Test #2 **(Spring 2025)** **Name:**

Do all problems and give the **process** of your solution.

1. (15 points). Find equation for the tangent plane to the surface $x^2+2y^2+z^2+yz=3$ at the point $(1,1,-1)$.

Solution. Let $F(x,y,z)=x^2+2y^2+z^2+yz$. $\nabla F(x,y,z) =< 2x, 4y+z, 2z+y >$. Thus, the normal vector is

$$\mathbf{n} = \nabla F(1,1,-1) =< 2, 3, -1 >$$

The equation of the tangent plane: $2(x-1)+3(y-1)-(z+1)=0$. Or

$$2x+3y-z=6$$

2. (20 points). Let $f(x,y,z) = \dfrac{1}{x^2+y^2+z^2}$.

(a). Find the maximum rate of change of f at the point $(1,1,2)$ and the direction in which it occurs.

Solution.

$$\nabla f = -\frac{1}{(x^2+y^2+z^2)^2}\langle 2x, 2y, 2z\rangle$$

The direction of the maximum rate

$$\nabla f(1,1,2) = -\frac{1}{36}\langle 2, 2, 4\rangle$$

and the maximal rate is

$$\|\nabla f(1,1,2)\| = \frac{1}{36}\sqrt{2^2+2^2+4^2} = \frac{\sqrt{24}}{36}$$

(b). Find the direction derivative of f at the same point $(1,1,2)$ in the direction of the vector $\mathbf{v} = \langle 1, 2, 2\rangle$

Solution. the unit vector $\mathbf{u}$ in the direction of $\mathbf{v}$:

$$\mathbf{u} = \frac{1}{\|\mathbf{v}\|}\mathbf{v} = \langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3}\rangle$$

Thus,

$$D_{\mathbf{u}}f(1,1,2) = \nabla f(1,1,2)\cdot \mathbf{u} = -\frac{1}{36}\langle 2,2,4\rangle \cdot \langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3}\rangle = -\frac{7}{54}$$

3. (15 points). Find the area of the part of the surface $z = x^2+y^2$ that lies under the plane $z=1$.

Solution. Let S be the requested area and $D = \{(x,y)|x^2+y^2 \le 1\}$.

$$\begin{aligned}
S &= \iint_D \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}dA = \iint_D \sqrt{1+4x^2+4y^2}dA \\
&= \int_0^{2\pi}\int_0^1 \sqrt{1+4r^2}rdrd\theta = 2\pi\int_0^1 \sqrt{1+4r^2}rdr \\
&\overset{u=1+4r^2}{=} \frac{\pi}{4}\int_1^5 \sqrt{u}du = \frac{\pi}{4}\cdot\frac{2}{3}u^{3/2}\Big|_1^5 = \frac{\pi}{6}(5^{3/2}-1)
\end{aligned}$$

4. (15 points). Find the volume of the solid under the surface $z = xy$ and above the region enclosed by the curves $y = x$ and $y = \sqrt{x}$.

Solution.

$$\begin{aligned}
V &= \iint_D xydA = \int_0^1\int_x^{\sqrt{x}} xydydx = \frac{1}{2}\int_0^1 xy^2\Big|_x^{\sqrt{x}}dx \\
&= \frac{1}{2}\int_0^1 x(x-x^2)dx = \frac{1}{2}\left(\frac{1}{3}-\frac{1}{4}\right) = \frac{1}{24}
\end{aligned}$$

5. (15 points). Evaluate $\iiint_E xdV$, where E is bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$.

Solution. Let $D = \{(x,y)|x+y \le 1,\ x,y \ge 0\}$. By Fubini theorem

$$\begin{aligned}
\iiint_E xdV &= \iint_D \left[\int_0^{1-x-y} xdz\right]dA = \iint_D x\Big[1-x-y\Big]dA \\
&= \int_0^1 x\left[\int_0^{1-x}\Big[1-x-y\Big]dy\right]dx = \int_0^1 x\left[(1-x)^2-\frac{1}{2}y^2\Big|_{y=0}^{y=1-x}\right]dx \\
&= \frac{1}{2}\int_0^1 x(1-x)^2dx = \frac{1}{2}\int_0^1\Big[x-2x^2+x^3\Big]dx = \frac{1}{2}\left[\frac{1}{2}-\frac{2}{3}+\frac{1}{4}\right] = \frac{1}{24}
\end{aligned}$$

6. (20 points). Use Lagrange multiplier to find the maximum and minimum values of the function $f(x,y,z) = 2x+4y+4z$ subject to the constraint $x^2+y^2+z^2 = 1$.

Solution. $\nabla f(x,y,z) =< 2,4,4 >$, $\nabla g(x,y,z) =< 2x,2y,2z >$. We solve

$$\begin{cases} \nabla f(x,y,z) = \lambda\nabla g(x,y,z) \\ x^2+y^2+z^2 = 1 \end{cases} \quad \text{or} \quad \begin{cases} 2 = \lambda 2x \\ 4 = \lambda 2y \\ 4 = \lambda 2z \\ x^2+y^2+z^2 = 1 \end{cases}$$

$$\begin{cases} x = \frac{1}{\lambda} \\ y = \frac{2}{\lambda} \\ z = \frac{2}{\lambda} \\ \left(\frac{1}{\lambda}\right)^2 + \left(\frac{2}{\lambda}\right)^2 + \left(\frac{2}{\lambda}\right)^2 = 1 \end{cases} \qquad \text{or} \qquad \begin{cases} x = \frac{1}{\lambda} \\ y = \frac{2}{\lambda} \\ z = \frac{2}{\lambda} \\ \lambda^2 = 9 \end{cases}$$

As $\lambda = 3$: $x = \frac{1}{3}$, $y = \frac{2}{3}$ and $z = \frac{2}{3}$

As $\lambda = -3$: $x = -\frac{1}{3}$, $y = -\frac{2}{3}$ and $z = -\frac{2}{3}$

$$f\left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right) == 2 \times \frac{1}{3} + 4 \times \frac{2}{3} + 4 \times \frac{2}{3} = 6$$

$$f\left(-\frac{1}{3}, -\frac{2}{3}, -\frac{2}{3}\right) == 2 \times \frac{1}{3} - 4 \times \frac{2}{3} - 4 \times \frac{2}{3} = -6$$

So the maximal value is 6, the minimal value is -6.