Test #1 **Spring 2025** **Name:**

Do all problems and give the **process** of your solution.

1. (20 points) Given two planes $3x + 2y - z = 0$ and $x - 2y + 3z = 0$,

(a). Find the angle between them.

Solution. $\mathbf{n_1} =< 3, 2, -1 >$ and $\mathbf{n_2} =< 1, -2, 3 >$

$$\cos\theta = \frac{\mathbf{n_1}\cdot\mathbf{n_2}}{|\mathbf{n_1}||\mathbf{n_2}|} = \frac{3\times 1 + 2\times(-2) + (-1)\times 3}{\sqrt{3^2+2^2+(-1)^2}\sqrt{1^2+(-2)^2+3^2}} = \frac{-2}{7}$$

$$\theta = \cos^{-1}\left(\frac{-2}{7}\right) \approx 106.6^\circ$$

(b). Find the equation of the intersection line between them.

Solution. A point: Let $z = 0$ and solve $3s + 2y = 0$ and $x - 2y = 0$ gives $x = y = 0$. So we get a point $(0, 0, 0)$ in the plane. To get a direction,

$$\mathbf{v} = \mathbf{n_1} \times \mathbf{n_2} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & -1 \\ 1 & -2 & 3 \end{vmatrix} = 4\mathbf{i} - 10\mathbf{j} - 8\mathbf{k}$$

So the parametric equation is $\gamma(t) = \langle 4t, -10t, -8t\rangle$.

2. (20 points) Given the three points $P(1, 1, 1)$, $Q(1, 3, 2)$, and $R(2, 1, 3)$ in space,

(a.)Find the area of the triangle with the vertecies P, Q, and R.
Solution. $\overrightarrow{PQ} =< 0, 2, 1 >$ and $\overrightarrow{PR} =< 1, 0, 2 >$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 2 & 1 \\ 1 & 0 & 2 \end{vmatrix} = 4\mathbf{i} + \mathbf{j} - 2\mathbf{k}$$

The area of the triangle is

$$\frac{1}{2}\|\overrightarrow{PQ} \times \overrightarrow{PR}\| = \frac{1}{2}\sqrt{4^2+1^2+2^2} = \frac{1}{2}\sqrt{21}$$

(b.) Find the equation of the plane that contains the points P, Q and R.

Solution.

$$\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = 4\mathbf{i} + \mathbf{j} - 2\mathbf{k}$$

$$4(x-1) + (y-1) - 2(z-1) = 0 \quad \text{or} \quad 4x + y - 2z = 3$$

3. (15 points) Find parametric equation for the tangent line to the curve $\mathbf{r}(t) = \langle t^2 - 1, t^2 + 1, t + 1\rangle$ at the point when $t = 0$.

Solution. When $t = 0$, $x = -1$, $y = 1$ and $z = 1$. $\mathbf{r}'(t) = \langle 2t, 2t, 1\rangle$. $\mathbf{r}'(0) = \langle 0, 0, 1\rangle$. The parametric equation of the tangent line is:

$$x = -1, \quad y = 1, \quad z = 1 + t$$

Or

$$\mathbf{r}(t) = \langle -1, 1, 1 + t\rangle$$

4. (15 points) Compute the arc-length of the curve $\mathbf{r}(t) = \langle \cos t, \sin t, t\rangle$ over $0 \le t \le 1$.

Solution. $\mathbf{r}'(t) = \langle -\sin t, \cos t, 1\rangle$.

$$\|\mathbf{r}'(t)\| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$$

$$s = \int_0^1 \|\mathbf{r}'(t)\| dt = \int_0^1 \sqrt{2} dt = \sqrt{2}$$

5 (15 points). Let $z = \ln(x^2 + xy + y^4)$. Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

Solution.

$$\frac{\partial z}{\partial x} = \frac{2x + y}{x^2 + xy + y^4}, \qquad \frac{\partial z}{\partial y} = \frac{x + 4y^3}{x^2 + xy + y^4}$$

6. (15 points). Find equation for the tangent plane to the surface $z = x^2 + xy + 3y^2$ at the point $(1, 1, 5)$.

Solution.

$$\frac{\partial f}{\partial x} = 2x + y \quad \text{and} \quad \frac{\partial f}{\partial y} = x + 6y$$

Hence

$$\frac{\partial f}{\partial x}(1,1) = 3 \quad \text{and} \quad \frac{\partial f}{\partial y}(1,1) = 7$$

The equation of the tangent plane:

$$z - 5 = 3(x - 1) + 7(y - 1) \quad \text{or} \quad 3x + 7y - z = 5$$