Test #3 **Spring 2025** **Name:**

Do all problems and give the **process** of your solution.

1 (20 points). Evaluate $\iiint_E z\sqrt{x^2+y^2}dV$, where $E = \{(x,y,z)|1 \le x^2+y^2 \le 4 \text{ and } 0 \le z \le 3\}$.

Solution. By cylindrical substitution

$$\iiint_E z\sqrt{x^2+y^2}dV = \int_0^{2\pi}\int_1^2\int_0^3 zr\cdot rdzdrd\theta = 2\pi\left(\int_1^2 r^2dr\right)\left(\int_0^3 zdz\right) = 21\pi$$

2. (20 points). Evaluate $\iiint_E (x^2+y^2+z^2)^{3/2}dV$, where $E = \{(x,y,z)|\ x^2+y^2+z^2 \le 1,\ x,y,z \ge 0\}$.

Solution.

$$\begin{aligned}\iiint_E (x^2+y^2+z^2)^{3/2}dV &= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^1 \rho^3\cdot\rho^2\sin\phi d\rho d\phi d\theta\\ &= \frac{\pi}{2}\left(\int_0^{\frac{\pi}{2}}\sin\phi d\phi\right)\left(\int_0^1\rho^5d\rho\right) = \frac{\pi}{2}\cdot 1\cdot\frac{1}{6} = \frac{\pi}{12}\end{aligned}$$

3. (20 points). Find the work done by the force field $\mathbf{F}(x,y,z) = y\mathbf{i}+z\mathbf{j}+x\mathbf{k}$ in moving a particle along the trajectory $\mathbf{r}(t) = t\mathbf{i}+t^2\mathbf{j}+t^3\mathbf{k}$ $(0 \le t \le 1)$.

Solution.

$$W = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C ydx+zdy+xdz = \int_0^1\left(t^2+t^3\cdot 2t+t\cdot 3t^2\right)dt = \frac{1}{3}+\frac{2}{5}+\frac{3}{4} = \frac{89}{60}$$

4. (20 points).Consider 3-dimensional field $\mathbf{F} = \langle yz,\ xz+y,\ xy\rangle$.

(a) Is $\mathbf{F}$ conservative? (To receive any credit, you have to provide the reason).

Solution.

$$\text{curl}(\mathbf{F}) == \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ yz & xz+y & xy\end{vmatrix} = (x-x)\mathbf{i}-(y-y)\mathbf{j}+(z-z)\mathbf{k} = \mathbf{0}$$

Therefore, $\mathbf{F}$ is conservative.

(b). Find a function $f(x,y,z)$ such that $\mathbf{F} = \nabla f$

Solution. We solve the system

$$\begin{cases} \frac{\partial f}{\partial x} = yz \\ \frac{\partial f}{\partial y} = xz + y \\ \frac{\partial f}{\partial z} = xy \end{cases}$$

for f. Taking anti-derivative in the first equation: $f = xyz + C(y, z)$. Then taking partial derivative: $\frac{\partial f}{\partial y} = xz + \frac{\partial C}{\partial y}$. Comparing it with the second equation: $\frac{\partial C}{\partial y} = y$. Therefore,

$$C(y, z) = \frac{1}{2}y^2 + C(z) \quad \text{or} \quad f = xyz + \frac{1}{2}y^2 + C(z)$$

Taking partial derivative: $\frac{\partial f}{\partial z} = xy + C'(z)$. Comparing it with the third equation: $C'(z) = 0$. We take $C(z) = 0$. Finally

$$f(x, y, z) = xyz + \frac{1}{2}y^2$$

(c). Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is a path in space from $(0, 0, 0)$ to $(1, 1, 1)$.

Solution. By fundamental theorem,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(1, 1, 1) - f(0, 0, 0) = 1 + \frac{1}{2} = \frac{3}{2}$$

5. (20points). Evaluate the line integral

$$\oint_C (y - e^{\sin x})dx + (2x + \sqrt{1 + y^2})dy$$

where C is the counterclockwise closed curve that appears as the boundary of the region enclosed by the curve $y = x^2$ and $y = \sqrt{x}$.

Solution. Let D be the xy-domain with the boundary C. By Green's theorem

$$\oint_C (y - e^{\sin x})dx + (2x + \sqrt{1 + y^2})dy = \iint_D \Big[\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\Big] dA$$
$$= \iint_D \Big[2 - 1\Big] dA = \int_0^1 \Big[\int_{x^2}^{\sqrt{x}} dy\Big] dx = \int_0^1 [\sqrt{x} - x^2] dx = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}.$$