Small Deviation Estimates for Some Additive Processes

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Abstract. We study the small deviation probabilities for real valued additive processes. This naturally leads to the small deviation for the corresponding range process. Our general results can be applied to a wide range of additive processes generated from fractional Brownian motions, stable processes, Brownian sheets, etc. As an application, limit inf type LIL are proved for additive stable processes.

1. Introduction

Let $X_j(t)$, $1 \le j \le d$, be independent copies of a given real valued stochastic process $\{X(t), t \in E\}$ with index set E and $X(t_0) = 0$ for some $t_0 \in E$. Define the corresponding additive process

$$\mathbb{X}(\mathbf{t}) = \mathbb{X}(t_1, \cdots, t_d) = \sum_{j=1}^{d} X_j(t_j), \quad \mathbf{t} = (t_1, \cdots, t_d) \in E^d.$$

There are various motivations for the study of the additive process $X(\mathbf{t})$, $\mathbf{t} \in E^d$, and it has been active investigated recently from different points of view, see Khoshnevisan, Xiao, and Zhong (2002a,b) for detailed discussion and the bibliography for further works in this area. First of all, additive processes play a role in the study of other more interesting multiparameter processes. For example, locally and with time suitable rescaled, the Brownian sheet closely resembles additive Brownian motion, see Dalang and Walsh (1993a,b), Dalang and Mountford (2002). They also arise in the theory of intersections and selfintersections of Brownian processes; see Khoshnevisan and Xiao (2001) and Chen and Li (2002). Moreover, recent progress has shown that additive processes are more amenable to analysis, as we will also see in this paper.

The main objective of this paper is a study of the small deviation probabilities for real valued additive processes. This naturally leads to the small deviation for

Received by the editors December 27, 2002.
1991 *Mathematics Subject Classification.* Primary 60G60; Secondary 60G17, 60F15.
Key words and phrases. Small deviation, Additive process, Stable process.
Supported in part by NSF Grant DMS-0102238 and DMS-0204513.

the corresponding range process. Our general results given in Theorem 2.1 and 3.1 can be applied to a wide range of additive processes generated from fractional Brownian motions, stable processes, Brownian sheets, etc. As an application, limit inf type LIL are proved for additive stable processes.

It is useful and important to view our main results of this paper, under the sup-norm for various additive processes, as a part of a bigger picture. The small deviation/ball probability studies the behavior of

$$\log \mu(x : \|x\| \le \varepsilon) = -\phi(\varepsilon) \quad \text{as} \quad \varepsilon \to 0 . \tag{1.1}$$

for a given measure μ and a norm $\|\cdot\|$. In the literature, small deviation probabilities of various types are studied and applied to many problems of interest under different names such as small ball probability, lower tail behaviors, two sided boundary crossing probability and exit time.

For a Gaussian measure and any norm on a separable Banach space, there is a precise link, discovered in Kuelbs and Li (1993) and completed in Li and Linde (1999), between the function $\phi(\varepsilon)$ and the metric entropy of the unit ball of the reproducing kernel Hilbert space generated by μ. This powerful connection allows the use of tools and results from functional analysis and approximation theory to estimate small ball probabilities. The survey paper of Li and Shao (2001) on small ball probabilities for Gaussian processes, together with its extended references, covers much of the recent progress in this area. In particular, various applications and connections with other areas of probability and analysis are discussed.

For many other important processes such as Markov processes and additive processes, there is no general result available unless the process and the norm have the correct scaling (or self-similar) property. In that case (1.1) can be rewritten in terms of the first exit time of certain region and certain general results are known. For example, in the case of stable processes, the problems are related to the large deviation for occupation measures developed by Donsker and Varadhan (1977).

It is somewhat surprising that we are able to find the exact small deviation constants for various additive processes since the main results in many works in this area determine only the asymptotic behavior in (1.1) up to some constant factor in front of the rate. As far as we know, this is the first time that explicit constants are found for non-trivial multiparameter processes under the sup-norm.

The remaining of the paper is organized as follows. Section 2 contains the small deviations for the range process. Various remarks and examples are also given.

In Section 3, we obtain the small deviation for additive processes. The proof we present is much simpler than our original one but it is strictly based on one dimensional structure.

Section 4 establishes, as an application of our probability estimates, limit inf type LIL for additive stable processes. The key idea for the proof of the upper bound essentially comes from Kuelbs (1981). For the additive fractional Brownian motion and the additive fractional integrated Brownian motion, limit inf type LIL's are formulated.

In Section 5, we first exam some related additive type multiparameter processes generated by a single copy. Amongst other implications, these results show that their small deviation constants are different from the additive case by a factor of d, which is the number of independent copies needed in the additive process. The corresponding limit inf type LIL is also given. Finally, we generalize our small deviation estimates to additive type processes with sums of independent processes which are not necessarily copies of each other.

Acknowledgment

The authors are grateful to Davar Khoshnevisan and Yimin Xiao for helpful discussions.

2. Small Deviations for Range

In this section, we first present a general relation between the small deviation behaviors of the sup-norm and the range. The basic observation is that the range is about twice the sup-norm when they take small values.

Theorem 2.1. *Let $X(t)$, $t \in E$, be a real valued stochastic process with index set E and $X(t_0) = 0$ for some $t_0 \in E$. Assume the process satisfies the shift inequality*

$$\mathbb{P}\left(\sup_{t\in E} |X(t) - x| \le \varepsilon\right) \le \mathbb{P}\left(\sup_{t\in E} |X(t)| \le \varepsilon\right) \tag{2.1}$$

for any $x \in \mathbb{R}$, $\varepsilon > 0$, and has the small ball property

$$\lim_{\varepsilon\to 0^+} \varepsilon^\beta \log \mathbb{P}\left(\sup_{t\in E} |X(t)| \le \varepsilon\right) = -A_\beta \tag{2.2}$$

where $\beta > 0$ and $0 < A_\beta < \infty$ is a constant. Then for the range

$$R = \sup_{s,t\in E} |X(t) - X(s)| = \sup_{t\in E} X(t) - \inf_{t\in E} X(t),$$

we have

$$\lim_{\varepsilon\to 0^+} \varepsilon^\beta \log \mathbb{P}\left(R \le \varepsilon\right) = -2^\beta A_\beta \tag{2.3}$$

More general, for any fixed constants $a, b \ge 0$,

$$\lim_{\varepsilon\to 0^+} \varepsilon^\beta \log \mathbb{P}\left(aR + bM \le \varepsilon\right) = -(2a + b)^\beta A_\beta \tag{2.4}$$

where $M = \sup_{t\in E} |X(t)|$.

Before we prove the result, a few remarks and examples are needed. First, the shift inequality (2.1) holds for all centered Gaussian processes since it is a special case of Anderson's inequality, and the small ball property (2.2) is satisfied by various Gaussian processes. See the survey of Li and Shao (2001) for more details. Of particular interests to us in this paper are fractional Brownian motion

$B_H(t)$ with $B_H(0) = 0$ and index parameter $H \in (0,1)$, and fractional integrated Brownian motion

$$W_\gamma(t) = \frac{1}{\Gamma(\gamma+1)} \int_0^t (t-s)^\gamma dW(s) \quad \gamma > -1/2.$$

where $W_0(t) = W(t) = B_{1/2}(t)$ is the standard Brownian motion. To be more precise, $\{B_H(t), t \geq 0\}$ is a Gaussian process with mean zero and covariance function

$$\mathbb{E}\, B_H(t)B_H(s) = \frac{1}{2}(|s|^{2H} + |t|^{2H} - |s-t|^{2H}), \quad 0 < H < 1.$$

The small ball property (2.2) was proved in Li and Linde (1998), namely,

$$\lim_{\varepsilon\to 0} \varepsilon^{2/(2\gamma+1)} \log \mathbb{P}\left(\sup_{0\leq t\leq 1} |W_\gamma(t)| \leq \varepsilon\right) = -k_\gamma \tag{2.5}$$

for any $\gamma > -1/2$ and

$$\lim_{\varepsilon\to 0} \varepsilon^{1/H} \log \mathbb{P}\left(\sup_{0\leq t\leq 1} |B_H(t)| \leq \varepsilon\right) = -C_H \tag{2.6}$$

where $k_\gamma, C_H \in (0,\infty)$ are given by

$$\begin{aligned} k_\gamma &= -\inf_{\varepsilon>0} \varepsilon^{2/(2\gamma+1)} \log \mathbb{P}\left(\sup_{0\leq t\leq 1} |W_\gamma(t)| \leq \varepsilon\right) \\ C_\gamma &= k_{H-1/2} \cdot (\Gamma(H+1/2))^{1/H} \\ &\quad \cdot \left((2H)^{-1} + \int_{-\infty}^0 ((1-s)^{H-1/2} - (-s)^{H-1/2})^2 ds\right)^{-1/(2H)} \end{aligned}$$

The existence of the constant in (2.6) is also proved in Shao (1999) by developing a weaker form of correlation inequality. In the Brownian motion case, i.e. $\gamma = 0$, it is well known that $k_0 = C_{1/2} = \pi^2/8$.

Second, the symmetric α-stable processes $S_\alpha(t)$ with $S_\alpha(0) = 0$, $0 < \alpha \leq 2$, is covered by Theorem 2.1. The shift inequality (2.1) for $S_\alpha(t)$ is easy to prove when the process is viewed as a mixture of Gaussian, see, e.g. Chen, Kuelbs and Li (2000). The small ball property (2.2) is well known and more precisely,

$$\lim_{\varepsilon\to 0^+} \varepsilon^{\alpha} \log \mathbb{P}\left(\sup_{0\leq t\leq 1} |S_\alpha(t)| \leq \varepsilon\right) = -\lambda_\alpha \tag{2.7}$$

where $\lambda_\alpha > 0$ is the principle Dirichlet eigenvalue for the fractional Laplacian operator associated with $S_\alpha(t)$ in the interval $[-1,1]$. It should be pointed out that (2.7) can be equivalently stated as

$$\lim_{t\to\infty} \frac{1}{t} \mathbb{P}(\tau > t) = -\lambda_\alpha$$

due to scaling property of $S_\alpha(t)$, where

$$\tau = \inf\{s : |S_\alpha(s)| \geq 1\}$$

is the first exit time of the interval domain $[-1,1]$. Little seems to be known concerning the explicit value of λ_α, $0<\alpha<2$ despite the often appearances of this constant in other problems. The best known bounds to date are

$$\Gamma(\alpha+1) \le \lambda_\alpha \le \Gamma(\frac{\alpha}{2}+1)\Gamma(\alpha+\frac{3}{2})\Big/\Gamma(\frac{\alpha+3}{2}), \quad 0<\alpha<2 \tag{2.8}$$

and it is a challenge to find more explicit expression for λ_α than the well known variation ones. For more information, see a recent survey of Li and Linde (2002). The existence of a constant λ_α in (2.7) and the result (2.3) are proved in Mogul'skii (1974) based on scaling and independent increment properties of symmetric α-stable processes.

Third, the exact distributions in infinite series forms for the range of Brownian motion and Brownian bridge were first found in Feller (1951) by observing the connection between the joint distribution function of the maximum and minimum of $W(t)$ for $0 \le t \le 1$ and a certain distribution function arising in the Kolmogorov-Smirnov theorem on empirical distributions. In particular, the following more precise results are known:

$$\mathbb{P}\left(\sup_{0\le s,t\le 1}|W(t)-W(s)|\le\varepsilon\right) \sim \frac{8}{\varepsilon^2}\exp\left(-\frac{\pi^2}{2\varepsilon^2}\right)$$

as $\varepsilon\to\infty$, and this should be compared with the well known fact

$$\mathbb{P}\left(\sup_{0\le t\le 1}|W(t)|\le\varepsilon\right) \sim \frac{4}{\pi}\exp\left(-\frac{\pi^2}{8\varepsilon^2}\right).$$

Fourth, in the case of standard two dimensional Brownian sheet $W(s,t)$ and its tied (pinned) down variants, the correct decay rate for the sup-norm is known but not the existence of the constant, see Li and Shao (2001) for a brief history of the problem and open conjectures in three and higher dimension. Nevertheless, the proof for Theorem 2.1 can be used to obtain, as $\varepsilon\to 0$,

$$\begin{aligned}
& \log\mathbb{P}\left(\sup_{0\le s,s',t,t'\le 1}|W(s,t)-W(s',t')|\le\varepsilon\right) \\
\approx\ & \log\mathbb{P}\left(\sup_{0\le s,t\le 1}|W(s,t)|\le\varepsilon\right) \\
\approx\ & \frac{1}{\varepsilon^2}\left(\log\frac{1}{\varepsilon}\right)^3
\end{aligned}$$

where $\approx$ means both side the ratio differ by at most some constants above and below.

Fifth, our results also apply to the range of additive processes discussed in the next section, see (3.2) in Theorem 3.1. Finally we can present the proof of Theorem 2.1.

Proof: The lower bound of (2.3) follows easily from (2.2) and the fact

$$R=\sup_{s,t\in E}|X(t)-X(s)|\le 2\sup_{t\in E}|X(t)|=2M.$$

For the upper bound, fix a large integer N, we have by using $X(t_0)=0$,

$$\begin{aligned}
\mathbb{P}\left(R \le \varepsilon\right) &= \sum_{k=0}^{N-1} \mathbb{P}\left(\sup_{t\in E} X(t) - \inf_{t\in E} X(t) \le \varepsilon,\ \frac{k}{N}\varepsilon \le \sup_{t\in E} X(t) \le \frac{k+1}{N}\varepsilon\right) \\
&\le \sum_{k=0}^{N-1} \mathbb{P}\left(\inf_{t\in E} X(t) \ge -\frac{N-k}{N}\varepsilon,\ \frac{k}{N}\varepsilon \le \sup_{t\in E} X(t) \le \frac{k+1}{N}\varepsilon\right) \\
&\le \sum_{k=0}^{N-1} \mathbb{P}\left(\sup_{t\in E} |X(t) - \frac{2k-N+1}{2N}| \le \frac{N+1}{2N}\varepsilon\right) \\
&\le N\cdot \mathbb{P}\left(\sup_{t\in E} |X(t)| \le \frac{N+1}{2N}\varepsilon\right) = N\cdot\mathbb{P}\left(M \le \frac{N+1}{2N}\varepsilon\right)
\end{aligned}$$

where the last step follows from the shift inequality (2.1). Thus from the small ball property (2.2),

$$\begin{aligned}
\limsup_{\varepsilon\to 0} \varepsilon^{\beta} \log \mathbb{P}\left(R\le\varepsilon\right) &\le \limsup_{\varepsilon\to 0} \varepsilon^{\beta}\log\mathbb{P}\left(\sup_{t\in E}|X(t)| \le \frac{N+1}{2N}\varepsilon\right) \\
&= -\left(\frac{2N}{N+1}\right)^{\beta} A_\beta
\end{aligned}$$

Taking $N\to\infty$, we obtain (2.3).

The result (2.4) follows from the relation

$$(\gamma+\beta/2)R \le \gamma R + \beta M \le (2\gamma+\beta)M$$

and the tight estimates (2.2) and (2.3).

3. Small Deviations for Additive Processes

We present our main result here. Some generalizations are given in the last section. An inspection of our arguments reveals that the special structure of additive processes plays a very important role in our derivations.

Theorem 3.1. *Let $X_j(t)$ be independent copies of a stochastic process $X(t)$ index by E. Assume the conditions in Theorem 2.1 are satisfied. Then for the additive process $\mathbb{X}(\mathbf{t}) = \sum_{j=1}^d X_j(t_j)$, $\mathbf{t}\in E^d$,*

$$\lim_{\varepsilon\to 0^+} \varepsilon^{\beta}\log\mathbb{P}\left(\sup_{\mathbf{t}\in E^d} |\mathbb{X}(\mathbf{t})| \le \varepsilon\right) = -d^{\beta+1}A_\beta \tag{3.1}$$

and

$$\lim_{\varepsilon\to 0^+} \varepsilon^{\beta}\log\mathbb{P}\left(\sup_{\mathbf{t},\mathbf{s}\in E^d} |\mathbb{X}(\mathbf{t})-\mathbb{X}(\mathbf{s})| \le \varepsilon\right) = -2^{\beta}d^{\beta+1}A_\beta \tag{3.2}$$

where A_β is the small ball constant for $X(t)$ given in (2.2).

Proof: We only need to show (3.1) since (3.2) follows from Theorem 2.1 and (3.1). First note that

$$\sup_{\mathbf{t}\in E^d} |\mathbb{X}(t)| \le \sup_{\mathbf{t}\in E^d} \sum_{j=1}^d |X_j(t_j)| = \sum_{j=1}^d \sup_{t\in E} |X_j(t)|.$$

Thus

$$\begin{aligned}
\mathbb{P}\left(\sup_{\mathbf{t}\in E^d} |\mathbb{X}(t)| \le \varepsilon\right) &\ge \mathbb{P}\left(\sum_{j=1}^d \sup_{t\in E} |X_j(t)| \le \varepsilon\right) \\
&\ge \mathbb{P}\left(\max_{1\le j\le d} \sup_{t\in E} |X_j(t)| \le d^{-1}\varepsilon\right) \\
&= \prod_{j=1}^d \mathbb{P}\left(\sup_{t\in E} |X_j(t)| \le d^{-1}\varepsilon\right) \\
&= \mathbb{P}^d\left(\sup_{t\in E} |X(t)| \le d^{-1}\varepsilon\right)
\end{aligned}$$

and we obtain the lower bound

$$\begin{aligned}
\liminf_{\varepsilon\to 0^+} \varepsilon^\beta \log \mathbb{P}\left(\sup_{\mathbf{t}\in E^d} |\mathbb{X}(t)| \le \varepsilon\right) &\ge d \cdot \liminf_{\varepsilon\to 0^+} \varepsilon^\beta \log \mathbb{P}\left(\sup_{t\in E} |X(t)| \le d^{-1}\varepsilon\right) \\
&= -d^{\beta+1} A_\beta
\end{aligned}$$

where the last line follows from (2.2).

For the upper bound, consider the range process $R_j(t)$ of X_j over the index set E, that is,

$$R_j = \sup_{s,t\in E} |X_j(t) - X_j(s)| = \sup_{t\in E} X_j(t) - \inf_{t\in E} X_j(t).$$

Then

$$\begin{aligned}
\sum_{j=1}^d R_j &= \sum_{j=1}^d \left(\sup_{t\in E} X_j(t) - \inf_{t\in E} X_j(t)\right) \\
&= \sum_{j=1}^d \sup_{t\in E} X_j(t) - \sum_{j=1}^d \inf_{t\in E} X_j(t) \\
&= \sup_{\mathbf{t}\in E^d} \sum_{j=1}^d X_j(t_j) + \sup_{\mathbf{t}\in E^d} \sum_{j=1}^d (-X_j(t_j)) \\
&\le \sup_{\mathbf{t}\in E^d} \left|\sum_{j=1}^d X_j(t_j)\right| + \sup_{\mathbf{t}\in E^d} \left|\sum_{j=1}^d X_j(t_j)\right| \\
&= 2 \sup_{\mathbf{t}\in E^d} |\mathbb{X}(t)|
\end{aligned}$$

Thus we have

$$\mathbb{P}\left(\sup_{\mathbf{t}\in E^d}|\mathbb{X}(t)|\le\varepsilon\right)\le\mathbb{P}\left(\sum_{j=1}^d R_j\le 2\varepsilon\right)$$

and by exponential Chebyshev inequality, for any $\lambda>0$,

$$\begin{aligned}\mathbb{P}\left(\sum_{j=1}^d R_j\le 2\varepsilon\right) &= \mathbb{P}\left(e^{-\lambda\sum_{j=1}^d R_j}\ge e^{-2\lambda\varepsilon}\right)\\ &\le e^{2\lambda\varepsilon}\cdot\mathbb{E}\,e^{-\lambda\sum_{j=1}^d R_j}\\ &= e^{2\lambda\varepsilon}\cdot\left(\mathbb{E}\,e^{-\lambda R}\right)^d. \qquad (3.3)\end{aligned}$$

Now it follows from (2.3) and Tauberian's theorem, see Li and Shao (2001, p547),

$$\lim_{\lambda\to\infty}\lambda^{-\beta/(\beta+1)}\log\mathbb{E}\,e^{-\lambda R}=-(\beta+1)(2/\beta)^{\beta/(\beta+1)}A_\beta^{1/(\beta+1)}$$

Taking $\lambda=2^{-1}\beta d^{\beta+1}A_\beta\varepsilon^{-(\beta+1)}$ in (3.3), we see that

$$\limsup_{\varepsilon\to0^+}\varepsilon^\beta\log\mathbb{P}\left(\sup_{\mathbf{t}\in E^d}|\mathbb{X}(t)|\le\varepsilon\right)\le-d^{\beta+1}A_\beta.$$

Note that we know in fact that

$$\lim_{\varepsilon\to0^+}\varepsilon^\beta\log\mathbb{P}\left(\sum_{j=1}^d R_j\le\varepsilon\right)=-d^{\beta+1}2^\beta A_\beta$$

from Lemma 2 in Li (2001). Here we try to stress the direct and relative easy upper bound estimates. Put out upper and lower bound together, we finish our proof of Theorem 3.1.

4. Limit Theorems for additive stable processes

Consider the additive α-stable processes

$$\mathbb{S}_\alpha(\mathbf{t})=\sum_{j=1}^d S_j(t_j)$$

on $[0,\infty)^d$ constructed from α-stable processes $S_j(t)$.

Theorem 4.1.

$$\liminf_{T\to\infty}\left(T^{-1}\log\log T\right)^{1/\alpha}\sup_{\mathbf{t}\in[0,T]^d}|\mathbb{S}_\alpha(\mathbf{t})|=(d^{\alpha+1}\lambda_\alpha)^{1/\alpha}\qquad a.s.$$

Proof: According to Theorem 3.1 and small ball estimate (2.7),

$$\lim_{\varepsilon\to 0} \varepsilon^\alpha \log \mathbb{P}\left(\sup_{\mathbf{t}\in[0,1]^d} |\mathbb{S}_\alpha(\mathbf{t})| \le \varepsilon \right) = -d^{\alpha+1}\lambda_\alpha.$$

Given $\theta > 1$, let $T_k = \theta^k$, $k \ge 1$. For any $\lambda < (d^{\alpha+1}\lambda_\alpha)^{1/\alpha}$, using the scaling property and the above estimate

$$\begin{aligned} & \sum_{k\ge 1} \mathbb{P}\left(\sup_{\mathbf{t}\in[0,T_k]^d} |\mathbb{S}_\alpha(\mathbf{t})| \le \lambda \left(T_k(\log\log T_k)^{-1}\right)^{1/\alpha} \right) \\ = & \sum_{k\ge 1} \mathbb{P}\left(\sup_{\mathbf{t}\in[0,1]^d} |\mathbb{S}_\alpha(\mathbf{t})| \le \lambda(\log\log T_k)^{-1/\alpha} \right) < \infty. \end{aligned}$$

Hence by the Borel-Cantelli lemma,

$$\liminf_{k\to\infty} \left(T_k^{-1}\log\log T_k\right)^{1/\alpha} \sup_{\mathbf{t}\in[0,T_k]^d} |\mathbb{S}_\alpha(\mathbf{t})| \ge \lambda \qquad a.s.$$

Now for any $T_k \le T \le T_{k+1}$,

$$\begin{aligned} & \left(T^{-1}\log\log T\right)^{1/\alpha} \sup_{\mathbf{t}\in[0,T]^d} |\mathbb{S}_\alpha(\mathbf{t})| \\ \ge & \left(\theta^{-1/\alpha} + o(1)\right) \left(T_k^{-1}\log\log T_k\right)^{1/\alpha} \sup_{\mathbf{t}\in[0,T_k]^d} |\mathbb{S}_\alpha(\mathbf{t})|. \end{aligned}$$

Thus

$$\liminf_{T\to\infty} \left(T^{-1}\log\log T\right)^{1/\alpha} \sup_{\mathbf{t}\in[0,T]^d} |\mathbb{S}_\alpha(\mathbf{t})| \ge \theta^{1/\alpha}\lambda \qquad a.s.$$

Letting $\lambda \to (d^{\alpha+1}\lambda_\alpha)^{1/\alpha}$ and $\theta \to 1$ proves the lower bound.

The idea for the proof of the upper bound essentially comes from Kuelbs (1981). We take $T_k = 2^k$. Let $\lambda > (d^{\alpha+1}\lambda_\alpha)^{1/\alpha}$ and $\delta > 0$ be fixed. Choose $j \ge 1$, independent of k, so that $T_{k+j} \ge \delta^{-1}T_k$ and

$$\left(T_{k+j}^{-1}\log\log T_{k+j}\right)^{1/\alpha} < \delta \left(T_k^{-1}\log\log T_{k+1}\right)^{1/\alpha}, \qquad \forall k \ge 1. \tag{4.1}$$

Next define the events

$$\begin{aligned} D_k \equiv & \Big\{ \left(T^{-1}\log\log T\right)^{1/\alpha} \sup_{\mathbf{t}\in[0,T]^d} |\mathbb{S}_\alpha(\mathbf{t})| > \lambda \text{ for all } T \ge T_{k+j}, \\ & \left(T_k^{-1}\log\log T_k\right)^{1/\alpha} \sup_{\mathbf{t}\in[0,T_k]^d} |\mathbb{S}_\alpha(\mathbf{t})| \le \lambda \Big\}. \end{aligned}$$

Then from (4.1),

$$\begin{aligned} & \Big\{ \left(T^{-1}\log\log T\right)^{1/\alpha} \sup_{\mathbf{t}\in[T_k,T]^d} |\mathbb{S}_\alpha(\mathbf{t}) - \mathbb{S}_\alpha(\mathbf{T_k})| > (1+\delta)\lambda \text{ for all } T \ge T_{k+j}, \\ & \left(T_k^{-1}\log\log T_k\right)^{1/\alpha} \sup_{\mathbf{t}\in[0,T_k]^d} |\mathbb{S}_\alpha(\mathbf{t})| \le \lambda \Big\} \subset D_k \end{aligned}$$

where $\mathbf{T_k} = (T_k, \cdots, T_k)$. Hence by independence of increment, stationarity, and scaling

$$\begin{aligned}
&\mathbb{P}(D_k) \\
\geq\ & p_k(\lambda) \cdot \mathbb{P}\left(\phi(T) \sup_{\mathbf{t}\in[0,T-T_k]^d} |\mathbb{S}_\alpha(\mathbf{t})| > (1+\delta)\lambda \text{ for all } T \geq T_{k+j}\right) \\
\geq\ & p_k(\lambda) \cdot \mathbb{P}\left(\phi(T) \sup_{\mathbf{t}\in[0,(1-\delta)T]^d} |\mathbb{S}_\alpha(\mathbf{t})| > (1+\delta)\lambda \text{ for all } T \geq T_{k+j}\right) \\
=\ & p_k(\lambda) \cdot \mathbb{P}\left(\phi(T) \sup_{\mathbf{t}\in[0,T]^d} |\mathbb{S}_\alpha(\mathbf{t})| > (1-\delta)^{-1/\alpha}(1+\delta)\lambda \text{ for all } T \geq T_{k+j}\right)
\end{aligned}$$

where

$$p_k(\lambda) = \mathbb{P}\left(\left(T_k^{-1}\log\log T_k\right)^{1/\alpha} \sup_{\mathbf{t}\in[0,T_k]^d} |\mathbb{S}_\alpha(\mathbf{t})| \leq \lambda\right)$$

and

$$\phi(T) = \left(T^{-1}\log\log T\right)^{1/\alpha}.$$

Hence, for any integer $N \geq 1$, as long as $T_k \geq N$, i.e. $k \geq \log N$,

$$\begin{aligned}
&\mathbb{P}(D_k) \geq p_k(\lambda) \\
&\cdot\mathbb{P}\left(\left(T^{-1}\log\log T\right)^{1/\alpha} \sup_{\mathbf{t}\in[0,T]^d} |\mathbb{S}_\alpha(\mathbf{t})| > (1-\delta)^{-1/\alpha}(1+\delta)\lambda \text{ for all } T \geq N\right).
\end{aligned}$$

On the other hand,

$$\sum_{k=1}^\infty \mathbb{P}(D_k) = \mathbb{E}\sum_{k=1}^\infty I_{D_k} \leq j$$

since among $\{D_k;\ \ k \geq 1\}$, at most j of them occur. Hence,

$$\begin{aligned}
j \ \geq\ & \sum_{k\geq\log N} p_k(\lambda) \\
&\cdot\mathbb{P}\left(\left(T^{-1}\log\log T\right)^{1/\alpha} \sup_{\mathbf{t}\in[0,T]^d} |\mathbb{S}_\alpha(\mathbf{t})| > (1-\delta)^{1/\alpha}(1+\delta)\lambda \text{ for all } T \geq N\right)
\end{aligned}$$

Notice that by scaling, for $\lambda > (d^{\alpha+1}\lambda_\alpha)^{1/\alpha}$,

$$\sum_{k\geq\log N} p_k(\lambda) = \mathbb{P}\left(\sup_{\mathbf{t}\in[0,T_k]^d} |\mathbb{S}_\alpha(\mathbf{t})| \leq \lambda\left(T_k(\log\log T_k)^{-1}\right)^{1/\alpha}\right) = \infty.$$

We must have for all N

$$\mathbb{P}\left(\left(T^{-1}\log\log T\right)^{1/\alpha} \sup_{\mathbf{t}\in[0,T]^d} |\mathbb{S}_\alpha(\mathbf{t})| > (1-\delta)^{-1/\alpha}(1+\delta)\lambda \text{ for all } T \geq N\right) = 0.$$

Hence

$$\liminf_{T\to\infty}\left(T^{-1}\log\log T\right)^{1/\alpha}\sup_{\mathbf{t}\in[0,T]^d}|\mathbb{S}_\alpha(\mathbf{t})|\le(1-\delta)^{-1/\alpha}(1+\delta)\lambda\qquad a.s.$$

Let $\delta\to 0$ and $\lambda\to(d^{\alpha+1}\lambda_\alpha)^{1/\alpha}$ we obtain the desired upper bound and finished the proof.

Next we formulate limiting behaviors for some additive Gaussian processes. First, consider the additive fractional Brownian motions

$$\mathbb{B}_H(\mathbf{t})=\sum_{j=1}^d B_j(t_j)$$

on $[0,\infty)^d$ constructed from fractional Brownian motions $B_j(t)$ with index parameter $H\in(0,1)$. It is nature to expect

$$\liminf_{T\to\infty}\left(T^{-1}\log\log T\right)^{H}\sup_{\mathbf{t}\in[0,T]^d}|\mathbb{B}_H(\mathbf{t})|=-\left(d^{1+1/H}C_H\right)^{H}\tag{4.2}$$

based on the small deviation estimate

$$\lim_{\varepsilon\to 0}\varepsilon^{1/H}\log\mathbb{P}\left(\sup_{\mathbf{t}\in[0,1]^d}|\mathbb{B}_H(\mathbf{t})|\le\varepsilon\right)=-d^{1+1/H}C_H.$$

from (2.6) and Theorem 3.1. Similarly, for the additive fractional integrated Brownian motions

$$\mathbb{W}_\gamma(\mathbf{t})=\sum_{j=1}^d W_j(t_j)$$

on $[0,\infty)^d$ constructed from fractional Brownian motions $W_j(t)$ with index parameter $\gamma>-1/2$. It is nature to expect

$$\liminf_{T\to\infty}\left(T^{-1}\log\log T\right)^{(2\gamma+1)/2}\sup_{\mathbf{t}\in[0,T]^d}|\mathbb{W}_\gamma(\mathbf{t})|=-\left(d^{1+2/(2\gamma+1)}k_\gamma\right)^{(2\gamma+1)/2}\tag{4.3}$$

based on the small deviation estimate

$$\lim_{\varepsilon\to 0}\varepsilon^{2/(2\gamma+1)}\log\mathbb{P}\left(\sup_{\mathbf{t}\in[0,1]^d}|\mathbb{W}_\gamma(\mathbf{t})|\le\varepsilon\right)=-d^{1+2/(2\gamma+1)}k_\gamma$$

from (2.5) and Theorem 3.1. The lower bounds for (4.2) and (4.3) follows easily from standard arguments given in the proof of Theorem 4.1. For the upper bounds in (4.2) and (4.3), we believe that detailed proofs can be obtained, but we will not go further in this direction since we do not have a nice and instructive arguments.

5. Some related multi-parameter processes

The additive process in the early sections requires copies of independent process $X(t)$ on E, and each addition sign can be changed to minus if $X(t)$ is symmetric. If the same process $X(t)$ is used with plus/minus sign, then we obtain some related multi-parameter processes. Using symmetry, these processes can be represented as

$$\mathbb{X}_{m,d}(\mathbf{t}) = \sum_{i=1}^{d-m} X(t_i) - \sum_{i=d-m+1}^{d} X(t_i), \quad \mathbf{t} \in E^d, \quad m \le d/2.$$

Next observe that the supremum norm of the process $\mathbb{X}_{m,d}(\mathbf{t})$ is simple. Indeed, it is easy to see

$$\sup_{\mathbf{t}\in E^d} |\mathbb{X}_{m,d}(\mathbf{t})| = \sup_{0\le s,t,u\le 1} |m(X(t)-X(s)) + (d-2m)X(u)| = mR + (d-2m)M.$$

and thus the small ball estimates follow from (2.4). More precisely,

$$\lim_{\varepsilon\to 0} \varepsilon^{\beta} \log \mathbb{P}\left(\sup_{\mathbf{t}\in E^d} |\mathbb{X}_{m,d}(\mathbf{t})| \le \varepsilon\right) = -d^{\beta} A_{\beta}. \tag{5.1}$$

Theorem 5.1. *Let $\mathbb{S}_{m,d}(\mathbf{t}), t \in [0,\infty)^d$ be the multi-parameter process generated by the same α-stable process $S_\alpha(t), t \ge 0$. Then*

$$\liminf_{T\to\infty} \left(T^{-1}\log\log T\right)^{1/\alpha} \sup_{\mathbf{t}\in[0,T]^d} |\mathbb{S}_{m,d}(\mathbf{t})| = -d\lambda_\alpha^{1/\alpha}.$$

In particular, for the range process $R_\alpha(T) = \sup_{s,t\in[0,T]} |S_\alpha(t) - S_\alpha(s)|$ of α-stable processes $S_\alpha(t)$,

$$\liminf_{T\to\infty} \left(T^{-1}\log\log T\right)^{1/\alpha} R_\alpha(T) = -2\lambda_\alpha^{1/\alpha}.$$

The proof of Theorem 5.1 follows from the same argument given in the proof of Theorem 4.1 and we omit the details.

Next we consider a generalization of the additive processes in the early sections by requiring additions of independent processes $Y_j(t)$ on E, which are not necessarily copies of each other. These cover processes such as $\sum_{j=1}^{d} \pm X_j(t_j)$ where $X_j(t)$ are independent copies given in Theorem 3.1. Note that $X(t)$ need not be symmetric.

Theorem 5.2. *Let $Y_j(t)$, $1 \le j \le d$, be independent stochastic processes index by E. Assume the conditions in Theorem 2.1 are satisfied for each $Y_j(t)$ with the small ball properties*

$$\lim_{\varepsilon\to 0^+} \varepsilon^{\beta_j} \log \mathbb{P}\left(\sup_{t\in E} |Y_j(t)| \le \varepsilon\right) = -A_j, \quad 0 < A_j < \infty, \quad 1 \le j \le d$$

and for a fixed m, $1 \le m \le d$,

$$\beta = \beta_1 = \beta_2 = \cdots = \beta_m > \beta_{m+1} \ge \beta_{m+2} \ge \cdots \ge \beta_d \ge 0.$$

Then for the additive type process $\mathbb{Y}(\mathbf{t}) = \sum_{j=1}^{d} Y_j(t_j)$, $\mathbf{t} \in E^d$,

$$\lim_{\varepsilon \to 0^+} \varepsilon^\beta \log \mathbb{P}\left(\sup_{\mathbf{t} \in E^d} |\mathbb{Y}(\mathbf{t})| \le \varepsilon \right) = -\left(\sum_{j=1}^{m} A_j^{1/(1+\beta)} \right)^{1+\beta} \tag{5.2}$$

and

$$\lim_{\varepsilon \to 0^+} \varepsilon^\beta \log \mathbb{P}\left(\sup_{\mathbf{t},\mathbf{s} \in E^d} |\mathbb{Y}(\mathbf{t}) - \mathbb{Y}(\mathbf{s})| \le \varepsilon \right) = -2^\beta \left(\sum_{j=1}^{m} A_j^{1/(1+\beta)} \right)^{1+\beta}. \tag{5.3}$$

Proof: We only need to show (5.2) since (5.3) follows from Theorem 2.1 and (5.2). The proof of (5.2) follows more or less the arguments given for (3.1). Here we only point out the additional differences. For the lower bound, fix $\delta > 0$ small and set $A = \sum_{j=1}^{m} A_j^{1/(1+\beta)}$. Then

$$\begin{aligned}
\mathbb{P}\left(\sup_{\mathbf{t} \in E^d} |\mathbb{Y}(t)| \le \varepsilon \right) &\ge \mathbb{P}\left(\sum_{j=1}^{d} \sup_{t \in E} |Y_j(t)| \le \varepsilon \right) \\
&\ge \mathbb{P}\Big(\max_{1 \le j \le m} \sup_{t \in E} |Y_j(t)| \le A_j^{1/(1+\beta)} (A + d\delta)^{-1} \varepsilon, \\
&\qquad \max_{m < j \le d} \sup_{t \in E} |Y_j(t)| \le \delta (A + d\delta)^{-1} \varepsilon \Big) \\
&= \prod_{j=1}^{m} \mathbb{P}\left(\sup_{t \in E} |Y_j(t)| \le A_j^{1/(1+\beta)} (A + d\delta)^{-1} \varepsilon \right) \\
&\qquad \cdot \prod_{j=m+1}^{d} \mathbb{P}\left(\sup_{t \in E} |Y_j(t)| \le \delta (A + d\delta)^{-1} \varepsilon \right)
\end{aligned}$$

and hence

$$\begin{aligned}
& \liminf_{\varepsilon \to 0^+} \varepsilon^\beta \log \mathbb{P}\left(\sup_{\mathbf{t} \in E^d} |\mathbb{Y}(t)| \le \varepsilon \right) \\
\ge\; & \sum_{j=1}^{m} \liminf_{\varepsilon \to 0^+} \varepsilon^\beta \log \mathbb{P}\left(\sup_{t \in E} |Y_j(t)| \le A_j^{1/(1+\beta)} (A + d\delta)^{-1} \varepsilon \right) \\
& + \sum_{j=m+1}^{d} \liminf_{\varepsilon \to 0^+} \varepsilon^\beta \log \mathbb{P}\left(\sup_{t \in E} |Y_j(t)| \le \delta (A + d\delta)^{-1} \varepsilon \right) \\
=\; & -\sum_{j=1}^{m} A_j \left(A_j^{1/(1+\beta)} (A + d\delta)^{-1} \right)^{-\beta} \\
=\; & -A (A + d\delta)^\beta.
\end{aligned}$$

The lower bound follows by taking $\delta \to 0$.

For the upper bound, we use

$$\mathbb{P}\left(\sup_{\mathbf{t}\in E^d}|\mathbb{Y}(t)|\le\varepsilon\right)\le\mathbb{P}\left(\sum_{j=1}^{d}R_j\le 2\varepsilon\right)\le\mathbb{P}\left(\sum_{j=1}^{m}R_j\le 2\varepsilon\right)$$

and then Lemma 2 in Li (2001).

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