \times J_j)
$$

and also

$$B_n = \sum_i B((J_i \times J_i) \cap (0,n]^2_<) + \sum_i B(\cup_{j=1}^{i-1} J_j) \times J_i).$$

Lemma 2.3 *There is a constant $c > 0$ such that*

$$\sup_n \mathbb{E} \exp\left\{ \frac{c}{n} | \overline{B}_n | \right\} < \infty. \tag{2.17}$$

Proof. We first prove that there is $c > 0$ such that

$$M \equiv \sup_n \mathbb{E} \exp\left\{ \frac{c}{2^n} | \overline{B}_{2^n} | \right\} < \infty. \tag{2.18}$$

We have

$$\begin{aligned} (2.19) \quad & B_{2^n} \\ &= \sum_{j=1}^{n} \sum_{k=1}^{2^{j-1}} B\Big(((2k-2)2^{n-j},\ (2k-1)2^{n-j}] \times ((2k-1)2^{n-j},\ (2k)2^{n-j}] \Big) \end{aligned}$$

Write

$$\begin{aligned} (2.20) \quad \alpha_{j,k} = & B\Big(((2k-2)2^{n-j},\ (2k-1)2^{n-j}] \times ((2k-1)2^{n-j},\ (2k)2^{n-j}] \Big) \\ & - \mathbb{E}\, B\Big(((2k-2)2^{n-j},\ (2k-1)2^{n-j}] \times ((2k-1)2^{n-j},\ (2k)2^{n-j}] \Big) \end{aligned}$$

For each $1 \le j \le n$, the random variables $\alpha_{j,k},\ \ k = 1, \cdots, 2^{j-1}$ are i.i.d. with common distribution $I_{2^{n-j}} - \mathbb{E}\, I_{2^{n-j}}$. By the previous lemma there exists $\delta > 0$ such that

$$\sup_n \sup_{j \le n} \mathbb{E} \exp\left\{ \delta \frac{1}{2^{n-j}} |\alpha_{j,1}| \right\} < \infty. \tag{2.21}$$

By [3, Lemma 1], there exists $\theta > 0$ such that

$$\begin{aligned} (2.22) \qquad C(\theta) \ & \equiv \ \sup_n \sup_{j \le n} \mathbb{E} \exp\left\{ \theta 2^{j/2} \frac{1}{2^n} \Big| \sum_{k=1}^{2^{j-1}} \alpha_{j,k} \Big| \right\} \\ & = \ \sup_n \sup_{j \le n} \mathbb{E} \exp\left\{ \theta 2^{-j/2} \frac{1}{2^{n-j}} \Big| \sum_{k=1}^{2^{j-1}} \alpha_{j,k} \Big| \right\} < \infty. \end{aligned}$$

Write

$$\lambda_N = \prod_{j=1}^{N}(1-2^{-j/2}) \quad\text{and}\quad \lambda_\infty = \prod_{j=1}^{\infty}(1-2^{-j/2}). \tag{2.23}$$

Using Hölder's inequality with $1/p = 1-2^{-n/2}$, $1/q = 2^{-n/2}$ we have

$$\begin{aligned} &\mathbb{E}\,\exp\Big\{\lambda_n\frac{\theta}{2^n}\Big|\sum_{j=1}^{n}\sum_{k=1}^{2^{j-1}}\alpha_{j,k}\Big|\Big\} \\ &\le \Big(\mathbb{E}\,\exp\Big\{\lambda_{n-1}\frac{\theta}{2^n}\Big|\sum_{j=1}^{n-1}\sum_{k=1}^{2^{j-1}}\alpha_{j,k}\Big|\Big\}\Big)^{1-2^{-n/2}} \\ &\qquad\times\Big(\mathbb{E}\,\exp\Big\{2^{n/2}\lambda_n\frac{\theta}{2^n}\Big|\sum_{k=1}^{2^{n-1}}\alpha_{n,k}\Big|\Big\}\Big)^{2^{-n/2}} \\ &\le \mathbb{E}\,\exp\Big\{\lambda_{n-1}\frac{\theta}{2^n}\Big|\sum_{j=1}^{n-1}\sum_{k=1}^{2^{j-1}}\alpha_{j,k}\Big|\Big\}C(\theta)^{2^{-n/2}} \end{aligned} \tag{2.24}$$

Repeating this procedure,

$$\begin{aligned} &\mathbb{E}\,\exp\Big\{\lambda_n\frac{\theta}{2^n}\Big|\sum_{j=1}^{n}\sum_{k=1}^{2^{j-1}}\alpha_{j,k}\Big|\Big\} \\ &\le C(\theta)^{2^{-1/2}+\cdots+2^{-n/2}} \le C(\theta)^{2^{-1/2}(1-2^{-1/2})^{-1}}. \end{aligned} \tag{2.25}$$

So we have

$$\sup_n \mathbb{E}\,\exp\Big\{\lambda_\infty\frac{\theta}{2^n}|\,\overline{B}_{2^n}\,|\Big\} < \infty. \tag{2.26}$$

We now prove our lemma for general n. Given an integer $n \ge 2$, we have the following unique representation:

$$n = 2^{m_1} + 2^{m_2} + \cdots + 2^{m_l} \tag{2.27}$$

where $m_1 > m_2 > \cdots m_l \ge 0$ are integers. Write

$$n_0 = 0 \quad\text{and}\quad n_i = 2^{m_1} + \cdots + 2^{m_i}, \qquad i = 1, \cdots, l. \tag{2.28}$$

Then

$$\sum_{1\le j<k\le n}\delta(S_j,S_k) \;=\; \sum_{i=1}^{l}\sum_{n_{i-1}<j<k\le n_i}\delta(S_j,S_k) + \sum_{i=1}^{l-1}B((n_{i-1},n_i]\times(n_i,n])$$

$$=\ : \sum_{i=1}^{l} B^{(i)}_{2^{m_i}} + \sum_{i=1}^{l-1} A_i. \tag{2.29}$$

By Hölder's inequality, with M as in (2.18)

$$\begin{aligned} &\mathbb{E}\exp\Big\{\frac{c}{n}\Big|\sum_{i=1}^{l}(B^{(i)}_{2^{m_i}} - \mathbb{E}\,B^{(i)}_{2^{m_i}})\Big|\Big\} \\ &\le \prod_{i=1}^{l}\Big(\mathbb{E}\exp\Big\{\frac{c}{2^{m_i}}|B^{(i)}_{2^{m_i}} - \mathbb{E}\,B^{(i)}_{2^{m_i}}|\Big\}\Big)^{\frac{2^{m_i}}{n}} \le \prod_{i=1}^{l} M^{2^{m_i}/n} = M. \end{aligned} \tag{2.30}$$

Using Hölder's inequality,

$$\mathbb{E}\exp\Big\{\frac{c}{n}\sum_{i=1}^{l-1} A_i\Big\} \le \prod_{i=1}^{l-1}\Big(\mathbb{E}\exp\Big\{\frac{c}{2^{m_i}}A_i\Big\}\Big)^{\frac{2^{m_i}}{n}}. \tag{2.31}$$

Notice that for each $1 \le i \le l-1$,

$$A_i \overset{d}{=} \sum_{j=1}^{2^{m_i}}\sum_{k=1}^{n-n_i}\delta(S_j, S'_k) \le \sum_{j=1}^{2^{m_i}}\sum_{k=1}^{2^{m_i}}\delta(S_j, S'_k), \tag{2.32}$$

where the inequality follows from

$$n - n_i = 2^{m_{i+1}} + \cdots + 2^{m_l} \le 2^{m_i}. \tag{2.33}$$

Using (2.32) and Lemma 2.1, we can take $c > 0$ so that

$$\mathbb{E}\exp\Big\{\frac{c}{2^{m_i}}A_i\Big\} \le \sup_n \mathbb{E}\exp\Big\{\frac{c}{n}I_n\Big\} \equiv N < \infty. \tag{2.34}$$

Consequently,

$$\mathbb{E}\exp\Big\{\frac{c}{n}\sum_{i=1}^{l-1} A_i\Big\} \le \prod_{i=1}^{l-1} N^{2^{m_i}/n} \le N. \tag{2.35}$$

In particular, this shows that

$$\mathbb{E}\Big\{\frac{c}{n}\sum_{i=1}^{l-1} A_i\Big\} \le N. \tag{2.36}$$

Combining (2.35) and (2.36) with (2.30) we have

$$\sup_n \mathbb{E}\exp\Big\{\frac{c}{2n}|\overline{B}_n|\Big\} < \infty. \tag{2.37}$$

□

Lemma 2.4

$$\mathbb{E}\, B_n = \frac{1}{2\pi\sqrt{\det \Gamma}} n \log n + o(n \log n), \tag{2.38}$$

and if $\mathbb{E}\,|S_1|^{2+2\delta} < \infty$ *for some* $\delta > 0$ *then*

$$\mathbb{E}\, B_n = \frac{1}{2\pi\sqrt{\det \Gamma}} n \log n + O(n). \tag{2.39}$$

Proof.

$$\mathbb{E}\, B_n = \mathbb{E} \sum_{1 \le j < k \le n} \delta(S_j, S_k) = \sum_{1 \le j < k \le n} p_{k-j}(0) \tag{2.40}$$

where $p_m(x) = \mathbb{E}\,(S_m = x)$. If $\mathbb{E}\,|S_1|^{2+2\delta} < \infty$, then by [12, Proposition 6.7],

$$p_m(0) = \frac{1}{2\pi\sqrt{\det \Gamma}} \frac{1}{m} + o\left(\frac{1}{m^{1+\delta}}\right). \tag{2.41}$$

Since the last term is summable, it will contribute $O(n)$ to (2.40). Also,

$$\sum_{1 \le j < k \le n} \frac{1}{k-j} = \sum_{m=1}^{n} \sum_{i=1}^{n-m} \frac{1}{m} = \sum_{m=1}^{n} \frac{n-m}{m} = n \sum_{m=1}^{n} \frac{1}{m} - n \tag{2.42}$$

and our lemma follows from the well known fact that

$$\sum_{m=1}^{n} \frac{1}{m} = \log n + \gamma + O\left(\frac{1}{n}\right) \tag{2.43}$$

where γ is Euler's constant.

If we only assume finite second moments, instead of (2.41) we use (2.7) and proceed as above. □

Lemma 2.5 *For any* $\theta > 0$

$$\sup_n \mathbb{E}\, \exp\left\{\frac{\theta}{n}(\mathbb{E}\, B_n - B_n)\right\} < \infty \tag{2.44}$$

and for any $\lambda > 0$

$$\lim_{n \to \infty} \frac{1}{b_n} \log \mathbb{P}\Big\{\mathbb{E}\, B_n - B_n \ge \lambda n b_n\Big\} = -\infty. \tag{2.45}$$

Proof. By Lemma 2.3 this is true for some $\theta_o > 0$. For any $\theta > \theta_o$, take an integer $m \geq 1$ such that $\theta m^{-1} < \theta_o$. We can write any n as $n = rm + i$ with $1 \leq i < m$. Then

$$\begin{aligned}(2.46)\quad & \mathbb{E}\, B_n - B_n \\ & \leq \sum_{j=1}^{m} \Big[\mathbb{E} \sum_{(j-1)r<k,l\leq jr} \delta(S_k, S_l) - \sum_{(j-1)r<k,l\leq jr} \delta(S_k, S_l) \Big] + \mathbb{E}\, B_n - m\mathbb{E}\, B_r.\end{aligned}$$

We claim that

$$\mathbb{E}\, B_n - m\mathbb{E}\, B_r = O(n). \tag{2.47}$$

To see this, write

$$\mathbb{E}\, B_n - m\mathbb{E}\, B_r = \mathbb{E}\, B_n - \sum_{l=1}^{m} \mathbb{E}\, B(((l-1)r, lr]^2_<) \tag{2.48}$$

Notice that

$$\begin{aligned}(2.49)\qquad & B_n - \sum_{l=1}^{m} B(((l-1)r, lr]^2_<) \\ & = \sum_{l=1}^{m} B(((l-1)r, lr] \times (lr, mr]) + B((mr, n]^2_<) \\ & \qquad\qquad + B((0, mr] \times (mr, n])\end{aligned}$$

Since

$$B(((l-1)r, lr] \times (lr, mr]) \stackrel{d}{=} I_{r,(m-l)r} \tag{2.50}$$

by (2.3) we have

$$\begin{aligned}(2.51)\quad & \mathbb{E}\, B(((l-1)r, lr] \times (lr, mr]) \\ & \leq C \Big\{ (m-(l-1))r) \log(m-(l-1))r) \\ & \qquad\qquad - ((m-l)r) \log((m-l)r) - r \log r \Big\}\end{aligned}$$

Therefore

$$(2.52)\quad \sum_{l=1}^{m} \mathbb{E}\, B(((l-1)r, lr] \times (lr, mr])$$

$$
\begin{aligned}
&\leq C \sum_{l=1}^{m} \Big\{ (m-(l-1))r) \log(m-(l-1))r) \\
&\qquad\qquad\qquad\qquad -((m-l)r) \log((m-l)r) - r \log r \Big\} \\
&= C \Big\{ mr \log mr - mr \log r \Big\} = Cmr \log m.
\end{aligned}
$$

Using (2.5) for $\mathbb{E}\, B((0, mr] \times (mr, n]) = \mathbb{E}\, I_{mr,i}$ and (2.38) for $\mathbb{E}\, B((mr, n]^2_<)$ then completes the proof of (2.47).

Note that the summands in (2.46) are independent. Therefore, for some constant $C > 0$ depending only on θ and m,

$$
\mathbb{E}\, \exp\left\{ \frac{\theta}{n} (\mathbb{E}\, B_n - B_n) \right\} \leq C \Big(\mathbb{E}\, \exp\left\{ \frac{\theta}{n} (\mathbb{E}\, B_r - B_r) \right\} \Big)^m \tag{2.53}
$$

which proves (2.44), since $\theta/n \leq \theta/mr < \theta_o/r$ and $r \to \infty$ as $n \to \infty$.

Then, by Chebychev's inequality, for any fixed $h > 0$

$$
\mathbb{P}\Big\{ \mathbb{E}\, B_n - B_n \geq \lambda n b_n \Big\} \leq e^{-h\lambda b_n} \mathbb{E}\, \exp\left\{ \frac{h}{n} (\mathbb{E}\, B_n - B_n) \right\} \tag{2.54}
$$

so that by (2.44)

$$
\limsup_{n \to \infty} \frac{1}{b_n} \log \mathbb{P}\Big\{ \mathbb{E}\, B_n - B_n \geq \lambda n b_n \Big\} \leq -h\lambda. \tag{2.55}
$$

Since $h > 0$ is arbitrary, this proves (2.45). $\square$

3 Proof of Theorem 1.1

By the Gärtner-Ellis theorem ([11, Theorem 2.3.6]), we need only prove

$$
\lim_{n \to \infty} \frac{1}{b_n} \log \mathbb{E}\, \exp\left\{ \theta \sqrt{\frac{b_n}{n}} |B_n - \mathbb{E}\, B_n|^{1/2} \right\} = \frac{1}{4} \kappa(2,2)^4 \theta^2 \det(\Gamma)^{-1/2}. \tag{3.1}
$$

Indeed, by the Gärtner-Ellis theorem the above implies that

$$
\lim_{n \to \infty} \frac{1}{b_n} \log \mathbb{P}\Big\{ |B_n - \mathbb{E}\, B_n| \geq \lambda n b_n \Big\} = -\lambda \sqrt{\det(\Gamma)} \kappa(2,2)^{-4}. \tag{3.2}
$$

Using (2.45) we will then have Theorem 1.1. It thus remains to prove (3.1).

Let f be a symmetric probability density function in the Schwartz space $\mathcal{S}(\mathbb{R}^2)$ of C^∞ rapidly decreasing functions. Let $\epsilon > 0$ be a small number and write

$$f_\epsilon(x) = \epsilon^{-2} f(\epsilon^{-1}x), \qquad x \in \mathbb{R}^2 \tag{3.3}$$

and

$$l(n,x) = \sum_{k=1}^n \delta(S_k,\, x), \qquad l(n,x,\epsilon) = \sum_{k=1}^n f_{\epsilon (b_n^{-1} n)^{1/2}}(S_k - x). \tag{3.4}$$

By [8, Theorem 3.1],

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{b_n} \log \mathbb{E} \exp\Big\{ \frac{\theta}{\sqrt{2}} \sqrt{\frac{b_n}{n}} \Big(\sum_{x\in\mathbb{Z}^2} l^2(n,x,\epsilon)\Big)^{1/2} \Big\} \\ &= \sup_{g\in\mathcal{F}_2} \Big\{ \frac{\theta}{\sqrt{2}} \Big(\int_{\mathbb{R}^2} |g^2 * f_\epsilon(x)|^2 dx\Big)^{1/2} - \frac{1}{2} \int_{\mathbb{R}^2} \langle \nabla g, \Gamma \nabla g\rangle dx \Big\} \end{aligned} \tag{3.5}$$

where

$$\mathcal{F}_2 = \Big\{ g \in W^{1,2}(\mathbb{R}^2) \,|\, \; \|g\|_2 = 1 \Big\}. \tag{3.6}$$

As in the proof of [10, Theorem 1], (3.1) will follow from (3.5) and the next Theorem.

Theorem 3.1 *For any $\theta > 0$,*

$$\lim_{\epsilon\to 0} \lim_{n\to\infty} \frac{1}{b_n} \log \mathbb{E} \exp \Big\{ \theta \sqrt{\frac{b_n}{n}} |B_n - \mathbb{E}\, B_n - \frac{1}{2} \sum_{x\in\mathbb{Z}^2} l^2(n,x,\epsilon)|^{1/2} \Big\} = 0.$$

Proof. Let $l > 1$ be a large but fixed integer. Divide $[1,n]$ into l disjoint subintervals $D_1, \cdots, D_l$, each of length $[n/l]$ or $[n/l]+1$. Write

$$D_i^* = \{(j,k) \in D_i^2; \;\; j < k\} \qquad i = 1, \cdots, l. \tag{3.7}$$

With the notation of (2.16) we have

$$B_n = \sum_{i=1}^l B(D_i^*) + \sum_{1\le j<k\le l} B(D_j \times D_k). \tag{3.8}$$

Define a_j, b_j so that $D_j = (a_j, b_j]$ $(1 \le j \le l)$. Notice that

$$\begin{aligned}
B(D_j \times D_k) &= \sum_{n_1 \in D_j, n_2 \in D_k} \delta(S_{n_1}, S_{n_2}) \\
&= \sum_{n_1 \in D_j, n_2 \in D_k} \delta((S_{n_1} - S_{b_j}) + S_{b_j}, S_{a_k} + (S_{n_2} - S_{a_k})) \\
&= \sum_{n_1 \in D_j, n_2 \in D_k} \delta((S_{n_1} - S_{b_j}), Z + (S_{n_2} - S_{a_k})) \qquad (3.9)
\end{aligned}$$

with $Z \overset{d}{=} S_{a_k} - S_{b_j}$, so that $Z, S_{n_1} - S_{b_j}, S_{n_2} - S_{a_k}$ are independent. Then as in (2.6)

$$\begin{aligned}
\mathbb{E}\, B(D_j \times D_k) &= \mathbb{E} \sum_{n_1 \in D_j, n_2 \in D_k} p_{b_j - n_1 + n_2 - a_k}(Z) \qquad (3.10) \\
&\le \sum_{n_1 \in D_j, n_2 \in D_k} p_{b_j - n_1 + n_2 - a_k}(0)
\end{aligned}$$

since $\sup_x p_j(x) = p_j(0)$ for a symmetric random walk. Then as in the proof of (2.4) we have that

$$\mathbb{E}\, B(D_j \times D_k) \le cn/l. \qquad (3.11)$$

Hence,

$$\begin{aligned}
&(3.12)\ B_n - \mathbb{E}\, B_n \\
&= \sum_{i=1}^{l} [B(D_i^*) - \mathbb{E}\, B(D_i^*)] + \sum_{1 \le j < k \le l} B(D_j \times D_k) - \mathbb{E} \sum_{1 \le j < k \le l} B(D_j \times D_k) \\
&= \sum_{i=1}^{l} [B(D_i^*) - \mathbb{E}\, B(D_i^*)] + \sum_{1 \le j < k \le l} B(D_j \times D_k) + O(n)
\end{aligned}$$

where the last line follows from (3.11).

Write

$$\xi_i(n, x, \epsilon) = \sum_{k \in D_i} f_{\epsilon (b_n^{-1} n)^{1/2}}(S_k - x). \qquad (3.13)$$

Then

$$\sum_{x \in \mathbb{Z}^2} l^2(n, x, \epsilon) = \sum_{i=1}^{l} \sum_{x \in \mathbb{Z}^2} \xi_i^2(n, x, \epsilon) + 2 \sum_{1 \le j \le k \le l} \sum_{x \in \mathbb{Z}^2} \xi_j(n, x, \epsilon) \xi_k(n, x, \epsilon).$$

(3.14)

Therefore, by (3.12)

$$
\begin{aligned}
(3.15)\qquad & \Big|(B_n - \mathbb{E}\, B_n) - \frac{1}{2}\sum_{x\in\mathbb{Z}^2} l^2(n,x,\epsilon)\Big| \\
&\le \sum_{i=1}^{l} |B(D_i^*) - \mathbb{E}\, B(D_i^*)| + \frac{1}{2}\sum_{i=1}^{l}\sum_{x\in\mathbb{Z}^2}\xi_i^2(n,x,\epsilon) \\
&+ \sum_{1\le j<k\le l}\Big|B(D_j\times D_k) - \sum_{x\in\mathbb{Z}^2}\xi_j(n,x,\epsilon)\xi_k(n,x,\epsilon)\Big| + O(n).
\end{aligned}
$$

The proof of Theorem 3.1 is completed in the next two lemmas.

Lemma 3.2 *For any $\theta > 0$,*

$$
\begin{aligned}
(3.16)\qquad & \limsup_{n\to\infty}\frac{1}{b_n}\log \mathbb{E}\,\exp\Big\{\theta\sqrt{\frac{b_n}{n}}\Big(\sum_{i=1}^{l}\sum_{x\in\mathbb{Z}^2}\xi_i^2(n,x,\epsilon)\Big)^{1/2}\Big\} \\
&\le l^{-1}\frac{1}{2}\kappa(2,2)^4\theta^2\det(\Gamma)^{-1/2}
\end{aligned}
$$

and

$$
(3.17)\quad \limsup_{n\to\infty}\frac{1}{b_n}\log\mathbb{E}\,\exp\Big\{\theta\sqrt{\frac{b_n}{n}}\Big(\sum_{i=1}^{l}|B(D_i^*) - \mathbb{E}\, B(D_i^*)|\Big)^{1/2}\Big\} \le l^{-1}H\theta^2,
$$

where

$$
(3.18)\quad H = \Big(\sup\Big\{\lambda > 0;\ \ \sup_n \mathbb{E}\,\exp\Big\{\lambda\frac{1}{n}|B_n - \mathbb{E}\, B_n|\Big\} < \infty\Big\}\Big)^{-1}.
$$

Proof. Replacing θ by $\theta/\sqrt{l}$, n by n/l, and b_n by $b_n^* = b_{ln}$ (notice that $b_{n/l}^* = b_n$)

$$
\begin{aligned}
(3.19)\qquad & \limsup_{n\to\infty}\frac{1}{b_n}\log\mathbb{E}\,\exp\Big\{\theta\sqrt{\frac{b_n}{n}}\Big(\sum_{x\in\mathbb{Z}^2}\xi_i^2(n,x,\epsilon)\Big)^{1/2}\Big\} \\
&= \limsup_{n\to\infty}\frac{1}{b_{n/l}^*}\log\mathbb{E}\,\exp\Big\{\frac{\theta}{\sqrt{l}}\sqrt{\frac{b_{n/l}^*}{n/l}}\Big(\sum_{x\in\mathbb{Z}^2}\xi_i^2(n,x,\epsilon)\Big)^{1/2}\Big\}.
\end{aligned}
$$

Applying Jensen's inequality on the right hand side of (3.5),

$$\begin{aligned}\int_{\mathbb{R}^2}|g^2*f_\varepsilon(x)|^2 &= \int_{\mathbb{R}^2}\Big[\int_{\mathbb{R}^2} g^2(x-y)f_\varepsilon(y)\,dy\Big]^2 dx\\ &\le \int\int g^4(x-y)f_\varepsilon(y)\,dy\,dx = \int f_\varepsilon(y)\Big[\int g^4(x-y)\,dx\Big]\,dy\\ &= \Big[\int g^4(x)\,dx\Big]\int f_\varepsilon(y)\,dy = \int_{\mathbb{R}^2} g^4(y)\,dy.\end{aligned}$$

Combining the last two displays with (3.5) we have that

$$\begin{aligned}(3.20)\quad &\limsup_{n\to\infty}\frac{1}{b_n}\log\mathbb{E}\,\exp\Big\{\theta\sqrt{\frac{b_n}{n}}\Big(\sum_{x\in\mathbb{Z}^2}\xi_i^2(n,x,\epsilon)\Big)^{1/2}\Big\}\\ &\le \sup_{g\in\mathcal{F}_2}\Big\{\frac{\theta}{\sqrt{l}}\Big(\int_{\mathbb{R}^2}|g(x)|^4dx\Big)^{1/2}-\frac{1}{2}\int_{\mathbb{R}^2}\langle\nabla g(x),\Gamma\nabla g(x)\rangle dx\Big\}\\ &= l^{-1}\theta^2\det(\Gamma)^{-1/2}\sup_{h\in\mathcal{F}_2}\Big\{\Big(\int_{\mathbb{R}^2}|h(x)|^4dx\Big)^{1/2}-\frac{1}{2}\int_{\mathbb{R}^2}|\nabla h(x)|^2dx\Big\}\\ &= \frac{1}{2}l^{-1}\det(\Gamma)^{-1/2}\kappa(2,2)^4\theta^2,\end{aligned}$$

where the third line follows from the substitution $g(x)=\sqrt{|\det(A)|}f(Ax)$ with a 2×2 matrix A satisfying

$$(3.21)\qquad A^\tau\Gamma A = l^{-1}\theta^2\det(\Gamma)^{-1/2}\mathbf{I}_2$$

and the last line of [8, Lemma A.2]; here $\mathbf{I}_2$ is the 2×2 identity matrix.

Given $\delta>0$, there exist $\overline{a}_1=(a_{1,1},\cdots,a_{1,l}),\cdots,\overline{a}_m=(a_{m,1},\cdots,a_{m,l})$ in $\mathbb{R}^l$ such that $|\overline{a}_1|=\cdots=|\overline{a}_m|=1$ and

$$(3.22)\qquad |z|\le(1+\delta)\max\{\overline{a}_1\cdot z,\cdots,\overline{a}_m\cdot z\},\qquad z\in\mathbb{R}^l.$$

In particular, with

$$(3.23)\qquad z=\Big(\Big(\sum_{x\in\mathbb{Z}^2}\xi_1^2(n,x,\epsilon)\Big)^{1/2},\dots,\Big(\sum_{x\in\mathbb{Z}^2}\xi_l^2(n,x,\epsilon)\Big)^{1/2}\Big)$$

we have

$$(3.24)\quad \Big(\sum_{i=1}^l\sum_{x\in\mathbb{Z}^2}\xi_i^2(n,x,\epsilon)\Big)^{1/2}\le(1+\delta)\max_{1\le j\le m}\sum_{i=1}^l a_{j,i}\Big(\sum_{x\in\mathbb{Z}^2}\xi_i^2(n,x,\epsilon)\Big)^{1/2}.$$

Hence

$$\begin{aligned}
(3.25)\qquad & \mathbb{E}\,\exp\Big\{\theta\sqrt{\frac{b_n}{n}}\Big(\sum_{i=1}^{l}\sum_{x\in\mathbb{Z}^2}\xi_i^2(n,x,\epsilon)\Big)^{1/2}\Big\}\\
&\le \sum_{j=1}^{m}\mathbb{E}\,\exp\Big\{\theta\sqrt{\frac{b_n}{n}}(1+\delta)\sum_{i=1}^{l}a_{j,i}\Big(\sum_{x\in\mathbb{Z}^2}\xi_i^2(n,x,\epsilon)\Big)^{1/2}\Big\}\\
&= \sum_{j=1}^{m}\prod_{i=1}^{l}\mathbb{E}\,\exp\Big\{\theta\sqrt{\frac{b_n}{n}}(1+\delta)a_{j,i}\Big(\sum_{x\in\mathbb{Z}^2}\xi_i^2(n,x,\epsilon)\Big)^{1/2}\Big\},
\end{aligned}$$

where the last line follows from independence of $\|\xi_i(n,x,\epsilon)\|_{L^2(\mathbb{Z}^2)}$, $i=1,\dots,l$. Therefore

$$\begin{aligned}
(3.26)\qquad & \limsup_{n\to\infty}\frac{1}{b_n}\log\mathbb{E}\,\exp\Big\{\theta\sqrt{\frac{b_n}{n}}\Big(\sum_{k=1}^{l}\sum_{x\in\mathbb{Z}^2}\xi_k^2(n,x,\epsilon)\Big)^{1/2}\Big\}\\
&\le \max_{1\le j\le m}\frac{1}{2}l^{-1}\kappa(2,2)^4(1+\delta)^2\theta^2\Big(\sum_{i=1}^{l}a_{j,i}^2\Big)\\
&= \frac{1}{2}l^{-1}\det(\Gamma)^{-1/2}\kappa(2,2)^4(1+\delta)^2\theta^2.
\end{aligned}$$

Letting $\delta\to 0^+$ proves (3.16).

By the inequality $ab\le a^2+b^2$ we have that

$$\begin{aligned}
(3.27)\qquad & \mathbb{E}\,\exp\Big\{\theta\sqrt{\frac{b_n}{n}}|B_n-\mathbb{E}\,B_n|^{1/2}\Big\}\\
&\le \exp\{c^2\theta^2 b_n\}\mathbb{E}\,\exp\Big\{c^{-2}\frac{1}{n}|B_n-\mathbb{E}\,B_n|\Big\},
\end{aligned}$$

and taking $c^{-2}\uparrow H^{-1}$ we see that for any $\theta>0$,

$$(3.28)\qquad \limsup_{n\to\infty}\frac{1}{b_n}\log\mathbb{E}\,\exp\Big\{\theta\sqrt{\frac{b_n}{n}}|B_n-\mathbb{E}\,B_n|^{1/2}\Big\}\le H\theta^2.$$

Notice that for any $1\le i\le l$,

$$(3.29)\qquad B(D_i^*)-\mathbb{E}\,B(D_i^*)\overset{d}{=}B_{\#(D_i)}-\mathbb{E}\,B_{\#(D_i)}.$$

We have

$$\mathbb{E}\,\exp\Big\{\theta\sqrt{\frac{b_n}{n}}|B(D_i^*)-\mathbb{E}\,B(D_i^*)|^{1/2}\Big\}=\mathbb{E}\,\exp\Big\{\frac{\theta}{\sqrt{l}}\sqrt{\frac{b_n}{n/l}}|B(D_i^*-\mathbb{E}\,B(D_i^*)|^{1/2}\Big\}.$$

Replacing θ by $\theta/\sqrt{l}$, n by n/l, and b_n by $b_n^*=b_{ln}$ (notice that $b_{n/l}^*=b_n$) gives

$$\limsup_{n\to\infty}\frac{1}{b_n}\log\mathbb{E}\,\exp\Big\{\theta\sqrt{\frac{b_n}{n}}|B(D_i^*)-\mathbb{E}\,B(D_i^*)|^{1/2}\Big\}\le l^{-1}H\theta^2. \tag{3.30}$$

Thus (3.17) follows by the same argument we used to prove (3.16). □

Lemma 3.3 *For any $\theta>0$ and any $1\le j<k\le l$,*

$$\limsup_{\epsilon\to 0^+}\limsup_{n\to\infty}\frac{1}{b_n}\log\mathbb{E}\,\exp\Big\{\theta\sqrt{\frac{b_n}{n}}\Big|B(D_j\times D_k)-\sum_{x\in\mathbb{Z}^2}\xi_j(n,x,\epsilon)\xi_k(n,x,\epsilon)\Big|^{1/2}\Big\}=0. \tag{3.31}$$

Proof. We now fix $1\le j<k\le l$ and estimate

$$B(D_j\times D_k)-\sum_{x\in\mathbb{Z}^2}\xi_j(n,x,\epsilon)\xi_k(n,x,\epsilon). \tag{3.32}$$

Without loss of generality we may assume that $v=:[n/l]=\#(D_j)=\#(D_k)$. For $y\in Z^2$ set

$$I_n(y)=\sum_{n_1,n_2=1}^{n}\delta(S_{n_1},S'_{n_2}+y). \tag{3.33}$$

Note that $I_n=I_n(0)$. By (3.9) we have that

$$B(D_j\times D_k)\stackrel{d}{=}I_v(Z) \tag{3.34}$$

with Z independent of S,S'.

Similarly, we have

$$\sum_{x\in\mathbb{Z}^2}\xi_j(n,x,\epsilon)\xi_k(n,x,\epsilon)$$

$$= \sum_{x\in\mathbb{Z}^2} \sum_{n_1\in D_j, n_2\in D_k} f_{\epsilon(b_n^{-1}n)^{1/2}}(S_{n_1}-x) f_{\epsilon(b_n^{-1}n)^{1/2}}(S_{n_2}-x)$$

$$= \sum_{x\in\mathbb{Z}^2} \sum_{n_1\in D_j, n_2\in D_k} f_{\epsilon(b_n^{-1}n)^{1/2}}(x) f_{\epsilon(b_n^{-1}n)^{1/2}}(S_{n_2}-S_{n_1}-x)$$

$$= \sum_{n_1\in D_j, n_2\in D_k} f_{\epsilon(b_n^{-1}n)^{1/2}} \circledast f_{\epsilon(b_n^{-1}n)^{1/2}}(S_{n_2}-S_{n_1})$$

$$= \sum_{n_1\in D_j, n_2\in D_k} f_{\epsilon(b_n^{-1}n)^{1/2}} \circledast f_{\epsilon(b_n^{-1}n)^{1/2}}((S_{n_2}-S_{a_k})-(S_{n_1}-S_{b_j})+Z) \tag{3.35}$$

where

$$f \circledast f(y) = \sum_{x\in\mathbb{Z}^2} f(x) f(y-x) \tag{3.36}$$

denotes convolution in $L^1(\mathbb{Z}^2)$. It is clear that if $f\in\mathcal{S}(\mathbb{R}^2)$ so is $f\circledast f$. For $y\in Z^2$, define the link

$$L_{n,\epsilon}(y) = \sum_{n_1,n_2=1}^{n} f_\epsilon \circledast f_\epsilon(S'_{n_2}-S_{n_1}+y). \tag{3.37}$$

By (3.35) we have that

$$\sum_{x\in\mathbb{Z}^2} \xi_j(n,x,\epsilon)\xi_k(n,x,\epsilon) \stackrel{d}{=} L_{v,(b_n^{-1}n)^{1/2}\epsilon}(Z) \tag{3.38}$$

with Z independent of S, S'.

Lemma 3.4 *Let $f\in\mathcal{S}(\mathbb{R}^2)$ with Fourier transform $\widehat{f}$ supported on $(-\pi,\pi)^2$. Then for any $r\geq 1$*

$$\int e^{-i\lambda y}(f_r\circledast f_r)(y)\,dy = (\widehat{f}(r\lambda))^2, \quad \forall\lambda\in\mathbb{R}^2. \tag{3.39}$$

Proof. We have

$$\begin{aligned}\int e^{-i\lambda y}(f\circledast f)(y)\,dy &= \sum_{x\in\mathbb{Z}^2} f(x)\int e^{-i\lambda y} f(y-x)\,dy \\ &= \widehat{f}(\lambda)\sum_{x\in\mathbb{Z}^2} f(x)e^{-i\lambda x} \\ &= \widehat{f}(\lambda)\sum_{x\in\mathbb{Z}^2}\left(\frac{1}{(2\pi)^2}\int e^{ipx}\widehat{f}(p)\,dp\right)e^{-i\lambda x}.\end{aligned} \tag{3.40}$$

For $x \in \mathbb{Z}^2$

$$\int e^{ipx}\widehat{f}(p)\,dp = \sum_{u\in\mathbb{Z}^2}\int_{[-\pi,\pi]^2} e^{ipx}\widehat{f}(p+2\pi u)\,dp \tag{3.41}$$

and using Fourier inversion

$$\begin{aligned} &\sum_{x\in\mathbb{Z}^2}\left(\int e^{ipx}\widehat{f}(p)\,dp\right)e^{-i\lambda x} \\ &= \sum_{u\in\mathbb{Z}^2}\sum_{x\in\mathbb{Z}^2}\left(\int_{[-\pi,\pi]^2} e^{ipx}\widehat{f}(p+2\pi u)\,dp\right)e^{-i\lambda x} \\ &= (2\pi)^2\sum_{u\in\mathbb{Z}^2}\widehat{f}(\lambda+2\pi u). \end{aligned} \tag{3.42}$$

Thus from (3.40) we find that

$$\int e^{-i\lambda y} f \circledast f(y)\,dy = \widehat{f}(\lambda)\sum_{u\in\mathbb{Z}^2}\widehat{f}(\lambda+2\pi u). \tag{3.43}$$

Since $\widehat{f_r}(\lambda) = \widehat{f}(r\lambda)$ we see that for any $r > 0$

$$\int e^{-i\lambda y}(f_r \circledast f_r)(y)\,dy = \widehat{f}(r\lambda)\sum_{u\in\mathbb{Z}^2}\widehat{f}(r\lambda+2\pi r u). \tag{3.44}$$

Then if $r \geq 1$, using the fact that $\widehat{f}(\lambda)$ is supported in $(-\pi,\pi)^2$, we obtain (3.39). $\square$

Taking $f \in \mathcal{S}(\mathbb{R}^2)$ with $\widehat{f}(\lambda)$ supported in $(-\pi,\pi)^2$, Lemma 3.3 will follow from Theorem 4.1 of the next section. $\square$

4 Intersections of Random Walks

Let $S_1(n), S_2(n)$ be independent copies of the symmetric random walk $S(n)$ in $\mathbb{Z}^2$ with a finite second moment.

Let f be a positive symmetric function in the Schwartz space $\mathcal{S}(\mathbb{R}^2)$ with $\int f\,dx = 1$ and $\widehat{f}$ supported in $(-\pi,\pi)^2$. Given $\epsilon > 0$, and with the notation of the last section, let us define the link

$$I_{n,\epsilon}(y) = \sum_{n_1,n_2=1}^{n} f_{(b_n^{-1}n)^{1/2}\epsilon} \circledast f_{(b_n^{-1}n)^{1/2}\epsilon}(S_2(n_2) - S_1(n_1) + y)) \tag{4.1}$$

with $I_{n,\epsilon} = I_{n,\epsilon}(0)$.

Theorem 4.1 *For any* $\lambda > 0$

$$\limsup_{\epsilon\to 0} \limsup_{n\to\infty} \sup_y \frac{1}{b_n} \log \mathbb{E}\left(\exp\left\{ \lambda \left| \frac{I_n(y) - I_{n,\epsilon}(y)}{b_n^{-1} n} \right|^{1/2} \right\} \right) = 0. \tag{4.2}$$

Proof of Theorem 4.1. We have

$$\begin{aligned} \frac{1}{b_n^{-1} n} I_n(y) &= \frac{1}{b_n^{-1} n} \sum_{n_1,n_2=1}^{n} \delta(S_1(n_1), S_2(n_2) + y) \\ &= \frac{1}{b_n^{-1} n (2\pi)^2} \sum_{n_1,n_2=1}^{n} \Big[\int_{[-\pi,\pi]^2} e^{ip\cdot(S_2(n_2)+y-S_1(n_1))} \, dp \Big] \end{aligned} \tag{4.3}$$

where from now on we work modulo $\pm\pi$. Then by scaling we have

$$\begin{aligned} \frac{1}{b_n^{-1} n} I_n(y) &= \frac{1}{(b_n^{-1} n)^2 (2\pi)^2} \sum_{n_1,n_2=1}^{n} \Big[\int_{(b_n^{-1} n)^{1/2}[-\pi,\pi]^2} e^{ip\cdot(S_2(n_2)+y-S_1(n_1))/(b_n^{-1} n)^{1/2}} \, dp \Big]. \end{aligned} \tag{4.4}$$

As in (4.3)-(4.4), using Lemma 3.4, the fact that $\epsilon(b_n^{-1} n)^{1/2} \geq 1$ for $\epsilon > 0$ fixed and large enough n, and abbreviating $\widehat{h} = (\,\widehat{f}\,)^2$

$$\begin{aligned} \frac{1}{b_n^{-1} n} I_{n,\epsilon}(y) &= \frac{1}{b_n^{-1} n (2\pi)^2} \sum_{n_1,n_2=1}^{n} \Big[\int_{\mathbb{R}^2} e^{ip\cdot(S_2(n_2)+y-S_1(n_1))} \, \widehat{h}(\epsilon (b_n^{-1} n)^{1/2} p) \, dp \Big] \\ &= \frac{1}{(b_n^{-1} n)^2 (2\pi)^2} \sum_{n_1,n_2=1}^{n} \Big[\int_{\mathbb{R}^2} e^{ip\cdot(S_2(n_2)+y-S_1(n_1))/(b_n^{-1} n)^{1/2}} \, \widehat{h}(\epsilon p) \, dp \Big]. \end{aligned} \tag{4.5}$$

Using our assumption that $\widehat{h}$ supported in $[-\pi,\pi]^2$, and that $\epsilon^{-1} \le (b_n^{-1}n)^{1/2}$ for $\epsilon > 0$ fixed and large enough n, we have that

$$\begin{aligned}
&\frac{1}{b_n^{-1}n} I_{n,\epsilon}(y) \\
&= \frac{1}{(b_n^{-1}n)^2(2\pi)^2} \sum_{n_1,n_2=1}^{n} \\
&\qquad \Big[\int_{\epsilon^{-1}[-\pi,\pi]^2} e^{ip\cdot(S_2(n_2)+y-S_1(n_1))/(b_n^{-1}n)^{1/2}}\, \widehat{h}(\epsilon p)\, dp\Big] \\
&= \frac{1}{(b_n^{-1}n)^2(2\pi)^2} \sum_{n_1,n_2=1}^{n} \\
&\qquad \Big[\int_{(b_n^{-1}n)^{1/2}[-\pi,\pi]^2} e^{ip\cdot(S_2(n_2)+y-S_1(n_1))/(b_n^{-1}n)^{1/2}}\, \widehat{h}(\epsilon p)\, dp\Big].
\end{aligned} \tag{4.6}$$

To prove (4.2) it suffices to show that for each $\lambda > 0$ we have

$$\begin{aligned}
&\sup_y \mathbb{E}\left(\exp\left\{\lambda\left|\frac{I_n(y) - I_{n,\epsilon}(y)}{b_n^{-1}n}\right|^{1/2}\right\}\right) \\
&\le C b_n (1 - C\lambda\epsilon^{m/4})^{-1}(1 + C\lambda\epsilon^{1/4}b_n^{1/2}) e^{C\lambda^2\epsilon^{1/2}b_n}.
\end{aligned} \tag{4.7}$$

for some $C < \infty$ and all $\epsilon > 0$ sufficiently small.

We begin by expanding

$$\begin{aligned}
&\mathbb{E}\left(\exp\left\{\lambda\left|\frac{I_n(y) - I_{n,\epsilon}(y)}{b_n^{-1}n}\right|^{1/2}\right\}\right) \\
&= \sum_{m=0}^{\infty} \frac{\lambda^m}{m!} \mathbb{E}\left(\left|\frac{1}{b_n^{-1}n}(I_n(y) - I_{n,\epsilon}(y))\right|^{m/2}\right) \\
&\le \sum_{m=0}^{\infty} \frac{\lambda^m}{m!} \left(\mathbb{E}\left(\left\{\frac{1}{b_n^{-1}n}(I_n(y) - I_{n,\epsilon}(y))\right\}^{2m}\right)\right)^{1/4}.
\end{aligned} \tag{4.8}$$

By (4.4), (4.6) and the symmetry of S_1 we have

$$\mathbb{E}\left(\left\{\frac{1}{b_n^{-1}n}(I_n(y) - I_{n,\epsilon}(y))\right\}^{m}\right) \tag{4.9}$$

$$= \frac{1}{(b_n^{-1}n)^{2m}(2\pi)^{2m}} \sum_{\substack{n_{1,j},n_{2,j}=1\\ j=1,\dots,m}}^{n} \int_{(b_n^{-1}n)^{1/2}[-\pi,\pi]^{2m}}$$

$$\mathbb{E}\left(e^{i\sum_{j=1}^m p_j\cdot(S_2(n_{2,j})+y+S_1(n_{1,j}))/(b_n^{-1}n)^{1/2}}\right)\prod_{j=1}^m(1-\widehat{h}(\epsilon p_j))\,dp_j.$$

Then

$$\left|\mathbb{E}\left(\left\{\frac{1}{b_n^{-1}n}(I_n(y)-I_{n,\epsilon}(y))\right\}^m\right)\right| \tag{4.10}$$

$$\le \frac{1}{(b_n^{-1}n)^{2m}(2\pi)^{2m}} \sum_{\substack{n_{1,j}=1\\ j=1,\dots,m}}^{n} \sum_{\substack{n_{2,j}=1\\ j=1,\dots,m}}^{n} \int_{(b_n^{-1}n)^{1/2}[-\pi,\pi]^{2m}}$$

$$\left|\mathbb{E}\left(e^{i\sum_{j=1}^m p_j\cdot S_1(n_{1,j})/(b_n^{-1}n)^{1/2}}\right)\right|$$

$$\left|\mathbb{E}\left(e^{i\sum_{j=1}^m p_j\cdot S_2(n_{2,j})/(b_n^{-1}n)^{1/2}}\right)\right|\prod_{j=1}^m|1-\widehat{h}(\epsilon p_j)|\,dp_j.$$

By the Cauchy-Schwarz inequality

$$\int_{(b_n^{-1}n)^{1/2}[-\pi,\pi]^{2m}}\left|\mathbb{E}\left(e^{i\sum_{j=1}^m p_j\cdot S_1(n_{1,j})/(b_n^{-1}n)^{1/2}}\right)\right| \tag{4.11}$$

$$\left|\mathbb{E}\left(e^{i\sum_{j=1}^m p_j\cdot S_2(n_{2,j})/(b_n^{-1}n)^{1/2}}\right)\right|\prod_{j=1}^m|1-\widehat{h}(\epsilon p_j)|\,dp_j$$

$$\le \prod_{i=1}^2\left\{\int_{(b_n^{-1}n)^{1/2}[-\pi,\pi]^{2m}}\right.$$

$$\left.\left|\mathbb{E}\left(e^{i\sum_{j=1}^m p_j\cdot S(n_{i,j})/(b_n^{-1}n)^{1/2}}\right)\right|^2\prod_{j=1}^m|1-\widehat{h}(\epsilon p_j)|\,dp_j\right\}^{1/2}. \tag{4.12}$$

Thus

$$\left|\mathbb{E}\left(\left\{\frac{1}{b_n^{-1}n}(I_n(y)-I_{n,\epsilon}(y))\right\}^m\right)\right|^{1/2} \tag{4.13}$$

$$\le \sum_{\substack{n_j=1\\ j=1,\dots,m}}^{n} \frac{1}{(b_n^{-1}n)^m(2\pi)^m}\left\{\int_{(b_n^{-1}n)^{1/2}[-\pi,\pi]^{2m}}\right.$$

$$\left|\mathbb{E}\left(e^{i\sum_{j=1}^m p_j\cdot S(n_j)/(b_n^{-1}n)^{1/2}}\right)\right|^2\prod_{j=1}^m|1-\widehat{h}(\epsilon p_j)|\,dp_j\Bigg\}^{1/2}.$$

For any permutation ψ of $\{1\dots,m\}$ let

$$D_m(\psi)=\{(n_1,\dots,n_m)|\,1\le n_{\psi(1)}\le\cdots\le n_{\psi(m)}\le n\}. \tag{4.14}$$

Using the (non-disjoint) decomposition

$$\{1,\dots,n\}^m=\bigcup_{\pi}D_m(\psi)$$

we have from (4.13) that

$$\begin{aligned}&\left|\mathbb{E}\left(\left\{\frac{1}{b_n^{-1}n}(I_n(y)-I_{n,\epsilon}(y))\right\}^m\right)\right|^{1/2} \\ &\le\sum_{\psi}\sum_{D_m(\psi)}\frac{1}{(b_n^{-1}n)^m(2\pi)^m}\Bigg\{\int_{(b_n^{-1}n)^{1/2}[-\pi,\pi]^{2m}} \\ &\qquad\left|\mathbb{E}\left(e^{i\sum_{j=1}^m p_j\cdot S(n_j)/(b_n^{-1}n)^{1/2}}\right)\right|^2\prod_{j=1}^m|1-\widehat{h}(\epsilon p_j)|\,dp_j\Bigg\}^{1/2}.\end{aligned} \tag{4.15}$$

where the first sum is over all permutations ψ of $\{1\dots,m\}$.

Set

$$\phi(u)=\mathbb{E}\left(e^{iu\cdot S(1)}\right). \tag{4.16}$$

It follows from our assumptions that $\phi(u)\in C^2$, $\frac{\partial}{\partial u_i}\phi(0)=0$ and $\frac{\partial^2}{\partial u_i\partial u_j}\phi(0)=-E\left(S_{(i)}(1)S_{(j)}(1)\right)$ where $S(1)=(S_{(1)}(1),S_{(2)}(1))$ so that for some $\delta>0$

$$\phi(u)=1-E\left((u\cdot S(1))^2\right)/2+o(|u|^2),\qquad |u|\le\delta. \tag{4.17}$$

Then for some $c_1>0$

$$\phi(u)\le e^{-c_1|u|^2},\qquad |u|\le\delta. \tag{4.18}$$

Strong aperiodicity implies that $|\phi(u)|<1$ for $u\ne0$ and $u\in[-\pi,\pi]^2$. In particular, we can find $b<1$ such that $|\phi(u)|\le b$ for $\delta\le|u|$ and $u\in[-\pi,\pi]^2$. But clearly we can choose $c_2>0$ so that $b\le e^{-c_2|u|^2}$ for $u\in[-\pi,\pi]^2$. Setting $c=\min(c_1,c_2)>0$ we then have

$$\phi(u)\le e^{-c|u|^2},\qquad u\in[-\pi,\pi]^2. \tag{4.19}$$

On $D_m(\psi)$ we can write

$$\sum_{j=1}^{m} p_j \cdot S(n_j) = \sum_{j=1}^{m} (\sum_{i=j}^{m} p_{\psi(i)})(S(n_{\psi(j)}) - S(n_{\psi(j-1)})). \tag{4.20}$$

Hence on $D_m(\psi)$

$$\mathbb{E}\left(e^{i\sum_{j=1}^{m} p_j \cdot S(n_j)/(b_n^{-1}n)^{1/2}}\right) = \prod_{j=1}^{m} \phi((\sum_{i=j}^{m} p_{\psi(i)})/(b_n^{-1}n)^{1/2})^{(n_{\psi(j)}-n_{\psi(j-1)})}. \tag{4.21}$$

Now it is clear that

$$\begin{aligned}
&\sum_{D_m(\psi)} \left\{ \int_{(b_n^{-1}n)^{1/2}[-\pi,\pi]^{2m}} \right. \\
&\qquad \left. \left| \prod_{j=1}^{m} \phi((\sum_{i=j}^{m} p_{\psi(i)})/(b_n^{-1}n)^{1/2})^{(n_{\psi(j)}-n_{\psi(j-1)})} \right|^2 \prod_{j=1}^{m} |1-\widehat{h}(\epsilon p_j)|\, dp_j \right\}^{1/2} \\
&= \sum_{1\le n_{\psi(1)}\le \cdots \le n_{\psi(m)}\le n} \left\{ \int_{(b_n^{-1}n)^{1/2}[-\pi,\pi]^{2m}} \right. \\
&\qquad \left. \left| \prod_{j=1}^{m} \phi((\sum_{i=j}^{m} p_{\psi(i)})/(b_n^{-1}n)^{1/2})^{(n_{\psi(j)}-n_{\psi(j-1)})} \right|^2 \prod_{j=1}^{m} |1-\widehat{h}(\epsilon p_j)|\, dp_j \right\}^{1/2}
\end{aligned} \tag{4.22}$$

is independent of the permutation ψ. Hence writing

$$u_j = \sum_{i=j}^{m} p_i \tag{4.23}$$

we have from (4.15) that

$$\begin{aligned}
&\left| \mathbb{E}\left(\left\{ \frac{1}{b_n^{-1}n} (I_n(y) - I_{n,\epsilon}(y)) \right\}^m \right) \right|^{1/2} \\
&\le m! \sum_{1\le n_1\le \cdots \le n_m\le n} \frac{1}{(b_n^{-1}n)^m (2\pi)^m} \left\{ \int_{(b_n^{-1}n)^{1/2}[-\pi,\pi]^{2m}} \right. \\
&\qquad \left. \left| \prod_{j=1}^{m} \phi(u_j/(b_n^{-1}n)^{1/2})^{(n_j-n_{j-1})} \right|^2 \prod_{j=1}^{m} |1-\widehat{h}(\epsilon p_j)|\, dp_j \right\}^{1/2}.
\end{aligned} \tag{4.24}$$

For each $A \subseteq \{2,3,\ldots,m\}$ we use $D_m(A)$ to denote the subset of $\{1 \le n_1 \le \cdots \le n_m \le n\}$ for which $n_j = n_{j-1}$ if and only if $j \in A$. Then we have

$$
\begin{aligned}
(4.25)\qquad & \left| \mathbb{E}\left(\left\{ \frac{1}{b_n^{-1}n} (I_n(y) - I_{n,\epsilon}(y)) \right\}^m \right) \right|^{1/2} \\
& \le m! \sum_{A \subseteq \{2,3,\ldots,m\}} \sum_{D_m(A)} \frac{1}{(b_n^{-1}n)^m (2\pi)^m} \left\{ \int_{(b_n^{-1}n)^{1/2}[-\pi,\pi]^{2m}} \right. \\
& \qquad \left. \left| \prod_{j=1}^m \phi(u_j/(b_n^{-1}n)^{1/2})^{(n_j - n_{j-1})} \right|^2 \prod_{j=1}^m |1 - \widehat{h}(\epsilon p_j)| \, dp_j \right\}^{1/2}.
\end{aligned}
$$

For any $u \in \mathbb{R}^d$ let $\widetilde{u}$ denote the representative of u mod $(b_n^{-1}n)^{1/2} 2\pi \mathbb{Z}^2$ of smallest absolute value. We note that

$$
(4.26)\qquad |\widetilde{-u}| = |\widetilde{u}|, \qquad \text{and} \qquad |\widetilde{u+v}| = |\widetilde{u} + \widetilde{v}| \le |\widetilde{u}| + |\widetilde{v}|.
$$

Using the periodicity of ϕ we see that (4.19) implies that for all u

$$
(4.27)\qquad |\phi(u/(b_n^{-1}n)^{1/2})| \le e^{-c|\widetilde{u}|^2/(b_n^{-1}n)}.
$$

Then we have that on $\{1 \le n_1 \le \cdots \le n_m \le n\}$

$$
(4.28)\qquad \left| \prod_{j=1}^m \phi(u_j/(b_n^{-1}n)^{1/2})^{(n_j - n_{j-1})} \right|^2 \le \prod_{j=1}^m e^{-c|\widetilde{u}_j|^2 (n_j - n_{j-1})/(b_n^{-1}n)}
$$

Using $|1 - \widehat{h}(\epsilon p_j)| \le c\epsilon^{1/2} |p_j|^{1/2}$ we bound the integral in (4.25) by

$$
(4.29)\qquad c^m \epsilon^{m/2} \int_{(b_n^{-1}n)^{1/2}[-\pi,\pi]^{2m}} \prod_{j=1}^m e^{-c|\widetilde{u}_j|^2 (n_j - n_{j-1})/(b_n^{-1}n)} |p_j|^{1/2} \, dp_j.
$$

Using (4.23) and (4.26) we have that

$$
(4.30)\qquad \prod_{j=1}^m |p_j|^{1/2} \le \prod_{j=1}^m (|\widetilde{u}_j|^{1/2} + |\widetilde{u}_{j+1}|^{1/2})
$$

and when we expand the right hand side as a sum of monomials we can be sure that no factor $|\widetilde{u}_k|^{1/2}$ appears more than twice. Thus we see that we can bound (4.29) by

$$
(4.31)\qquad C^m \epsilon^{m/2} \max_{h(j)} \int_{(b_n^{-1}n)^{1/2}[-\pi,\pi]^{2m}} \prod_{j=1}^m e^{-c|\widetilde{u}_j|^2 (n_j - n_{j-1})/(b_n^{-1}n)} |\widetilde{u}_j|^{h(j)/2} \, dp_j
$$

where the max runs over the the set of functions $h(j)$ taking values $0, 1$ or 2 and such that $\sum_j h(j) = m$. Here we used the fact that the number of ways to choose the $\{h(j)\}$ is bounded by the number of ways of dividing m objects into 3 groups, which is 3^m. Changing variables, we thus need to bound

$$\int_{\Lambda_n} \prod_{j=1}^{m} e^{-c|\widetilde{u}_j|^2 (n_j - n_{j-1})/(b_n^{-1} n)} |\widetilde{u}_j|^{h(j)/2} \, du_j \tag{4.32}$$

where, see (4.23),

$$\Lambda_n = \{(u_1, \ldots, u_m) \,|\, u_j - u_{j+1} \in (b_n^{-1} n)^{1/2} [-\pi, \pi]^2, \, \forall j\}. \tag{4.33}$$

Let C_n denote the rectangle $(b_n^{-1} n)^{1/2} [-\pi, \pi]^2$ and let us call any rectangle of the form $2\pi k + C_n$, where $k \in \mathbb{Z}^2$, an elementary rectangle. Note that any rectangle of the form $v + C_n$, where $v \in \mathbb{R}^2$, can be covered by 4 elementary rectangles. Hence for any $v \in \mathbb{R}^2$ and $1 \leq s \leq n$

$$\begin{aligned} &\int_{v+C_n} e^{-c \frac{s}{b_n^{-1} n} |\widetilde{u}|^2} |\widetilde{u}|^{h/2} \, du \\ &\leq 4 \int_{R^2} e^{-c \frac{s}{(b_n^{-1} n)} |u|^2} |u|^{h/2} \, du \\ &\leq C \left(\frac{s}{b_n^{-1} n} \right)^{-(1+h/4)} . \end{aligned} \tag{4.34}$$

Similarly

$$\int_{v+C_n} |\widetilde{u}|^{h/2} \, du \leq C (b_n^{-1} n)^{(1+h/4)}. \tag{4.35}$$

We now bound (4.32) by bounding successively the integration with respect to $u_1, \ldots, u_m$. Consider first the du_1 integral, fixing $u_2, \ldots, u_m$. By (4.33) the du_1 integral is over the rectangle $u_2 + C_n$, hence the factors involving u_1 can be bounded using (4.34). Proceeding inductively, using (4.33) when $n_j - n_{j-1} > 0$ and (4.35) when $n_j = n_{j-1}$, leads to the following bound of (4.32), and hence of (4.29) on $D_m(A)$:

$$\begin{aligned} &c^m \epsilon^{m/2} \int_{(b_n^{-1} n)^{1/2} [-\pi,\pi]^{2m}} \prod_{j=1}^{m} e^{-c|\widetilde{u}_j|^2 (n_j - n_{j-1})/(b_n^{-1} n)} |p_j|^{1/2} \, dp_j \\ &\leq C^m \epsilon^{m/2} \prod_{j \in A} (b_n^{-1} n)^{(1+h(j)/4)} \prod_{j \in A^c} \left(\frac{(n_j - n_{j-1})}{b_n^{-1} n} \right)^{-(1+h(j)/4)} . \end{aligned} \tag{4.36}$$

Here A^c means the complement of A in $\{1,\dots,m\}$, so that A^c always contains 1. If $A^c = \{i_1,\dots,i_k\}$ where $i_1 < \cdots < i_k$ we then obtain for the sum in (4.25) over $D_m(A)$, the bound

$$C^m \epsilon^{m/4} \max_{h(j)} \frac{1}{(b_n^{-1}n)^m} \prod_{j\in A} (b_n^{-1}n)^{(1+h(j)/4)} \sum_{1\le n_{i_1}<\cdots<n_{i_k}\le n} \prod_{j\in A^c} \left(\frac{(n_j-n_{j-1})}{b_n^{-1}n}\right)^{-(1+h(j)/4)/2} \tag{4.37}$$

Note that

$$(b_n^{-1}n)^{(1+h(j)/4)/2} \frac{1}{b_n^{-1}n} \to 0 \text{ as } n\to\infty. \tag{4.38}$$

Using this to bound the product over $j\in A$, and then bounding the sum by an integral, we can bound (4.37) by

$$\begin{aligned} &C^m\epsilon^{m/4} \max_{h(j)} \sum_{1\le n_{i_1}<\cdots<n_{i_k}\le n} \prod_{j\in A^c} \left(\frac{(n_j-n_{j-1})}{b_n^{-1}n}\right)^{-(1+h(j)/4)/2} \frac{1}{b_n^{-1}n} \\ &\le C^m\epsilon^{m/4} \max_{h(j)} \int_{0\le r_{i_1}<\cdots<r_{i_k}\le b_n} \prod_{j\in A^c} (r_j-r_{j-1})^{-(1/2+h(j)/8)}\, dr_j \\ &\le C^m\epsilon^{m/4} \max_{h(j)} \frac{b_n^{\sum_{j\in A^c}(1/2-h(j)/8)}}{\Gamma(\sum_{j\in A^c}(1/2-h(j)/8))} \end{aligned} \tag{4.39}$$

Using this together with (4.25), but with m replaced by $2m$, and the fact that $(2m!)^{1/2}/m! \le 2^m$, we see that (4.7) is bounded by

$$\sum_{m=0}^{\infty} C^m\lambda^m\epsilon^{m/4} \left(\sum_{A\subseteq\{2,3,\dots,2m\}} \max_{h(j)} \frac{b_n^{\sum_{j\in A^c}(1/2-h(j)/8)}}{\Gamma(\sum_{j\in A^c}(1/2-h(j)/8))}\right)^{1/2}. \tag{4.40}$$

We have $\sum_{A\subseteq\{1,2,3,\dots,2m\}} 1 = 2^{2m}$. Then noting that $\sum_{j\in A^c}(1/2-h(j)/8)$ is an integer multiple of $1/8$ which is always less than m, we can bound the last line by

$$\sum_{l=0}^{\infty} \left(\sum_{m=l}^{\infty} C^m\lambda^m\epsilon^{m/4}\right) \sum_{j=0}^{7} \left(\frac{b_n^{l+j/8}}{\Gamma(l+j/8)}\right)^{1/2} \tag{4.41}$$

$$
\begin{aligned}
&\le C b_n \sum_{l=0}^{\infty} \left(\sum_{m=l}^{\infty} C^m \lambda^m \epsilon^{m/4} \right) \left(\frac{b_n^l}{\Gamma(l)} \right)^{1/2} \\
&\le C b_n (1 - C\lambda\epsilon^{m/4})^{-1} \sum_{l=0}^{\infty} C^l \lambda^l |\epsilon|^{l/4} b_n^{l/2} \left(\frac{1}{\Gamma(l)} \right)^{1/2}
\end{aligned}
$$

for $\epsilon > 0$ sufficiently small.

(4.7) then follows from the fact that for any $a > 0$

$$
\begin{aligned}
& \sum_{l=0}^{\infty} a^l \left(\frac{1}{\Gamma(l)} \right)^{1/2} \\
&= \sum_{m=0}^{\infty} \left(a^{2m} \left(\frac{1}{\Gamma(2m)} \right)^{1/2} + a^{2m+1} \left(\frac{1}{\Gamma(2m+1)} \right)^{1/2} \right) \\
&\le C(1+a) \sum_{m=0}^{\infty} a^{2m} \left(\frac{1}{\Gamma(2m)} \right)^{1/2} \\
&\le C(1+a) e^{Ca^2}.
\end{aligned}
\tag{4.42}
$$

□

Remark 4.2 It follows from the proof that in fact for $\rho > 0$ sufficiently small, for any $\lambda > 0$

$$
\begin{aligned}
& \limsup_{\epsilon \to 0} \limsup_{n \to \infty} \sup_y \\
& \frac{1}{b_n} \log \mathbb{E} \left(\exp \left\{ \lambda \left| \frac{I_n(y) - I_{n,\epsilon}(y)}{\epsilon^\rho \, b_n^{-1} n} \right|^{1/2} \right\} \right) = 0.
\end{aligned}
\tag{4.43}
$$

5 Theorem 1.2: Upper bound for $\mathbb{E}\, B_n - B_n$

Proof of Theorem 1.2.

We prove (1.5) for $\theta = 1$:

$$
\begin{aligned}
-C_1 \quad &\le \quad \liminf_{n \to \infty} b_n^{-1} \log \mathbb{P} \Big\{ \mathbb{E}\, B_n - B_n \ge (2\pi)^{-1} \det(\Gamma)^{-1/2} n \log b_n \Big\} \\
&\le \quad \limsup_{n \to \infty} b_n^{-1} \log \mathbb{P} \Big\{ \mathbb{E}\, B_n - B_n \ge (2\pi)^{-1} \det(\Gamma)^{-1/2} n \log b_n \Big\} \le -C_2
\end{aligned}
\tag{5.1}
$$

for any $\{b_n\}$ satisfying (1.3). The case of general θ follows by replacing b_n by b_n^θ in (5.1).

In this section we prove the upper bound for (5.1). Let $t > 0$ and write $K = [t^{-1}b_n]$. Divide $[1, n]$ into $K > 1$ disjoint subintervals $(n_0, n_1], \cdots, (n_{K-1}, n_K]$, each of length $[n/K]$ or $[n/K] + 1$. Notice that

$$\mathbb{E}\,B_n - B_n \le \sum_{i=1}^{K} \Big[\mathbb{E}\,B((n_{i-1}, n_i]_<^2) - B((n_{i-1}, n_i]_<^2)\Big] + \mathbb{E}\,B_n - \sum_{i=1}^{K} \mathbb{E}\,B((n_{i-1}, n_i]_<^2) \tag{5.2}$$

By (2.39),

$$\begin{aligned}\sum_{i=1}^{K} \mathbb{E}\,B((n_{i-1}, n_i]_<^2) &= \sum_{i=1}^{K} \mathbb{E}\,B_{n_i - n_{i-1}} \\ &= \sum_{i=1}^{K} \Big[\frac{1}{(2\pi)\sqrt{\det\Gamma}}(n/K)\log(n/K) + O(n/K)\Big] \\ &= \frac{1}{(2\pi)\sqrt{\det\Gamma}} n\log(n/K) + O(n)\end{aligned} \tag{5.3}$$

With $K > 1$, the error term can be taken to be independent of t and $\{b_n\}$. Thus, by (2.39), there is constant $\log a > 0$ independent of t and $\{b_n\}$ such that

$$\begin{aligned}&\mathbb{E}\,B_n - \sum_{j=1}^{K} \mathbb{E}\,B((n_{i-1}, n_i]_<^2) \\ &\le \frac{1}{(2\pi)\sqrt{\det\Gamma}} n\Big(\log(t^{-1}b_n) + \log a\Big).\end{aligned} \tag{5.4}$$

It is here that we use the condition that $\mathbb{E}\,|S_1|^{2+\delta} < \infty$ for some $\delta > 0$, needed for (2.39).

By first using Chebyshev's inequality, then using (5.2), (5.4) and the independence of the $B((n_{i-1}, n_i]_<^2)$, for any $\phi > 0$,

$$\mathbb{P}\Big\{\mathbb{E}\,B_n - B_n \ge (2\pi)^{-1}\det(\Gamma)^{-1/2} n\log b_n\Big\} \tag{5.5}$$

$$\leq \exp\Big\{-\phi b_n \log b_n\Big\}\mathbb{E}\exp\Big\{-2\pi\phi\sqrt{\det\Gamma}\frac{b_n}{n}(B_n-\mathbb{E}\,B_n)\Big\}$$
$$\leq \exp\Big\{\phi b_n(\log a-\log t)\Big\}\Big(\mathbb{E}\exp\Big\{-2\pi\phi\sqrt{\det\Gamma}\frac{b_n}{n}(B_{[n/K]}-\mathbb{E}\,B_{[n/K]})\Big\}\Big)^K$$

By [16, Theorem 1.2],

$$\sqrt{\det\Gamma}\frac{b_n}{n}(B_{[n/K]}-\mathbb{E}\,B_{[n/K]}) \xrightarrow{d} \gamma_t, \quad (n\to\infty) \tag{5.6}$$

where γ_t is the renormalized self-intersection local time of planar Brownian motion $\{W_s\}$ up to time t. By Lemma 2.5 and the dominated convergence theorem,

$$\mathbb{E}\exp\Big\{-2\pi\phi\sqrt{\det\Gamma}\frac{b_n}{n}(B_{[n/K]}-\mathbb{E}\,B_{[n/K]})\Big\} \longrightarrow \mathbb{E}\exp\Big\{-2\pi\phi t\gamma_1\Big\}, \quad (n\to\infty) \tag{5.7}$$

where we used the scaling $\gamma_t \stackrel{d}{=} t\gamma_1$.

Thus,

$$\begin{aligned}\limsup_{n\to\infty} b_n^{-1}\log\mathbb{P}\Big\{\mathbb{E}\,B_n - B_n \geq (2\pi)^{-1}\det(\Gamma)^{-1/2} n\log b_n\Big\}\\ \leq \phi(\log a-\log t)+\frac{1}{t}\log\mathbb{E}\exp\Big\{-2\pi\phi t\gamma_1\Big\}\\ = \phi\log(a\phi)+\frac{1}{t}\log\mathbb{E}\exp\Big\{-(\phi t)\log(\theta t)-2\pi(\phi t)\gamma_1\Big\}\end{aligned} \tag{5.8}$$

By [2, p. 3233], the limit

$$C \equiv \lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}\exp\Big\{-t\log t-2\pi t\gamma_1\Big\} \tag{5.9}$$

exists. Hence

$$\limsup_{n\to\infty} b_n^{-1}\log\mathbb{P}\Big\{\mathbb{E}\,B_n - B_n \geq (2\pi)^{-1}\det(\Gamma)^{-1/2} n\log b_n\Big\} \leq \phi\log(a\phi)+C\phi. \tag{5.10}$$

Taking the minimizer $\phi = a^{-1}e^{-(1+C)}$ we have

$$\limsup_{n\to\infty} b_n^{-1}\log\mathbb{P}\Big\{\mathbb{E}\,B_n - B_n \geq (2\pi)^{-1}\det(\Gamma)^{-1/2} n\log b_n\Big\} \leq -a^{-1}e^{-(1+C)}. \tag{5.11}$$

This proves the upper bound for (5.1).

6 Theorem 1.2: Lower bound for $\mathbb{E}\, B_n - B_n$

In this section we complete the proof of Theorem 1.2 by proving the lower bound for (5.1).

Let $B(x,r)$ be the ball of radius r centered at x. Let $\mathcal{F}_k = \sigma\{X_i : i \le k\}$. Let us assume for simplicity that the covariance matrix for the random walk is the identity; routine modifications are all that are needed for the general case. We write Θ for $(2\pi)^{-1} \det(\Gamma)^{-1/2} = (2\pi)^{-1}$. We write $D(x,r)$ for the disc of radius r in $\mathbb{Z}^2$ centered at x.

Let $K = [b_n]$ and $L = n/K$. Let us divide $\{1,2,\dots,n\}$ into K disjoint contiguous blocks, each of length strictly between $L/2$ and $3L/2$. Denote the blocks $J_1,\dots,J_K$. Let $v_i = \#(J_i)$, $w_i = \sum_{j=1}^i v_j$. Let

$$B_{v_i}^{(i)} = \sum_{j,k\in J_i, j<k} \delta(S_j,S_k), \qquad A_i = \sum_{j\in J_{i-1}, k\in J_i} \delta(S_j,S_k). \tag{6.1}$$

Define the following sets:

$$\begin{aligned}
F_{i,1} &= \{S_{w_i} \in D(i\sqrt{L}, \sqrt{L}/16)\},\\
F_{i,2} &= \{S(J_i) \subset [(i-1)\sqrt{L} - \sqrt{L}/8, i\sqrt{L} + \sqrt{L}/8] \times [-\sqrt{L}/8, \sqrt{L}/8]\},\\
F_{i,3} &= \{B_{v_i}^{(i)} - \mathbb{E}\, B_{v_i}^{(i)} \le \kappa_1 L\},\\
F_{i,4} &= \{\sum_{j\in J_i} 1_{D(x, r\sqrt{L})}(S_j) \le \kappa_2 r L \text{ for all } x \in D(i\sqrt{L}, 3\sqrt{L}), 1/\sqrt{L} < r < 2\},\\
F_{i,5} &= \{A_i < \kappa_3 L\},
\end{aligned}$$

where $\kappa_1, \kappa_2, \kappa_3$ are constants that will be chosen later and do not depend on K or L. Let

$$C_i = F_{i,1} \cap F_{i,2} \cap F_{i,3} \cap F_{i,4} \cap F_{i,5} \tag{6.2}$$

and

$$E = \cap_{i=1}^K C_i. \tag{6.3}$$

We want to show

$$\mathbb{P}(C_i \mid \mathcal{F}_{w_{i-1}}) \ge c_1 > 0 \tag{6.4}$$

on the event $C_1 \cap \cdots \cap C_{i-1}$. Once we have (6.4), then

$$\mathbb{P}(\cap_{i=1}^m C_i) = \mathbb{E}\left(\mathbb{P}(C_m \mid \mathcal{F}_{w_{m-1}}); \cap_{i=1}^{m-1} C_i\right) \ge c_1 \mathbb{P}(\cap_{i=1}^{m-1} C_i), \tag{6.5}$$

and by induction

$$\mathbb{P}(E) = \mathbb{P}(\cap_{i=1}^K C_i) \ge c_1^K = e^{K \log c_1} = e^{-c_2 K}. \tag{6.6}$$

On the set E, we see that $S(B_i) \cap S(B_j) = \emptyset$ if $|i - j| > 1$. So we can write

$$B_n = \sum_{k=1}^K (B_{v_k}^{(k)} - \mathbb{E}\, B_{v_k}^{(k)}) + \sum_{k=1}^K \mathbb{E}\, B_{v_k}^{(k)} + \sum_{k=1}^K A_k. \tag{6.7}$$

On the event E, each $B_{v_k}^{(k)} - \mathbb{E}\, B_{v_k}^{(k)}$ is bounded by $\kappa_1 L$ and each A_k is bounded by $\kappa_3 L$. By (2.38), each $\mathbb{E}\, B_{v_k}^{(k)} = \Theta v_k \log v_k + O(L) = \Theta v_k \log L + O(v_k)$. Therefore

$$B_n \le \kappa_1 K L + \Theta K L \log L + O(n) + \kappa_3 K L, \tag{6.8}$$

and using (2.38) again,

$$\begin{aligned} \mathbb{E}\, B_n - B_n &\ge \Theta n \log n - c_3 n - \Theta n \log(n/b_n) \\ &= \Theta n \log b_n - c_3 n \end{aligned} \tag{6.9}$$

on the event E. We conclude that

$$\mathbb{P}(\mathbb{E}\, B_n - B_n \ge \Theta n \log b_n - c_3 n) \ge e^{-c_2 b_n}. \tag{6.10}$$

We apply (6.10) with b_n replaced by $b'_n = c_4 b_n$, where $\Theta \log c_4 = c_3$. Then

$$\Theta n \log b'_n - c_3 n = \Theta n \log b_n + \Theta n \log c_4 - c_3 n = \Theta n \log b_n. \tag{6.11}$$

We then obtain

$$\mathbb{P}(\mathbb{E}\, B_n - B_n \ge \Theta n \log b_n) = \mathbb{P}(\mathbb{E}\, B_n - B_n \ge \Theta n \log b'_n - c_3 n) \ge e^{-c_2 b'_n}, \tag{6.12}$$

which would complete the proof of the lower bound for (5.1), hence of Theorem 1.2.

So we need to prove (6.4). By scaling and the support theorem for Brownian motion (see [1, Theorem I.6.6]), if W_t is a planar Brownian motion and $|x| \le \sqrt{L}/16$, then

$$\begin{aligned} \mathbb{P}^x \Big(& W_{v_i} \in D(\sqrt{L}, \sqrt{L}/16) \text{ and} \\ & \{W_s; 0 \le s \le v_i\} \subset [-\sqrt{L}/8, 9\sqrt{L}/8] \times [-\sqrt{L}/8, \sqrt{L}/8] \Big) > c_5, \end{aligned} \tag{6.13}$$

where c_5 does not depend on L. Using Donsker's invariance principle for random walks with finite second moments together with the Markov property,

$$\mathbb{P}(F_{i,1} \cap F_{i,2} \mid F_{w_{i-1}}) > c_6. \tag{6.14}$$

By Lemma 2.3, for $L/2 \le \ell \le 3L/2$

$$\mathbb{P}(B_\ell - \mathbb{E}\, B_\ell > \kappa_1 L) \le c_6/2 \tag{6.15}$$

if we choose κ_1 large enough. Again using the Markov property,

$$\mathbb{P}(F_{i,1} \cap F_{i,2} \cap F_{i,3} \mid F_{w_{i-1}}) > c_6/2. \tag{6.16}$$

Now let us look at $F_{i,4}$. By [17, p. 75], $\mathbb{P}(S_j = y) \le c_7/j$ with c_7 independent of $y \in \mathbb{Z}^2$ so that

$$\mathbb{P}(S_j \in D(x, r\sqrt{L})) = \sum_{y \in D(x, r\sqrt{L})} \mathbb{P}(S_j = y) \le \frac{c_8 r^2 L}{j}. \tag{6.17}$$

Therefore

$$\begin{aligned} \mathbb{E} \sum_{j \in J_1} 1_{D(x,r\sqrt{L})}(S_j) &\le \sum_{j=1}^{[2L]} \mathbb{P}(S_j \in D(x, r\sqrt{L})) \\ &\le r^2 L + \sum_{j=r^2 L}^{[2L]} \frac{c_9 r^2 L}{j} \\ &\le r^2 L + c_{10} L r^2 \log(1/r) \le c_{11} L r^2 \log(1/r) \end{aligned} \tag{6.18}$$

if $1/\sqrt{L} \le r \le 2$. Let $C_m = \sum_{j<m} 1_{D(x,r\sqrt{L})}(S_j)$ for $m \le [2L]+1$ and let $C_m = C_{[2L]+1}$ for $m > L$. By the Markov property and independence,

$$\begin{aligned} \mathbb{E}\,[C_\infty - C_m \mid \mathcal{F}_m] &\le 1 + \mathbb{E}\,[C_\infty - C_{m+1} \mid \mathcal{F}_m] \\ &\le 1 + \mathbb{E}^{\,S_m} C_\infty \le c_{12} L r^2 \log(1/r). \end{aligned} \tag{6.19}$$

By [1, Theorem I.6.11], we have

$$\mathbb{E}\, \exp\left(c_{13} \frac{C_{[2L]+1}}{c_{12} L r^2 \log(1/r)} \right) \le c_{14} \tag{6.20}$$

with c_{13}, c_{14} independent of L or r. We conclude that for $V > 0$

$$\mathbb{P}\Big(\sum_{j\in J_1} 1_{D(x,r\sqrt{L})}(S_j) > VLr^2\log(1/r)\Big) \le c_{15}e^{-c_{16}V}. \tag{6.21}$$

Suppose $2^{-s} \le r < 2^{-s+1}$ for some $s \ge 0$. If $x \in D(0, 3\sqrt{L})$, then each point in the disc $D(x, r\sqrt{L})$ will be contained in $D(x_i, 2^{-s+3}\sqrt{L})$ for some x_i, where each coordinate of x_i is an integer multiple of $2^{-s-2}\sqrt{L}$. There are at most $c_{17}2^{2s}$ such balls, and $Lr^2\log(1/r) \le c_{18}2^{s/2}Lr$, so for $V > 0$

$$\mathbb{P}\Big(\sup_{x\in D(0,3\sqrt{L}),2^{-s}\le r<2^{-s+1}} \sum_{j\in J_1} 1_{D(x,r\sqrt{L})}(S_j) > VrL\Big) \le c_{19}2^{2s}e^{-c_{20}V2^{s/2}}. \tag{6.22}$$

If we now sum over positive integers s and take κ_2 large enough, we see that

$$\mathbb{P}(F_{1,4}^c) \le c_6/4. \tag{6.23}$$

By the Markov property, we then obtain

$$\mathbb{P}(F_{i,1} \cap F_{i,2} \cap F_{i,3} \cap F_{i,4} \mid F_{w_{i-1}}) > c_6/4. \tag{6.24}$$

Finally, we examine $F_{i,5}$. We will show

$$\mathbb{P}(F_{i,5}^c \mid \mathcal{F}_{w_{i-1}}) \le c_6/8 \tag{6.25}$$

on the set $\cap_{j=1}^{i-1} C_j$ if we take κ_3 large enough. By the Markov property, it suffices to show

$$\mathbb{P}\Big(\sum_{j=1}^{[2L]} 1_{(S_j\in G)} \ge \kappa_3 L\Big) \le c_6/8 \tag{6.26}$$

whenever $G \in \mathbb{Z}^2$ is a fixed nonrandom set consisting of $[2L]$ points satisfying the property that

$$\#(G \cap D(x, r\sqrt{L})) \le \kappa_2 rL, \qquad x \in D(0, 3\sqrt{L}), \quad 1/\sqrt{L} \le r \le 2. \tag{6.27}$$

We compute the expectation of

$$\sum_{j=1}^{[2L]} 1_{(S_j\in G\cap(D(0,2^{-k}\sqrt{L})\setminus D(0,2^{-k+1}\sqrt{L})))}. \tag{6.28}$$

When $j \le 2^{-2k}L$, then the fact that the random walk has finite second moments implies that the probability that $|S_j|$ exceeds $2^{-k+1}\sqrt{L}$ is bounded by $c_{21}j/(2^{-2k+2}L)$. When $j > 2^{-2k}L$, we use [17, p. 75], and obtain

$$\mathbb{P}(S_j \in G \cap (D(0, 2^{-k}\sqrt{L}) \le c_{22}\frac{\kappa_2 2^{-k}L}{j}. \tag{6.29}$$

So

$$\begin{aligned} &\mathbb{E}\sum_{j=1}^{[2L]} 1_G(S_j) \\ &\le \sum_k \sum_{[2L]\ge j > 2^{-2k}L} c_{22}\frac{\kappa_2 2^{-k}L}{j} + \sum_k \sum_{j\le 2^{-2k}L} c_{21}\frac{j}{2^{-2k+2}L} \\ &\le \sum_k (c_{23}\kappa_2 k 2^{-k}L + c_{24}2^{-2k}L) \le c_{25}L. \end{aligned} \tag{6.30}$$

So if take κ_3 large enough, we obtain (6.26).

This completes the proof of (6.4), hence of Theorem 1.2. □

7 Laws of the iterated logarithm

7.1 Proof of the LIL for $B_n - \mathbb{E}\, B_n$

First, let S_j, S'_j be two independent copies of our random walk. Let

$$\ell(n,x) = \sum_{i=1}^n \delta(S_i\,, x), \qquad \ell'(n,x) = \sum_{i=1}^n \delta(S'_i\,, x) \tag{7.1}$$

and note that

$$I_{k,n} = \sum_{i=1}^k \sum_{j=1}^n \delta(S_i, S'_j) = \sum_{x\in\mathbb{Z}^2} \ell(k,x)\ell'(n,x). \tag{7.2}$$

Lemma 7.1 *There exist constants c_1, c_2 such that*

$$\mathbb{P}(I_{k,n} > \lambda\sqrt{kn}) \le c_1 e^{-c_2\lambda}. \tag{7.3}$$

Proof. Clearly

$$(7.4)\qquad (I_{k,n})^m = \sum_{x_1\in\mathbb{Z}^2}\cdots\sum_{x_m\in\mathbb{Z}^2}\Big(\prod_{i=1}^m \ell(k,x_i)\Big)\Big(\prod_{i=1}^m \ell'(n,x_i)\Big)$$

Using the independence of S and S',

$$(7.5)\qquad \mathbb{E}\left((I_{k,n})^m\right) = \sum_{x_1\in\mathbb{Z}^2}\cdots\sum_{x_m\in\mathbb{Z}^2}\mathbb{E}\Big(\prod_{i=1}^m \ell(k,x_i)\Big)\mathbb{E}\Big(\prod_{i=1}^m \ell'(n,x_i)\Big).$$

By Cauchy-Schwarz, this is less than

$$(7.6)\qquad \Big[\sum_{x_1\in\mathbb{Z}^2}\cdots\sum_{x_m\in\mathbb{Z}^2}\Big(\mathbb{E}\Big(\prod_{i=1}^m \ell(k,x_i)\Big)\Big)^2\Big]^{1/2}$$
$$\Big[\sum_{x_1\in\mathbb{Z}^2}\cdots\sum_{x_m\in\mathbb{Z}^2}\Big(\mathbb{E}\Big(\prod_{i=1}^m \ell'(n,x_i)\Big)\Big)^2\Big]^{1/2}$$
$$=: J_1^{1/2}J_2^{1/2}.$$

We can rewrite

$$(7.7)\quad J_1 = \sum_{x_1\in\mathbb{Z}^2}\cdots\sum_{x_m\in\mathbb{Z}^2}\mathbb{E}\Big(\prod_{i=1}^m \ell(k,x_i)\Big)\mathbb{E}\Big(\prod_{i=1}^m \ell'(k,x_i)\Big) = \mathbb{E}\left((I_k)^m\right),$$

and similarly $J_2 = \mathbb{E}\left((I_n)^m\right)$.

Therefore,

$$(7.8)\qquad \begin{aligned} & \mathbb{E}\,\exp(aI_{k,n}/\sqrt{kn}) \\ =\ & \sum_{m=0}^{\infty}\frac{a^m}{k^{m/2}n^{m/2}m!}\mathbb{E}\left((I_{k,n})^m\right) \\ \le\ & \sum_m \frac{a^m}{k^{m/2}n^{m/2}m!}(\mathbb{E}\left((I_k)^m\right))^{1/2}(\mathbb{E}\left((I_n)^m\right))^{1/2} \\ \le\ & \Big(\sum\frac{a^m}{m!}\mathbb{E}\Big(\frac{I_k}{k}\Big)^m\Big)^{1/2}\Big(\sum\frac{a^m}{m!}\mathbb{E}\Big(\frac{I_n}{n}\Big)^m\Big)^{1/2} \\ \le\ & \Big(\mathbb{E}\,e^{aI_k/k}\Big)^{1/2}\Big(\mathbb{E}\,e^{aI_n/n}\Big)^{1/2}. \end{aligned}$$

By Lemma 2.2 this can be bounded independently of k and n if a is taken small, and our result follows. □

We are now ready to prove the upper bound for the LIL for $B_n - \mathbb{E}\, B_n$. Write Ξ for $\sqrt{\det \Gamma}\ \kappa(2,2)^{-4}$. Recall that for any integrable random variable Z we let $\overline{Z}$ denote $Z - \mathbb{E}\, Z$. Let $\varepsilon > 0$ and let $q > 1$ be chosen later. Our first goal is to get an upper bound on

$$\mathbb{P}(\max_{n/2 \le k \le n} \overline{B}_k > (1+\varepsilon)\Xi^{-1} n \log\log n).$$

Let $m_0 = 2^N$, where N will be chosen later to depend only on ε and n. Let $\mathcal{A}_0$ be the integers of the form $n - km_0$ that are contained in $\{n/4, \ldots, n\}$. For each i let $\mathcal{A}_i$ be the set of integers of the form $n - km_0 2^{-i}$ that are contained in $\{n/4, \ldots, n\}$. Given an integer k, let k_j be the largest element of $\mathcal{A}_j$ that is less than or equal to k. For any $k \in \{n/2, \ldots, n\}$, we can write

$$\overline{B}_k = \overline{B}_{k_0} + (\overline{B}_{k_1} - \overline{B}_{k_0}) + \cdots + (\overline{B}_{k_N} - \overline{B}_{k_{N-1}}). \tag{7.9}$$

If $\overline{B}_k \ge (1+\varepsilon)\Xi^{-1} n \log\log n$ for some $n/2 \le k \le n$, then either
(a) $\overline{B}_{k_0} \ge (1 + \frac{\varepsilon}{2})\Xi^{-1} n \log\log n$ for some $k_0 \in \mathcal{A}_0$; or else
(b) for some $i \ge 1$ and some pair of consecutive elements $k_i, k_i' \in \mathcal{A}_i$, we have

$$\overline{B}_{k_i'} - \overline{B}_{k_i} \ge \tfrac{\varepsilon}{40 i^2} \Xi^{-1} n \log\log n. \tag{7.10}$$

For each k_0, using Theorem 1.1 and the fact that $k_0 \ge n/4$, the probability in (a) is bounded by

$$\exp(-(1 + \tfrac{\varepsilon}{4}) \log\log k_0) \le c_1 (\log n)^{-(1+\frac{\varepsilon}{4})}. \tag{7.11}$$

There are at most n/m_0 elements of $\mathcal{A}_0$, so the probability in (a) is bounded by

$$\frac{n}{m_0} \frac{c_1}{(\log n)^{1+\frac{\varepsilon}{4}}}. \tag{7.12}$$

Now let us examine the probability in (b). Fix i for the moment. Any two consecutive elements of $\mathcal{A}_i$ are $2^{-i} m_0$ apart. Recalling the notation (2.16) we can write

$$\overline{B}_k - \overline{B}_j = \overline{B}([j+1,k]^2_<) + \overline{B}([1,j] \times [j+1,k]), \tag{7.13}$$

So

$$\begin{aligned} \mathbb{P}(\overline{B}_k - \overline{B}_j \ge \tfrac{\varepsilon}{40 i^2}\Xi^{-1} n \log\log n) \quad &\le \quad \mathbb{P}(\overline{B}([j+1,k]^2_<) \ge \tfrac{\varepsilon}{80 i^2}\Xi^{-1} n \log\log n) \\ &\quad + \mathbb{P}\left(B([1,j] \times [j+1,k]) \ge \tfrac{\varepsilon}{80 i^2}\Xi^{-1} n \log\log n\right). \end{aligned} \tag{7.14}$$

We bound the first term on the right by Lemma 2.3, and get the bound

$$\exp\Big(-\frac{c\varepsilon}{80i^2}\frac{n\log\log n}{2^{-i}m_0}\Big)\le\exp\Big(-\frac{c\varepsilon}{80i^2}2^i(n/m_0)\log\log n\Big) \tag{7.15}$$

if j and k are consecutive elements of $\mathcal{A}_i$. Note that $B([1,j]\times[j+1,k])$ is equal in law to $I_{j-1,k-j}$. Using Lemma 7.1, we bound the second term on the right hand side of (7.14) by

$$\begin{aligned}&c_1\exp\Big(-c_2\frac{\varepsilon}{80i^2}\frac{n\log\log n}{\sqrt{2^{-i}m_0}\sqrt{j}}\Big)\\ &\le c_1\exp\Big(-c_2\frac{\varepsilon}{80i^2}2^{i/2}(n/m_0)^{1/2}\log\log n\Big).\end{aligned} \tag{7.16}$$

The number of pairs of consecutive elements of $\mathcal{A}_i$ is less than $2^{i+1}(n/m_0)$. So if we add (7.15) and (7.16) and multiply by the number of pairs, the probability of (b) occurring for a fixed i is bounded by

$$c_3\frac{n}{m_0}2^i\exp\Big(-c_4 2^{i/2}(n/m_0)^{1/2}\log\log n/(80i^2)\Big). \tag{7.17}$$

If we now sum over $i\ge 1$, we bound the probability in (b) by

$$c_5\frac{n}{m_0}\exp\Big(-c_6(n/m_0)^{1/2}\log\log n\Big). \tag{7.18}$$

We now choose m_0 to be the largest power of 2 so that $c_6(n/m_0)^{1/2}>2$; recall n is big.

Let us use this value of m_0 and combine (7.12) and (7.18). Let $n_\ell=q^\ell$ and

$$C_\ell=\{\max_{n_{\ell-1}\le k\le n_\ell}\overline{B}_k\ge(1+\varepsilon)\Xi^{-1}n_\ell\log\log n_\ell\}. \tag{7.19}$$

By our estimates, $\mathbb{P}(C_\ell)$ is summable, so for ℓ large, by Borel-Cantelli we have

$$\max_{n_{\ell-1}\le k\le n_\ell}\overline{B}_k\le(1+\varepsilon)\Xi^{-1}n_\ell\log\log n_\ell. \tag{7.20}$$

By taking q sufficiently close to 1, this implies that for k large we have $\overline{B}_k\le(1+2\varepsilon)\Xi^{-1}k\log\log k$. Since ε is arbitrary, we have our upper bound. $\square$

The lower bound for the first LIL is easier. Let $\delta>0$ be small and let $n_\ell=[e^{\ell^{1+\delta}}]$. Let

$$D_\ell=\{\overline{B}([n_{\ell-1}+1,n_\ell]^2_<)\ge(1-\varepsilon)\Xi^{-1}n_\ell\log\log n_\ell\}. \tag{7.21}$$

Using Theorem 1.1, and the fact that $n_\ell/(n_\ell - n_{\ell-1})$ is of order 1, we see that $\sum_\ell \mathbb{P}(D_\ell) = \infty$ if $\delta < \epsilon/(1-\epsilon)$. The D_ℓ are independent, so by Borel-Cantelli

$$\overline{B}([n_{\ell-1}+1, n_\ell]^2_<) \geq (1-\varepsilon)\Xi^{-1} n_\ell \log\log n_\ell \tag{7.22}$$

infinitely often with probability one. Note that as in (7.13) we can write

$$\overline{B}_{n_\ell} = \overline{B}([n_{\ell-1}+1, n_\ell]^2_<) + \overline{B}_{n_{\ell-1}} + \overline{B}([1, n_{\ell-1}] \times [n_{\ell-1}+1, n_\ell]). \tag{7.23}$$

By the upper bound,

$$\limsup_{\ell\to\infty} \frac{\overline{B}_{n_{\ell-1}}}{n_{\ell-1}\log\log n_{\ell-1}} \leq \Xi^{-1}$$

almost surely, which implies

$$\limsup_{\ell\to\infty} \frac{\overline{B}_{n_{\ell-1}}}{n_\ell \log\log n_\ell} = 0. \tag{7.24}$$

Since $B([1, n_{\ell-1}] \times [n_{\ell-1}+1, n_\ell]) \geq 0$ and by (2.5)

$$\mathbb{E}\, B([1, n_{\ell-1}] \times [n_{\ell-1}+1, n_\ell]) \leq c_1 \sqrt{n_{\ell-1}}\sqrt{n_\ell - n_{\ell-1}} = o(n_\ell \log\log n_\ell), \tag{7.25}$$

using (7.22)-(7.25) yields the lower bound. □

7.2 LIL for $\mathbb{E}\, B_n - B_n$

Let $\Delta = 2\pi\sqrt{\det\Gamma}$. Let us write $J_n = \mathbb{E}\, B_n - B_n$.

First we do the upper bound. Let m_0, $\mathcal{A}_i$, and k_j be as in the previous subsection. We write, for $n/2 \leq k \leq n$,

$$J_k = J_{k_0} + (J_{k_1} - J_{k_0}) + \cdots + (J_{k_N} - J_{k_{N-1}}). \tag{7.26}$$

If $\max_{n/2\leq k\leq n} J_k \geq (1+\varepsilon)\Delta^{-1} n \log\log\log n$, then either
(a) $J_{k_0} \geq (1+\frac{\varepsilon}{2})\Delta^{-1} n\log\log\log n$ for some $k_0 \in \mathcal{A}_0$, or else
(b) for some $i \geq 1$ and k_i, k_i' consecutive elements of $\mathcal{A}_i$ we have

$$J_{k_i'} - J_{k_i} \geq \frac{\varepsilon}{40 i^2}\Delta^{-1} n \log\log\log n. \tag{7.27}$$

There are at most n/m_0 elements of $\mathcal{A}_0$. Using Theorem 1.2, the probability of (a) is bounded by

$$c_1 \frac{n}{m_0} e^{-(1+\frac{\varepsilon}{4})\log\log n}. \tag{7.28}$$

To estimate the probability in (b), suppose j and k are consecutive elements of $\mathcal{A}_i$. There are at most $2^{i+1}(n/m_0)$ such pairs. We have

$$\begin{aligned} J_k - J_j &= -\overline{B}([j+1,k]^2_<) - \overline{B}([1,j]\times[j+1,k]) \\ &\le -\overline{B}([j+1,k]^2_<) + \mathbb{E}\, B([1,j]\times[j+1,k]) \\ &\le -\overline{B}([j+1,k]^2_<) + c_2\sqrt{j}\sqrt{k-j}, \end{aligned} \tag{7.29}$$

as in the previous subsection. Provided n is large enough, $c_2\sqrt{j}\sqrt{k-j} = c_2\sqrt{j}\sqrt{2^{-i}m_0}$ will be less than $\frac{\varepsilon}{80i^2}\Delta^{-1}n\log\log\log n$ for all i. So in order for $J_k - J_j$ to be larger than $\frac{\varepsilon}{40i^2}\Delta^{-1}n\log\log\log n$, we must have $-\overline{B}([j+1,k]^2_<)$ larger than $\frac{\varepsilon}{80i^2}\Delta^{-1}n\log\log\log n$. We use Theorem 1.2 to bound this. Then multiplying by the number of pairs and summing over i, the probability is (b) is bounded by

$$\sum_{i=1}^{\infty} 2^{i+1}\frac{n}{m_0} e^{-\frac{\varepsilon}{80i^2}\frac{n}{2^{-i}m_0}\log\log n} \le c_3\frac{n}{m_0}e^{-c_4(n/m_0)\log\log n}. \tag{7.30}$$

We choose m_0 to be the largest possible power of 2 such that $c_4(n/m_0) > 2$.

Combining (7.28) and (7.30), we see that if we set $q > 1$ close to 1, $n_\ell = [q^\ell]$, and

$$E_\ell = \{\max_{n_\ell/2\le k\le n_\ell} J_k \ge (1+\varepsilon)\Delta^{-1}n_\ell\log\log\log n_\ell\}, \tag{7.31}$$

then $\sum_\ell \mathbb{P}(E_\ell)$ is finite. So by Borel-Cantelli, the event E_ℓ happens for a last time, almost surely. Exactly as in the previous subsection, taking q close enough to 1 and using the fact that ε is arbitrary leads to the upper bound. $\square$

The proof of the lower bound is fairly similar to the previous subsection. Let $n_\ell = [e^{\ell^{1+\delta}}]$. Theorem 1.2 and Borel-Cantelli tell us that F_ℓ will happen infinitely often, where

$$F_\ell = \{-\overline{B}([n_{\ell-1}+1, n_\ell]^2_<) \ge (1-\varepsilon)\Delta^{-1}n_\ell\log\log\log n_\ell\}. \tag{7.32}$$

We have

$$J_{n_\ell} \geq -\overline{B}([n_{\ell-1}+1, n_\ell]^2_<) + J_{n_{\ell-1}} - A(1, n_{\ell-1}; n_{\ell-1}, n_\ell). \tag{7.33}$$

By the upper bound,

$$J_{n_{\ell-1}} = O(n_{\ell-1} \log\log\log n_{\ell-1}) = o(n_\ell \log\log\log n_\ell). \tag{7.34}$$

By Lemma 7.1,

$$\mathbb{P}(B([1, n_{\ell-1}] \times [n_{\ell-1}+1, n_\ell]) \geq \varepsilon n_\ell \log\log\log n_\ell) \leq c_1 \exp\Big(-c_2 \frac{\varepsilon n_\ell \log\log\log n_\ell}{\sqrt{n_{\ell-1}}\sqrt{n_\ell - n_{\ell-1}}}\Big). \tag{7.35}$$

This is summable in ℓ, so

$$\limsup_{\ell\to\infty} \frac{B([1, n_{\ell-1}] \times [n_{\ell-1}+1, n_\ell])}{n_\ell \log\log\log n_\ell} \leq \varepsilon \tag{7.36}$$

almost surely. This is true for every ε, so the limsup is 0. Combining this with (7.34) and substituting in (7.33) completes the proof. $\square$

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