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Original article

Spectral collocation method for stochastic Burgers equation driven by additive noise

Minoo Kamrani, S. Mohammad Hosseini *

Department of Applied Mathematics, Faculty of Mathematical Sciences, Tarbiat Modares University, P.O. Box 14115-175, Tehran, Iran

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Abstract

Almost nothing decisive has been said about collocation methods for solving SPDEs. Among the best of such SPDEs the Burgers equation shows a prototypical model for describing the interaction between the reaction mechanism, convection effect, and diffusion transport. This paper discusses spectral collocation method to reduce stochastic Burgers equation to a system of stochastic ordinary differential equations (SODEs). The resulting SODEs system is then solved by an explicit 3-stage stochastic Runge-Kutta method of strong order one. The convergence rate of Fourier collocation method for Burgers equation is also obtained. Some numerical experiments are included to show the performance of the method.

Keywords: Spectral collocation method; Stochastic ordinary differential equation; Stochastic partial differential equation; System of SODEs

1. Introduction

In recent years the study of stochastic partial differential equations (SPDEs) has been an important area of research. Many phenomena in science and engineering that may have been modeled by deterministic partial differential equations, have some uncertainty, due to existence of different stochastic perturbations. Therefore to represent a more accurate detail of behavior of such phenomena they usually should be modeled by SPDEs. SPDEs have many applications in continuum physics [3,4], finance, for example for contingent claim, bond pricing problem, interest rate of option and forward caps [6]. Several authors investigated numerical solutions of SPDEs. Some authors have used finite differences, finite elements or spectral Galerkin methods for spatial variable discertization and then solved the resulting system of SODEs via Euler method or the Crank–Nicolson scheme [2,14,21,25,26,32–34].

The Monte-Carlo method is another relatively straight-forward method with a long history of applications in finance and physics that can also be used. But this method is by far the most popular in simulating the effects modeled by SPDEs. In Monte-Carlo method the SPDE problems by generating suitable random numbers are formulated as deterministic PDEs and then are solved by standard numerical methods. However, this method has some limitations in applications with complex stochastic term as many realizations have to be carried out in order to obtain reliable estimates of various statistical properties. Hence, the Monte-Carlo method is generally computationally expensive and only recommended as the last resort [9,22,31]. For optimization problems governed by SPDEs we refer the reader to

* Corresponding author. Tel.: +98 21 82883454; fax: +98 21 82883454.
E-mail addresses: hosseini.gm@gmail.com, hossei_m@modares.ac.ir (S.M. Hosseini).

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[24] in which the result of a comparison between Monte-Carlo and stochastic Galerkin and collocation methods is in favor of collocation. Wiener Chaos expansion is another method that we can use for the solution of SPDE, but this method is also computationally expensive because a large number of chaos coefficients in the expansions need to be accurately computed, and many realizations have to be performed to obtain accurate estimates of the required statistical characteristics [18,28]. In 1998 Machiels and Deville investigated Fourier spectral method for SPDE [27]. Some other authors used Galerkin approximation for SPDEs [15,19]. But to our knowledge collocation methods have not been yet studied for parabolic SPDEs of the type of Burgers equation. Obviously, the Galerkin and finite difference, or other above-mentioned methods, like any other numerical method, have their own advantages and disadvantages, but here we just mention two major advantages of the collocation methods:

(i) Since no integration is required, the construction of the final system of equations is very efficient.
(ii) The functions must be evaluated only at the collocation nodes in contrast to other methods.

In addition to these advantages with the collocation method the computational cost of calculating nonlinear terms and incorporating general boundary conditions (Dirichlet, Neumann, and mixed) is reasonably low with good numerical accuracy. It should also be mentioned that a vast majority of the researches have considered SPDEs in which the drift functions satisfy global Lipschitz condition (see [16,20,23,30]), while in many applications, such as Burgers equation, the drift function fails to satisfy global Lipschitz condition. Therefore, in this paper, we study spectral collocation method to approximate numerical solution of stochastic Burgers equation with additive noise such as

$$\frac{\partial u}{\partial t}(t,x)=\frac{\partial^2 u}{\partial x^2}(t,x)+u(t,x)\frac{\partial u}{\partial x}(t,x)+\frac{\partial^2 W}{\partial t\partial x}(t,x) \tag{1.1}$$

with the initial condition

$$u(0,x)=u_0(x), \tag{1.2}$$

and boundary conditions

$$u(t,0)=0,\quad u(t,\pi)=0, \tag{1.3}$$

where $\partial^2 W/\partial t\partial x(t,x)$ is a space-time white noise.

The solvability and the properties of its solution have been intensively studied in the literature, see [10,11,13]. For spatial discretization of Burgers equation Blömker and Jentzen [5] used spectral Galerkin approximation and Alabert and Gyöngy [1] introduced and theoretically investigated the following finite difference method

$$\begin{aligned} du_i^N(t)&=N^2\sum_{k=1}^{N-1}D_{ik}u_k^N(t)dt+\frac{1}{6}(|u_{i+1}^N|^2(t)-|u_{i-1}^N|^2(t)+u_{i+1}^N(t)u_i^N(t)-u_i^N(t)u_{i-1}^N(t))dt+\sqrt{N}dW_i^N(t),\\ &\quad i=1,2,\ldots,N-1, \end{aligned} \tag{1.4}$$

where $u^N(t):=(u_i^N(t))$ is column vector in R^{N-1}, and $W^N(t):=(W_i^N(t))$ is an $(N-1)$-dimensional Wiener process, and $D_{ii}=-2$, $D_{ik}=1$, for $|i-k|=1$, and $D_{ik}=0$, for $|i-k|>1$, $i=1,2,\ldots,N-1$.

As a test of numerical performance, we also apply the method (1.4) to support the efficiency and robustness of spectral collocation method presented here. It should also be emphasized that our main aim here is to investigate an analysis of the collocation method for the space discretization. Hence, for solving the resulting system of SODEs for time variable one could use any of the well-established and appropriate methods like Euler or some other stochastic Runge-Kutta methods. We have used an explicit 3-stage stochastic Runge-Kutta method with optimal strong order one [7].

The paper is organized as follows. In Section 2 we explain Fourier spectral collocation method briefly. In Section 3 the formulations of this method for stochastic Burgers equation are obtained, then the formulas of an explicit 3 stage stochastic Runge-Kutta are recalled to be used for time discretization. In Section 4, analytical investigation of convergence and the rate of convergence of the Fourier spectral collocation method for this equation is carried out. Finally in the numerical section, the confirmation of theoretical rate of convergence is illustrated. Also some numerical comparison is made with the results of well-established method (1.4) for the support of the numerical performance of collocation method.

2. Spatial discretization by spectral collocation method

Spectral methods are a class of spatial discretization for differential equations. The key components for their formulations are the trial functions (or approximating functions) and test functions (also known as weight functions).

The choice of test functions distinguishes between the three earliest types of spectral schemes, namely, the Galerkin, collocation and tau versions.

The collocation approach appears to have been first used by Slater and by Kantorovic (1934) in specific applications. This approach is especially attractive whenever it applies to variable-coefficient and even nonlinear problems. One of the trial functions that we can use is the Fourier sine series, that are advisable for problems with appropriate boundary conditions as the Burgers equation considered here. These are defined on $[0, \pi]$ by $e_k(x) = \sqrt{2/\pi}\sin(kx)$, $k = 1, 2, \ldots$ Thus for approximating function $f(x)$ in the form of a sine series we choose the following interpolation points in the interval $[0, \pi]$:

$$x_j = \frac{\pi j}{N} \quad (j = 1, \ldots, N-1). \tag{2.1}$$

In order to construct the interpolant of $f(x)$ at these points we first define the polynomials [8]

$$C_j(x) = \frac{2}{N}\sum_{m=1}^{N-1} \sin(mx_j)\sin(mx), \tag{2.2}$$

to satisfy $C_j(x_i) = \delta_{ij}$, $i, j = 1, \ldots, N-1$. For a given function $f(x)$ which vanishes at $x = 0$ and π, we consider

$$P_N(x) = \sum_{j=1}^{N-1} f(x_j)C_j(x), \tag{2.3}$$

as its interpolating projection trigonometric polynomial. In this case the projection space is $\tilde{B}_N = \text{span}\{\sin(kx) : k = 1, \ldots, N-1\}$ on which some further properties are explained in Appendix B. The second step in the spectral collocation approximation is to express the derivative of $f(x)$ at the collocation points x_j, $j = 0, \ldots, N$. By differentiating (2.3), after extending to include $j = 0$ and $j = N$, we obtain

$$\frac{d^k P_N(x)}{dx^k} = \sum_{j=0}^{N} f(x_j)\frac{d^k}{dx^k}C_j(x), \tag{2.4}$$

so that

$$\frac{d^k P_N(x_i)}{dx^k} = \sum_{j=0}^{N} f(x_j)(D_k)_{i,j}, \quad i = 0, \ldots, N, \tag{2.5}$$

$$(D_k)_{i,j} = \frac{d^k}{dx^k}C_j(x_i), \quad i, j = 0, \ldots, N, \tag{2.6}$$

where

$$(D_1)_{ij} = \begin{cases} 0, & j = 0 \text{ or } N; \\ -0.5\cot(x_j), & i = j, \\ \dfrac{(-1)^{i+j+1}\sin(x_j)}{\cos(x_i) - \cos(x_j)}, & \text{otherwise.} \end{cases}$$

and $D_2 = D'D_1$, where

$$(D')_{ij} = \begin{cases} 0, & i = 0 \text{ or } i = N, \text{ all } j, \\ 0.5 \cot(x_j), & i = j;\ j = 1, \ldots, N-1, \\ \dfrac{(-1)^{j+1} \sin(x_i) \cos(Nx_i)}{[\cos(x_i) - \cos(x_j)]}, & \text{otherwise.} \end{cases}$$

Obviously, using these approximate matrix representations one can approximate function $f(x)$ and its derivatives.

3. Formulation of stochastic collocation method

Let us assume that $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ is a filtered probability space and the process $W = (W(t, x))$, a Brownian sheet on $[0, \pi] \times [0, T]$, is adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$, where $[0, \pi]$ is the spatial domain. Recall that W is a zero mean Gaussian process such that $\mathbb{E}(W(t, x)W(s, y)) = (x \wedge y)(t \wedge s)$, and $W(t, x) - W(s, x) + W(s, y) - W(t, y)$ is independent of $\mathcal{F}_s$ for all $x, y \in [0, \pi]$ and $0 \leq s \leq t$.

We will describe spectral collocation method for the approximation of solution of stochastic Burgers equation

$$\frac{\partial u}{\partial t}(t, x) = \frac{\partial^2 u}{\partial x^2}(t, x) + u(t, x)\frac{\partial u}{\partial x}(t, x) + \frac{\partial^2 W}{\partial t \partial x}(t, x), \tag{3.1}$$

with the initial condition

$$u(0, x) = u_0(x), \tag{3.2}$$

and boundary conditions

$$u(t, 0) = 0, \quad u(t, \pi) = 0, \tag{3.3}$$

where u_0 is a square integrable function over $[0, \pi]$, and $\partial^2 W/\partial t \partial x(t, x)$ is a space-time white noise.

These equations are important due to their role in modeling of certain turbulent effects. Now we apply the spectral collocation method to the stochastic Burgers equation.

Let N be a non-negative integer and $x_i = \pi i/N$, $i = 0, 1, \ldots, N$, the points in the interval $[0, \pi]$. We discretize the equation in its space variable by spectral collocation method.

We approximate $u(t, x)$, the solution of (1.1), by $u^N(t, x)$ defined as

$$u^N(t, x) = \sum_{i=0}^{N} u(t, x_i) C_i(x), \tag{3.4}$$

where $C_i(x)$ was introduced in (2.2). From (2.5) we have

$$\frac{\partial u^N}{\partial x}(t, x_i) = \sum_{j=0}^{N} (D_1)_{i,j} u^N(t, x_j), \tag{3.5}$$

and

$$\frac{\partial^2 u^N}{\partial x^2}(t, x_i) = \sum_{j=0}^{N} (D_2)_{i,j} u^N(t, x_j), \quad i = 0, 1, 2, \ldots, N, \tag{3.6}$$

where

$D_k = [(D_k)_{i,j}]_{i,j=0,}^N$, are spectral differentiation matrices of order k for $k = 1, 2, \ldots$ that were already introduced in (2.6). Then from (1.1) by using finite difference method for space discretization of the stochastic term and bearing in mind that $u^N(t, x_0) = 0$ and $u^N(t, x_N) = 0$, we have

$$du^N(t, x_i) = \left(\sum_{j=1}^{N-1} (D_2)_{i,j} u^N(t, x_j) + u^N(t, x_i) \sum_{j=1}^{N-1} (D_1)_{i,j} u^N(t, x_j) \right) dt + \frac{N}{\pi} d(W(t, x_{i+1}) - W(t, x_i)). \tag{3.7}$$

Using the notation $u_i^N(t) = u^N(t, x_i)$, $W_i^N(t) = \sqrt{N/\pi}(W(t, x_{i+1}) - W(t, x_i))$, for $i = 1, 2, \ldots, N-1$, the system (3.7), can be rewritten as

$$du_i^N(t) = \left(\sum_{j=1}^{N-1} (D_2)_{i,j} u_j^N(t) + u_i^N(t) \sum_{j=1}^{N-1} (D_1)_{i,j} u_j^N(t) \right) dt + \sqrt{\frac{N}{\pi}} dW_i^N(t). \tag{3.8}$$

for $i = 1, 2, \ldots, N-1$. Notice that $W_i^N(t)$ is a Wiener process, see Appendix A.

Therefore for the stochastic Burgers equation (1.1) the following Itô stochastic differential system

$$du^N(t) = F(u^N(t))dt + GdW^N(t), \tag{3.9}$$

represents (3.8), where $u^N(t)$ denotes the unknown semi-discretized solution at time t, i.e., $u^N(t) := (u_i^N(t))$, $F(u^N(t))$ and G denote the semi-discretized forms of deterministic and stochastic parts in (3.8), respectively. Here we have

$$F(u^N(t)) := (F_i(u^N(t))), \tag{3.10}$$

$$F_i(u^N(t)) \quad = \sum_{j=1}^{N-1} (D_2)_{i,j} u_j^N(t) + u_i^N(t) \sum_{j=1}^{N-1} (D_1)_{i,j} u_j^N(t) \tag{3.11}$$

and G is a $N-1 \times N-1$ diagonal matrix that its ith entry is $\sqrt{N/\pi}$, and $W^N(t) := (W_i^N(t))$ is a $N-1$ dimensional Wiener process and

$$u_i^N(0) = u_0(x(i)), \quad i = 1, 2, \ldots, N-1. \tag{3.12}$$

3.1. Stochastic Runge-Kutta schemes for time integration

In order to solve the system of SODEs (3.9) we apply an explicit 3-stage SRK of the class of stochastic Runge-Kutta methods. This is known as one of the best 3-stage explicit methods that has the strong convergence of order 1. Recall that an approximate solution $\overline{Y}(t)$ approximates the solution $Y(t)$ with order of accuracy p in the strong sense, if the following inequality holds:

$$\mathbb{E}|\overline{Y}_N - Y(t_N)| \leq Ch^p,$$

where $\overline{Y}_N$ is the numerical approximation to $Y(t_N)$, and constant $C > 0$ is independent of h.

Here we apply a class of stochastic Runge-Kutta methods proposed in [7] as follows

$$\begin{aligned} \mathrm{H}_i &= u^N(n\Delta t) + \Delta t \sum_{j=1}^{s} a_{ij} F(\mathrm{H}_j) + \sum_{k=1}^{n-1} \sum_{j=1}^{s} b_{ij} G^k \xi_k (\Delta t)^{1/2}, \quad i = 1, \ldots, s, \\ u^N((n+1)\Delta t) &= u^N(n\Delta t) + \Delta t \sum_{j=1}^{s} \alpha_j F(\mathrm{H}_j) + \sum_{k=1}^{n-1} \sum_{j=1}^{s} \gamma_j G^k \xi_k (\Delta t)^{1/2}, \end{aligned} \tag{3.13}$$

where $u^N(0)$ is defined in (3.12), G^k denotes the kth column of the matrix G, and ξ_k are random variables with normal distribution $N(0, 1)$.

In this formulation, $A = (a_{ij})$ and $B = (b_{ij})$ are $s \times s$ matrices with real elements while $\alpha^T = (\alpha_1, \ldots, \alpha_s)$ and $\gamma^T = (\gamma_1, \ldots, \gamma_s)$ are row vectors in R^s. According to the Butcher tableau

$$\begin{array}{c|c|c} & A & B \\ \hline & \alpha^T & \gamma^T \end{array}$$

for the members of the SRK family (3.13), the implemented $s = 3$ stage explicit SRK method is as follows:

$$\begin{array}{c|ccc|ccc} & 0 & 0 & 0 & 0 & 0 & 0 \\ & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 \\ & 0 & \frac{3}{4} & 0 & 0 & \frac{3}{4} & 0 \\ \hline & \frac{2}{9} & \frac{3}{9} & \frac{4}{9} & \frac{2}{9} & \frac{3}{9} & \frac{4}{9} \end{array}$$

It should be mentioned that for solving the stochastic system (3.9) we could also use another methods such as those proposed in [12,29], but as the analysis of collocation space discretization method has been the main purpose of this paper, taking any suitable well-established scheme to solve the system of SODEs will suffice. Therefore, for the system of SODEs, appearing here after spatial discretization which has more than one Wiener process, using the simple 3-stage explicit SRK method (3.13), of first order strong convergence, could be appropriate [7].

4. Convergence and order of convergence of the spectral collocation method

In this section we assume that the system (3.7) has a solution u^N. Because the analysis of space discretization method has been the main purpose of this paper we investigate the error of spectral collocation method.

We rewrite Eq. (3.7) for the solution u^N in the form

$$u^N(t) = e^{tD_2}a^N + \int_0^t e^{(t-s)D_2}(H(u^N(s))ds + \sqrt{\frac{N}{\pi}}\int_0^t e^{(t-s)D_2}dW^N(s), \tag{4.1}$$

where H is a function from R^{N-1} to R^{N-1} with $H_k(x_1, x_2, \ldots, x_{N-1}) = x_k\sum_{j=1}^{N-1}(D_1)_{k,j}x_j$, $k = 1, 2, \ldots, N-1$. Consider the R^{N-1}-valued random processes

$$\eta^N(t) = \sqrt{\frac{N}{\pi}}\int_0^t e^{(t-s)D_2}dW^N(s). \tag{4.2}$$

The vectors $e_1, \ldots, e_{N-1}$ defined by

$$e_j = (e_j(k)) = \left(\sqrt{\frac{2}{N}}\sin\left(j\frac{k\pi}{N}\right)\right), \quad k = 1, 2, \ldots, N-1, \tag{4.3}$$

form an orthonormal basis in R^{N-1}, and they are eigenvectors of the matrix D_2, with eigenvalues

$$\lambda_j = -j^2, \quad j = 1, 2, \ldots, N-1. \tag{4.4}$$

Therefore, for the random field $\{\eta^N(t, x), t \geq 0, x \in [0, \pi]\}$ defined by

$$\eta^N(t, x_k) := \eta_k^N := \sqrt{\frac{N}{\pi}}\int_0^t e^{(t-s)D_2}dW^N(s), \tag{4.5}$$

for $x_k := k\pi/N$, $k = 1, \ldots, N-1$, and

$$\eta^N(t, 0) = \eta^N(t, \pi) = 0, \tag{4.6}$$

$$\eta^N(t, x) := \eta^N(t, \kappa_N(x)), \quad x \in (0, \pi), \tag{4.7}$$

$\kappa_N(x) := \pi\frac{[Nx/\pi]}{N}$, we have

$$\eta^N(t, x) = \int_0^t\int_0^\pi G^N(t-s, x, y)dW(s, y), \tag{4.8}$$

for all $t \geq 0$, $x \in [0, \pi]$, where

$$G^N(t, x, y) := \sum_{k=1}^{N-1}\exp(\lambda_k t)\varphi_k^N(\kappa_N(x))\varphi_k(\kappa_N(y)), \tag{4.9}$$

with $\varphi_k(x) = \sqrt{2/\pi}\sin(kx)$, $\varphi_k^N(x) = \varphi_k(l\pi/N)$ for $x = l\pi/N$ and

$$\varphi_k^N(x) = \varphi_k\left(\frac{l\pi}{N}\right) + \frac{Nx - l\pi}{\pi}\left(\varphi_k\left(\frac{(l+1)\pi}{N}\right) - \varphi_k\left(\frac{l\pi}{N}\right)\right), \tag{4.10}$$

for $x \in (l\pi/N, ((l+1)\pi)/N)$.

Theorem 4.1. *Let $N \geq 2$ be an integer. For u^N, the solution of system (3.7), with initial condition*

$$u_k^N(0) = a_k^N, \quad k = 1, 2, \ldots, N-1, \tag{4.11}$$

there is a finite random variable ξ *such that*

$$\sup_{t\le T}\frac{2}{N}\sum_{k=1}^{N-1}(u_k^N(t)^2)\le \xi\left(\frac{2}{N}\sum_{k=1}^{N-1}|a_k^N|^2+1\right)\ (a.s.) \tag{4.12}$$

Proof. Assume

$$v(t):=v^N(t)=u^N(t)-\eta^N(t). \tag{4.13}$$

Then from Eqs. (3.7) and (4.1), v satisfies

$$\begin{aligned} &dv^N(t)=(D_2v^N(t)+H(v^N(t)+\eta^N(t)))dt,\\ &v^N(0)=a^N, \end{aligned} \tag{4.14}$$

where $a^N:=(a_k^N)$ is a $N-1$-dimensional vector in R^{N-1} . This means

$$dv^N(t,x_k)=\left(\sum_{j=1}^{N-1}(D_2)_{k,j}v^N(t,x_j)+(v^N(t,x_k)+\eta^N(t,x_k))\sum_{j=1}^{N-1}(D_1)_{k,j}(v^N(t,x_j)+\eta^N(t,x_j))\right)dt, \tag{4.15}$$

From (3.5) and (3.6) we conclude

$$\frac{\partial v^N}{\partial t}(t,x_k)=\frac{\partial^2 v^N}{\partial x^2}(t,x_k)+(v(t,x_k)+\eta^N(t,x_k))\frac{\partial(v^N+\eta^N)}{\partial x}(t,x_k), \tag{4.16}$$

for $k=1,\ldots,N-1$. Let us multiply the kth equation of (4.16) by $(2/N)v^N(t,x_k)$ and sum over k. We get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\frac{2}{N}\sum_{k=1}^{N-1}v^N(t,x_k)^2&=\frac{2}{N}\sum_{k=1}^{N-1}\frac{\partial^2 v^N}{\partial x^2}(t,x_k)v^N(t,x_k)+\frac{2}{N}\sum_{k=1}^{N-1}v^N(t,x_k)(v^N(t,x_k)\\ &+\eta^N(t,x_k))\frac{\partial(v^N+\eta^N)}{\partial x}(t,x_k).\end{aligned} \tag{4.17}$$

Now we use the discrete inner product

$$(f,g)_N=\frac{2}{N}\sum_{k=1}^{N-1}f(x_k)g(x_k)$$

for f and $g\in\tilde{B}_N$. Then from the trapezoidal quadrature rule and because each element of $\tilde{B}_N$ vanishes at $x=0\,\&\,\pi$, it is seen that the discrete inner product equals to the continuous inner product, see Appendix B, that is

$$(f,g)_N=\frac{2}{\pi}\int_0^\pi f\,g\,dx,\quad \text{for}\quad f,g\in\tilde{B}_N. \tag{4.18}$$

Therefore, from (4.18) and the exactness of the quadrature formula, since v^N and $\partial^2 v^N/\partial x^2$ belong to $\tilde{B}_N$ and

$$\int_0^\pi\frac{\partial^2 v^N}{\partial x^2}(t,x)v^N(t,x)dx=-\int_0^\pi\left(\frac{\partial v^N(t,x)}{\partial x}\right)^2dx, \tag{4.19}$$

Eq. (4.17) is rewritten as

$$\frac{1}{2}\frac{d}{dt}\int_0^\pi v^N(t,x)^2dx=-\int_0^\pi\left(\frac{\partial v^N(t,x)}{\partial x}\right)^2dx+\frac{\pi}{N}\sum_{k=1}^{N-1}v^N(t,x_k)(v^N(t,x_k)+\eta^N(t,x_k))\frac{\partial(v^N+\eta^N)}{\partial x}(t,x_k). \tag{4.20}$$

The second term on the right side of (4.20), which is denoted by $\mathcal{C}$ for simplicity, is trapezoidal quadrature approximation of the integral value $(1/2)\mathcal{I}$ where

$$\mathcal{I} = \int_0^\pi v^N(t,x)\frac{\partial(v^N(t,x)+\eta^N(t,x))^2}{\partial x}dx.$$

Hence for any $\varepsilon>0$ there exists N such that

$$|\mathcal{C}| \le \frac{1}{2}|\mathcal{I}| + \varepsilon. \tag{4.21}$$

On the other hand

$$\begin{aligned}\mathcal{I} &= v^N(t,x)(v^N(t,x)+\eta^N(t,x))^2|_0^\pi - \int_0^\pi \frac{\partial v^N(t,x)}{\partial x}(v^N(t,x)+\eta^N(t,x))^2dx\\ &= -\int_0^\pi \frac{\partial v^N(t,x)}{\partial x}v^N(t,x)^2dx - \int_0^\pi \frac{\partial v^N(t,x)}{\partial x}\eta^N(t,x)^2dx - 2\int_0^\pi \frac{\partial v^N(t,x)}{\partial x}v^N(t,x)\eta^N(t,x)dx\\ &= -\int_0^\pi \frac{\partial v^N(t,x)}{\partial x}\eta^N(t,x)^2dx - 2\int_0^\pi \frac{\partial v^N(t,x)}{\partial x}v^N(t,x)\eta^N(t,x)dx,\end{aligned}$$

and

$$\begin{aligned}|\mathcal{I}| &\le \left(\int_0^\pi\left(\frac{\partial v^N(t,x)}{\partial x}\right)^2dx\right)^{1/2}\cdot\left(\int_0^\pi \eta^N(t,x)^4dx\right)^{1/2} + 2|\overline{\eta}^N|\left(\int_0^\pi\left(\frac{\partial v^N(t,x)}{\partial x}\right)^2dx\right)^{1/2}\left(\int_0^\pi v^N(t,x)^2dx\right)^{1/2}\\ &\le 2\left(\int_0^\pi\left(\frac{\partial v^N(t,x)}{\partial x}\right)^2dx\right)^{1/2}\left(\left(\int_0^\pi \eta^N(t,x)^4dx\right)^{1/2} + |\overline{\eta}^N|\left(\int_0^\pi v^N(t,x)^2dx\right)^{1/2}\right)\\ &\le \int_0^\pi\left(\frac{\partial v^N(t,x)}{\partial x}\right)^2dx + \left(\left(\int_0^\pi \eta^N(t,x)^4dx\right)^{1/2} + |\overline{\eta}^N|\left(\int_0^\pi v^N(t,x)^2dx\right)^{1/2}\right)^2\\ &\le \int_0^\pi\left(\frac{\partial v^N(t,x)}{\partial x}\right)^2dx + 2\int_0^\pi \eta^N(t,x)^4dx + 2|\overline{\eta}^N|^2\int_0^\pi v^N(t,x)^2dx\\ &\le \int_0^\pi\left(\frac{\partial v^N(t,x)}{\partial x}\right)^2dx + 2\pi|\overline{\eta}^N|^4 + 2|\overline{\eta}^N|^2\int_0^\pi v^N(t,x)^2dx,\end{aligned} \tag{4.22}$$

where $\overline{\eta}^N = \underset{0\le x\le\pi}{sup}\ \underset{0\le t\le T}{sup}|\eta^N(t,x)|$.

Therefore, from (4.20), (4.21), and (4.22) we have

$$\frac{d}{dt}\int_0^\pi v^N(t,x)^2dx \le -\int_0^\pi\left(\frac{\partial v^N(t,x)}{\partial x}\right)^2dx + 2\epsilon + 2\pi|\overline{\eta}^N|^4 + 2|\overline{\eta}^N|^2\int_0^\pi v^N(t,x)^2dx, \tag{4.23}$$

Gronwall inequality yields

$$\underset{0\le t\le T}{sup}\int_0^\pi v^N(t,x)^2dx \le \left(\int_0^\pi v^N(0,x)^2dx + 2\pi|\overline{\eta}^N|^4T + 2\epsilon T\right)e^{2|\overline{\eta}^N|^2T}, \tag{4.24}$$

hence

$$\underset{0\le t\le T}{sup}\frac{2}{N}\sum_{k=1}^{N-1}v^N(t,x_k)^2 \le \left(\frac{2}{N}\sum_{k=1}^{N-1}|a_k^N|^2 + 2|\overline{\eta}^N|^4T + 2\epsilon/\pi T\right)e^{2|\overline{\eta}^N|^2T}. \tag{4.25}$$

Therefore

$$\underset{0\le t\le T}{sup}\frac{2}{N}\sum_{k=1}^{N-1}u_k^N(t)^2 \le \xi^N\left(\frac{2}{N}\sum_{k=1}^{N-1}|a_k^N|^2 + 1\right), \tag{4.26}$$

where from Theorem 2.1 in [1], it can easily be proved that $\xi = \sup_{N\geq 2} \xi^N = e^{2|\overline{\eta}^N|^2 T} + 2|\overline{\eta}^N|^4 T + 2\epsilon/\pi T$ is a finite random variable. □

Theorem 4.2. *Assume $u_0 \in C([0, \pi])$ almost surely. Then u^N almost surely converges in $L_2([0, \pi])$ to $u(t)$, the solution of (1.1), uniformly in t in bounded intervals and for each $\alpha < 1/2$, $T > 0$ there exists a finite random variable ζ_α such that*

$$P\left[\sup_{0\leq t\leq T}\left(\int_0^\pi |u^N(t, x) - u(t, x)|^2 dx\right)^{1/2} \leq \zeta_\alpha N^{-\alpha}\right] = 1, \tag{4.27}$$

for all integers $N \geq 2$.

Proof. To prove this theorem we shall first obtain some estimates. Let G denote the Green function for the heat equation with Dirichlet boundary conditions, i.e.,

$$G(t, x, y) = \sum_{k=1}^{\infty} \exp(-k^2 t)\varphi_k(x)\varphi_k(y), \quad \varphi_k(x) = \sqrt{\frac{2}{\pi}}\sin(kx), \tag{4.28}$$

and

$$G_y(t, x, y) = \sum_{k=1}^{\infty} k \exp(-k^2 t)\varphi_k(x)\psi_k(y), \quad \psi_k(x) = \sqrt{\frac{2}{\pi}}\cos(kx). \tag{4.29}$$

It is well known that problem (1.1) has a unique solution u, which satisfies the integral equation

$$u(t, x) = \int_0^\pi G(t, x, y)u(0, y)dy + \int_0^t\int_0^\pi G_y(t-s, x, y)u^2(s, y)dyds + \int_0^t\int_0^\pi G(t-s, x, y)dW(s, y). \tag{4.30}$$

Define

$$G_y^N(t, x, y) = N\left(G^N\left(t, x, y+\frac{1}{N}\right) - G^N(t, x, y)\right) = \sum_{k=1}^{N-1} \exp(-k^2 t)\varphi_k(\kappa_N(x))N(\varphi_k(\kappa_N^+(y)) - \varphi_k(\kappa_N(y))), \tag{4.31}$$

for $t \geq 0$, $x, y \in [0, \pi]$, where $\kappa_N^+(y) = \kappa_N(y) + (1/N)$. On the other hand, by an argument similar to the proof of Lemma 4.2 in [1] it can be shown that for each $T > 0$ there exists a constant K such that

$$\int_0^T\left(\int_0^\pi |G_y^N - G_y|^2(s, x, y)dx\right)^{1/2} ds \leq KN^{-1/2}, \tag{4.32}$$

for all $y \in [0, \pi]$. Then obviously u^N satisfies the equation

$$u^N(t, x) = \int_0^\pi G^N(t, x, y)u^N(0, \kappa_N(y))dy + \int_0^t\int_0^\pi G_y^N(t-s, x, y)u^N(\kappa_N(y))^2 dyds$$
$$+ \int_0^t\int_0^\pi G^N(t-s, x, y)dW(s, y). \tag{4.33}$$

Let $\|\cdot\|$ denotes the $L_2([0, \pi])$-norm in the x-variable. Therefore, from (4.30) and (4.33) we obtain

$$\|u^N(t, \cdot) - u(t, \cdot)\| \leq A(t) + B(t) + C(t), \tag{4.34}$$

with

$$A(t) = \left\|\int_0^\pi G^N(t, \cdot, y)u^N(0, y)dy - \int_0^\pi G(t, \cdot, y)u(0, y)dy\right\|, \tag{4.35}$$

$$B(t) = \| \int_0^t \int_0^\pi G_y^N(t-s, \cdot, y) u^N(\kappa_N(y))^2 dyds - \int_0^t \int_0^\pi G_y(t-s, \cdot, y) u(s, y)^2 dyds \|, \tag{4.36}$$

$$C(t) = \| \int_0^t \int_0^\pi G^N(t-s, \cdot, y) dW(s, y) - \int_0^t \int_0^\pi G(t-s, \cdot, y) dW(s, y) \|. \tag{4.37}$$

From the proof of Theorem 2.2 in [1] by taking $u^N(0, y) = u(0, y)$ we deduce

$$\sup_{0 \le t \le T} |A(t)|^2 \le \xi_1^2 N^{-2}, \tag{4.38}$$

for a finite random variable ξ_1, and for all $t \in [0, T]$.

For $B(t)$ we have $B \le B_1 + B_2$, where

$$B_1^2(t) = \int_0^\pi \left(\int_0^t \int_0^\pi (G_y^N - G_y)(t-s, x, y) u^N(y, s)^2 dyds \right)^2 dx, \tag{4.39}$$

$$B_2^2(t) = \int_0^\pi \left(\int_0^t \int_0^\pi G_y(t-s, x, y)(u^N(y, s)^2 - u(y, s)^2) dyds \right)^2 dx. \tag{4.40}$$

For B_1 by Minkowski's inequality, (4.32) and Theorem 4.1 we obtain

$$\begin{aligned} B_1^2(t) &\le \left(\int_0^\pi \int_0^t \left(\int_0^\pi (G_y^N - G_y)^2(s, x, y) dx \right)^{1/2} u^N(y, s)^2 dsdy \right)^2 \\ &\le KN^{-1} \left(\int_0^t \int_0^\pi u^N(y, s)^2 dyds \right)^2 \le \xi_2 N^{-1}, \end{aligned} \tag{4.41}$$

for all $t \in [0, T]$, where ξ_2 is a finite random variable independent of t and N.

For B_2 by Lemma 3.1 (i) from [13], we have

$$B_2^2(t) \le K \left(\int_0^t (t-s)^{-3/4} \| u^N(s, .)^2 - u(s, .)^2 \|_1 ds \right)^2. \tag{4.42}$$

where $\| \cdot \|_1$ is $L_1([0, \pi])$-norm.

$$\begin{aligned} \| u(s, \cdot)^2 - u^N(s, \cdot)^2 \|_1 &= \int_0^\pi (u^N(s, y)^2 - u(s, y)^2) dy \\ &= \int_0^\pi (u^N(s, y) - u(s, y))(u^N(s, y) + u(s, y)) dy \\ &\le \left(\int_0^\pi (u^N(s, y) - u(s, y))^2 dy \right)^{1/2} \left(\int_0^\pi (u^N(s, y) + u(s, y))^2 dy \right)^{1/2} \\ &\le \| u^N(s, \cdot) - u(s, \cdot) \| \ \| u^N(s, \cdot) + u(s, \cdot) \| \\ &\le \| u^N(s, \cdot) - u(s, \cdot) \| \ (\| u^N(s, \cdot) \| + \| u(s, \cdot) \|). \end{aligned} \tag{4.43}$$

From Theorem 2.1 in [13], and Theorem 4.1 we conclude that there is a finite random variable ξ_3 such that almost surely

$$\| u(s, \cdot) \|^2 \le \xi_3, \quad \| u^N(s, \cdot) \|^2 \le \xi_3, \tag{4.44}$$

for all $s \in [0, T]$.

Thus by Cauchy-Schwarz inequality we obtain

$$|B_2(t)|^2 \le \xi_3 \left(\int_0^t (t-s)^{-3/4} \| u^N(s, .) - u(s, .) \|^2 ds \right), \tag{4.45}$$

for all $t \in [0, T]$.

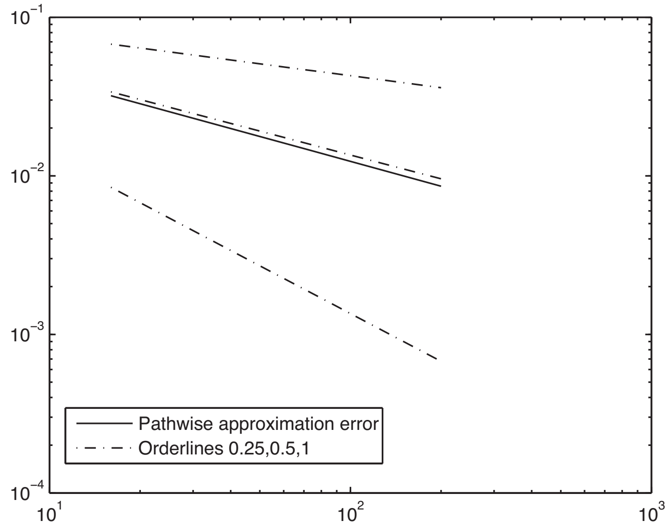

Fig. 1. Convergence rate: pathwise error of the collocation method versus N.

For $C(t)$ by similar computations like the proof of Theorem 2.2 of [1] we can conclude for any $\alpha \in (0, 1)$

$$\sup_{t\leq T}|C(t)|^2 \leq \xi_\alpha N^{-\alpha}. \tag{4.46}$$

Therefore from (4.38), (4.41), (4.45), (4.46) we have

$$\|u^N(t,.)-u(t,.)\|^2 \leq \xi_1^2 N^{-2} + \xi N^{-1} + \xi_3\left(\int_0^t (t-s)^{-3/4}\|u^N(s,.)-u(s,.)\|^2 ds\right) + \xi_\alpha N^{-\alpha}. \tag{4.47}$$

From Gronwall lemma we can conclude there exists a finite random variable ζ_α such that

$$\sup_{t\leq T}\|u^N(t,.)-u(t,.)\|^2 \leq \zeta_\alpha N^{-\alpha}. \quad \square \tag{4.48}$$

5. Numerical results

The theoretical and numerical performances of the presented collocation method for the Burgers equation are discussed here.

To illustrate the theoretical rate of convergence, the related plot, obtained with different numbers of collocation points $N = 16, 32, 64, 100, 200$, has been shown in Fig. 1. It confirms that, as we expected from Theorem 4.2, the order of convergence is 1/2 .

To test the numerical performance of the presented method we compare the following pathwise error for $T = M\Delta t$

$$\max_{0\leq n\leq M}\|u(t_n,\cdot)-\overline{u}(t_n,\cdot)\|, \tag{5.1}$$

of the numerical solutions of the Burgers equation from the two space discretization methods (1.4) and the collocation method of this paper. $\overline{u}(t,x)$, $u(t,x)$ are, respectively, the exact and the numerical solutions at point x and time t. The exact solution $\overline{u}(t,x)$ has been approximated with a very high accuracy using a reasonably small time step size.

Example 1. Consider the stochastic Burgers equation

$$\frac{\partial u}{\partial t}(t,x) = \frac{\partial^2 u}{\partial x^2}(t,x) + u(t,x)\frac{\partial u}{\partial x}(t,x) + \frac{\partial^2 W}{\partial t\partial x}(t,x), \tag{5.2}$$

Table 1
Pathwise error (5.1) of the stochastic Burgers equation with space discretizations (1.4) and collocation method.

$N = 64$	$T = 3/200$	$T = 1/20$	$T = 0.2$	$T = 1$
Collocation method	4.969e−05	8.662e−05	4.670e−04	4.422e−04
Finite difference method (1.4)	5.119e−05	3.182e−05	2.985e−04	1.231e−04

with the initial condition

$$u(0, x) = \frac{6}{5}\sin(x),$$

and boundary conditions

$$u(t, 0) = 0, \quad u(t, \pi) = 0.$$

In Table 1, we present the pathwise errors for the Burgers equation with $N = 64$ at four different times $T = 3/200$, 1/20, 0.2 and $T = 1$ with 100 realizations. For time variable, the time discretization (3.13) with $\Delta t = T/N^2$ has been used. In Fig. 2 we present the approximation of the solution of Eq. (5.2) computed by spectral collocation method for $T \in \{3/200, 1/20, 0.2, 1\}$.

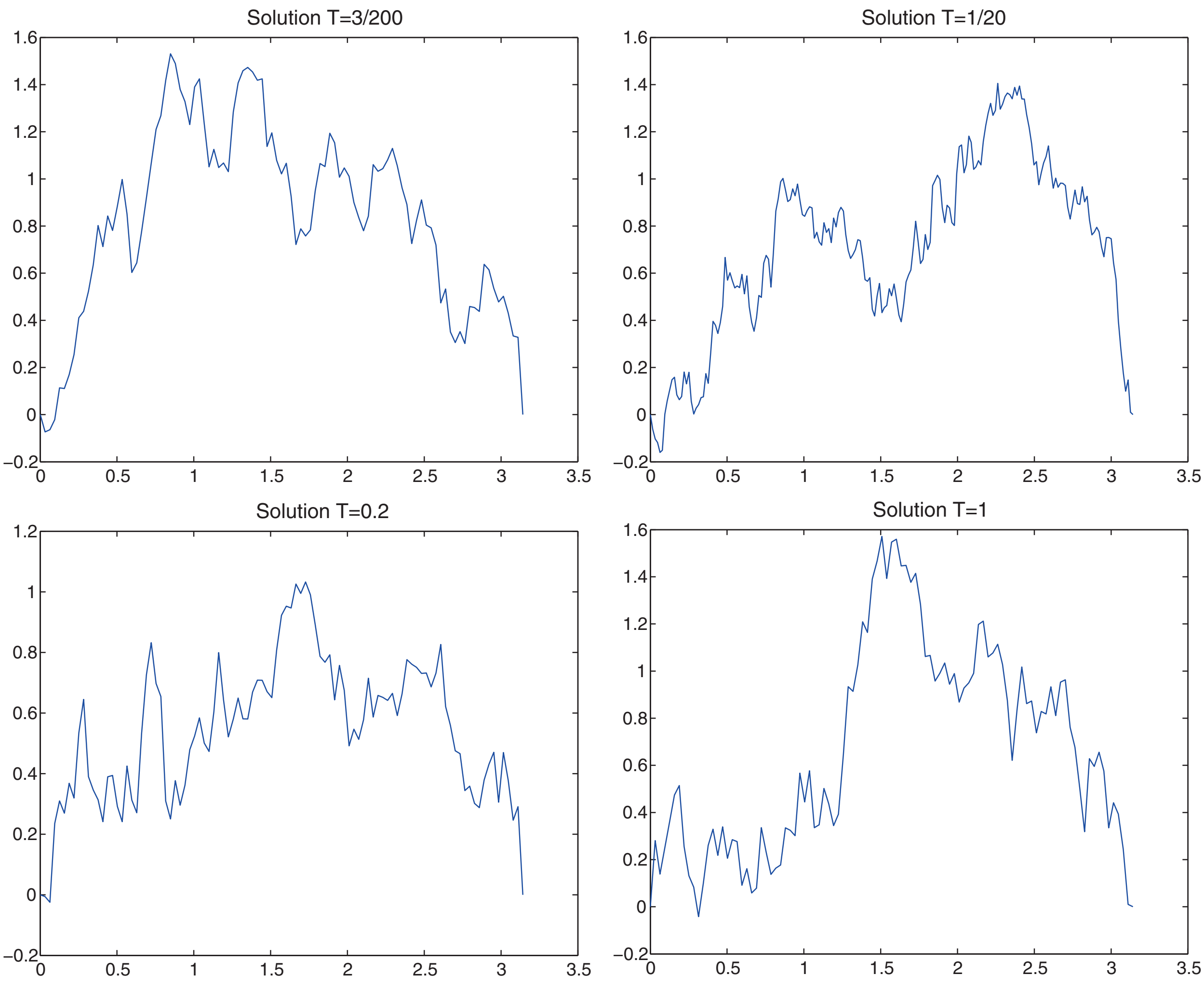

Fig. 2. Solution paths, $u(t, x)$, of stochastic Burgers equation for $x \in [0, \pi]$ and $T \in \{3/200, 1/20, 0.2, 1\}$ with one random value $\omega \in \Omega$.

6. Conclusion

The main purpose of this work was to study the analysis of spectral collocation method as a suitable numerical algorithm for solving SPDEs, particularly stochastic Burgers equation. We have studied theoretically and numerically the spectral Fourier collocation method for this equation. We have proved the convergence rate and illustrated it numerically. Some numerical experiments have been reported to illustrate the performance of the collocation method.

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Appendix A.

We here prove that $W_i^N(t)$ as was defined by $\sqrt{N/\pi}(W(t, x_{i+1}) - W(t, x_i))$ is a Wiener process:

1. $W_i^N(0) = \sqrt{N/\pi}(W(0, x_{i+1}) - W(0, x_i)) = 0,\quad$ with probability 1, because of properties of the Brownian sheet.
2. For $i = 1, \ldots, N-1$, and arbitrary t and h, $W_i^N(t+h) - W_i^N(t)$ has a Gaussian distribution with mean 0 and variance h, as the following analysis shows:

$$\mathbb{E}(W_i^N(t+h) - W_i^N(t)) = \sqrt{\frac{N}{\pi}}\mathbb{E}(W(t+h, x_{i+1}) - W(t+h, x_i) - W(t, x_{i+1}) + W(t, x_i)) = 0. \tag{A.1}$$

Thus its mean was shown to be zero.
For its variance, we use the fact that

$$\mathbb{E}(W(t, x)W(s, y)) = (x \wedge y)(t \wedge s),$$

to obtain

$$\begin{aligned}\mathbb{E}(W_i^N(t))^2 &= \frac{N}{\pi}\mathbb{E}(W(t, x_{i+1}) - W(t, x_i))^2 \\ &= \frac{N}{\pi}(\mathbb{E}(W(t, x_{i+1}))^2 + \mathbb{E}(W(t, x_i))^2 - 2\mathbb{E}(W(t, x_i))(W(t, x_{i+1}))) \\ &= \frac{N}{\pi}(tx_{i+1} + tx_i - 2tx_i) = \frac{N}{\pi}(t(x_{i+1} - x_i)) = t,\end{aligned} \tag{A.2}$$

which implies

$$\mathbb{E}(W_i^N(t))^2 = t, \tag{A.3}$$

and also

$$\begin{aligned}\mathbb{E}(W_i^N(s)W_i^N(t)) &= \frac{N}{\pi}\mathbb{E}((W(s, x_{i+1}) - W(s, x_i)).(W(t, x_{i+1}) - W(t, x_i))) \\ &= \frac{N}{\pi}(\min\{s, t\}x_{i+1} - \min\{s, t\}x_i - \min\{s, t\}x_i + \min\{s, t\}x_i) \\ &= \frac{N}{\pi}\min\{s, t\}(x_{i+1} - x_i) = \min\{s, t\}.\end{aligned} \tag{A.4}$$

Hence,

$$\mathbb{E}(W_i^N(s)W_i^N(t)) = \min\{s, t\}. \tag{A.5}$$

Therefore from (A.3) and (A.5) we conclude that

$$\begin{aligned}\mathbb{E}(W_i^N(t+h)-W_i^N(t))^2 &= \mathbb{E}(W_i^N(t+h))^2+\mathbb{E}(W_i^N(t))^2-2\mathbb{E}((W_i^N(t+h))(W_i^N(t)))\\ &= t+h+t-2t=h.\end{aligned} \tag{A.6}$$

3. By using (A.5), one can easily show that the increments of $W_i^N(t_1)-W_i^N(t_0),\ W_i^N(t_2)-W_i^N(t_1),\ \dots,\ W_i^N(t_n)-W_i^N(t_{n-1})$ are also independent.

   It is obviously seen, from (A.5), that

$$\begin{aligned}\mathbb{E}(W_1^N(t).W_2^N(t)) &= \sqrt{\frac{N}{\pi}}\mathbb{E}((W(t,x_2)-W(t,x_1)).\sqrt{\frac{N}{\pi}}\mathbb{E}((W(t,x_3)-W(t,x_2))\\ &= \frac{N}{\pi}\mathbb{E}(W(t,x_2)W(t,x_3))-\mathbb{E}(W(t,x_2))^2-\mathbb{E}(W(t,x_1)W(t,x_3))+\mathbb{E}(W(t,x_1)W(t,x_2))\\ &= \frac{N}{\pi}(tx_2-tx_2-tx_1+tx_1)=0.\end{aligned} \tag{A.7}$$

   Therefore for $i \neq j$,

$$\mathbb{E}(W_i^N(t).W_j^N(t))=0.$$

Appendix B.

Thus for approximating function $f(x)$ in the form of a sine series we choose the following interpolation points in the interval $[0, \pi]$:

$$x_j=\frac{\pi j}{N} \quad (j=1,\dots,N-1). \tag{B.1}$$

In order to construct the interpolant of $f(x)$ at these points we first define the polynomials [8]

$$C_j(x)=\frac{2}{N}\sum_{m=1}^{N-1}\sin(mx_j)\sin(mx), \tag{B.2}$$

to satisfy $C_j(x_i)=\delta_{ij}$, $i, j=1, \dots, N-1$. For a given function $f(x)$ which vanishes at $x=0$ and π, we consider

$$P_N(x)=\sum_{j=1}^{N-1}f(x_j)C_j(x), \tag{B.3}$$

as its interpolating projection trigonometric polynomial. In this case the projection space is $\tilde{B}_N=\text{span}\{\ \sin(kx): k=1,\dots,N-1\}$ with $dim(\tilde{B}_N)=N-1$. Now we use the discrete inner product

$$(f,g)_N=\frac{2}{N}\sum_{k=1}^{N-1}f(x_k)g(x_k)$$

for f and $g\in\tilde{B}_N$. Then from the trapezoidal quadrature rule and because each element of $\tilde{B}_N$ vanishes at $x=0$ & π, it is seen that the discrete inner product equals to the continuous inner product, that is

$$(f,g)_N=\frac{2}{\pi}\int_0^{\pi} f\,g\,dx, \quad \text{for} \quad f,g\in\tilde{B}_N.$$

This is easily shown because $(\sin(kx),\sin(mx))_N=(2/\pi)\int_0^{\pi}\sin(kx)\sin(mx)dx=\delta_{k,m},\ k,m=1,\dots,N-1$.

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