QUASI-REDIRECTING BOUNDARIES OF NON-POSITIVELY CURVED GROUPS

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Abstract. In this paper, we show that the quasi-redirecting boundary (QR boundary) is well-defined as a topological space for several classes of groups with nonpositive curvature: admissible groups that act geometrically on CAT(0) spaces, relatively hyperbolic groups relative to groups whose QR boundaries are well defined, right-angled Coxeter groups whose flag complexes are planar, and the fundamental groups of non-geometric 3-manifolds. Secondly, we give a complete description of the QR boundaries of admissible groups that act geometrically on CAT(0) spaces, which are non-Hausdorff and are one-point compactifications of the Morse-like directions in the associated Bass-Serre tree. Lastly, we prove that if G is a hyperbolic group relative to groups whose QR boundaries are well-defined, then the QR boundary of G maps surjectively onto the Bowditch boundary of G.

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1. Introduction

Gromov introduced hyperbolic groups in [Gro97] characterizing a large class of infinite groups with solvable word problems. The class of Gromov-hyperbolic groups is closed under quasi-isometry. Among the many tools developed by Gromov is an equivariant and compact bordification of the Cayley graph of a group, now known as the Gromov boundary. A basic property of the boundary is that quasi-isometries on the group extend equivariantly to the Gromov boundaries.

Gromov asked the question whether this property holds if the hyperbolicity assumption is dropped. Croke and Kleiner provided an example of a group to answer the question in the negative [CK00]. Since then, various methods have been developed to address this issue. One breakthrough idea involves defining a boundary using only geodesic rays that satisfy a Morse property[CS15], rather than considering all geodesic rays. The Morse property ensures that these rays have properties similar to geodesic rays in hyperbolic spaces. This approach was first used to develop a quasi-isometrically invariant boundary for CAT(0) spaces by Charney and Sultan [CS15], and then extended to general metric spaces by Cordes [Cor17]. The topology used by Charney and Sultan is quite fine, but there is also a coarser topology that resembles the visual topology and is still quasi-isometrically invariant [CM19] by Cashen-Mackay. Notably, even though the Morse boundary is usually an uncountable set, the Morse boundary can still consists of only a trivially small fraction of all directions, from the point of view of random walk on groups [CDG20]. Furthermore, the Morse boundary is usually non-compact.

More recently, Qing, Rafi and Tiozzo ([QRT22], [QRT23]) developed sublinearly Morse boundary, including geodesic rays whose Morse-ness can decay sublinearly with distance from the base point. These boundaries are group invariants and metrizable topological spaces. They resemble the Gromov boundary of hyperbolic spaces and offer insights into groups containing hyperbolic-like features. A key new property of sublinearly Morse boundaries is its connection with simple random walks on groups. In many important classes of groups, such as right-angled Artin groups, relatively hyperbolic groups, mapping class groups of surfaces of finite type, and hierarchical hyperbolic groups, a sublinear function can be chosen appropriately. The sublinearly Morse boundary has been shown to be large enough to be used as a topological model for the Poisson boundaries of the group (with mild assumptions) (see [QRT22], [QRT23], [NQ24]). Furthermore, genericity of sublinearly Morse directions is also evidenced from the point of view of Patterson-Sullivan measure on the sphere at infinity [GQR22, QY24].

Rafi-Qing recently introduced a new boundary for metric spaces called *quasi-redirecting boundary* [QR24]. The quasi-redirecting boundary (or QR boundary, for short) contains sublinearly Morse boundaries as topological subspaces and is often compact. It identifies a new and large QI-invariant boundary. The QR boundary is also shown to serve as a topological model for suitable random walks. It is shown when the QR boundary contains 3 or more points, the sublinearly Morse boundaries are dense subsets of the QR boundary [GQV]. It is also established in [GQV] that when X is either a rank-one CAT(0) space, the QR boundary, when it exists, is a visibility space; and when X is a proper CAT(0) cube complex with cocompact action, the QR boundary, when exists and is not mono-directional, contains a Morse element. These properties provide evidence of the Gromov-like nature of the QR boundary.

It is worth pointing out that in cases that are studied closely, there are new, QI-invariant and Morse-like directions in the QR boundary. These new directions are not sublinearly Morse and to our understanding are not previously identified in boundary theories. This indicates that the QR boundary encodes more information than the sublinearly Morse boundary. Now we give the definition.

Definition 1.1. Let $\alpha, \beta \colon [0, \infty) \to X$ be two quasi-geodesic rays in a metric space X. We say α can be *quasi-redirected* to β (and write $\alpha \preceq \beta$) if there exists a pair of constant (q, Q) such that for every $r > 0$, there exists a (q, Q)–quasi-geodesic ray γ that is identical to α inside the ball $B(\alpha(0), r)$ and eventually γ becomes identical to β. We say $\alpha \sim \beta$ if $\alpha \preceq \beta$ and $\beta \preceq \alpha$. The resulting set of equivalence classes forms a poset, denoted by $P(X)$. The post $P(X)$ comes together with a "cone-like topology" is called *quasi-redirecting boundary* (QR boundary) of X and denoted by ∂X.

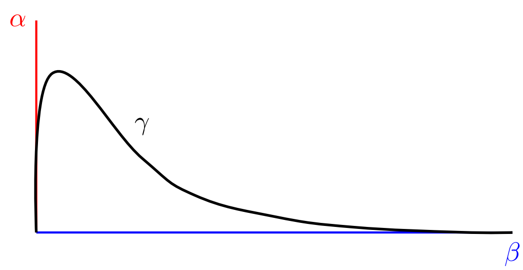


FIGURE 1. The ray α can be quasi-redirected to β at radius r.

We remark here that in [QR24], to define a "cone-like topology" on $P(X)$, three QR-Assumptions need to be satisfied (see Section 2.1) for X. However, it is unknown which groups satisfy all three QR-Assumptions, and consequently, whether the QR boundary is well-defined. In fact in [QR24, Question D], it is asked that do all finitely generated group satisfy all three QR-Assumptions? On the one hand, there is no known example of a finitely generated group which does not satisfy QR-Assumptions. On the other hand, very few class of groups have been verified to satisfy all three QR-Assumptions. In this paper, we answer [QR24, Question D] in the affirmative for several classes of groups. These class of groups include:

(1) Admissible groups that act geometrically on CAT(0) spaces.
(2) Relatively hyperbolic groups.
(3) The fundamental groups of non-geometric 3-manifolds.
(4) Right-angled Coxeter groups whose flag complexes are planar.

Therefore, we provide evidence that the theory of QR-boundaries applies in a variety of concrete contexts.

1.1. **CAT(0) admissible groups.** In [CK02], Croke and Kleiner study a particular class of graph of groups with edge groups $\mathbb{Z}^2$ which they call *admissible groups* and generalize fundamental groups of 3–dimensional graph manifolds and torus complexes (see [CK00]). In this paper, an admissible group G is called *CAT(0) admissible* group if it acts geometrically on a Hadamard space X. Such action $G \curvearrowright X$ is called CKA action, and the space X is called a *CAT(0) admissible space*.

The admissible groups are modeled on the JSJ structure of graph manifolds where (the fundamental groups of the) Seifert fibered pieces are replaced by the following central extensions general hyperbolic groups H

$$1 \to Z(G) = \mathbb{Z} \to G \to H \to 1 \tag{1}$$

In some sense, admissible groups are the simplest interesting groups constructed algebraically from any finite number of hyperbolic groups.

We give a complete description of the QR-boundary of admissible groups that act geometrically on CAT(0) spaces in the following theorem.

Theorem A. (Theorem 4.23) Let G be an admissible group that acts properly discontinuous, cocompactly and by isometries on a complete proper CAT(0) space. Then the following properties hold.

(1) G satisfies all three QR-Assumptions. Thus ∂G is a quasi-invariant topological space.
(2) The boundary ∂G is non-Hausdorff.

Let G be an admissible group that acts properly discontinuous, cocompactly and by isometries on a complete proper CAT(0)

Let Γ be a finite tree. If Γ is a segment, then A_Γ is isomorphic to $\mathbb{Z}^2$, and thus its QR-boundary consists of only one point. If Γ contains at least one vertex of degree ≥ 2 then it is a well-known fact that the associated right-angled Artin group A_Γ is the fundamental group of a non-posiitvely curve graph manifold M. In particular, A_Γ is a CAT(0) admissible group. The following corollary is an immediate consequence of Theorem A.

Corollary B. Let Γ be a finite tree, then the associated right-angled Artin group A_Γ satisfies all three QR-Assumptions and hence ∂A_Γ is well-defined.

1.2. **Relatively hyperbolic groups.** The notion of relatively hyperbolic groups can be formulated from a number of equivalent ways. Here we shall present a quick definition due to Bowditch [Bow12].

Let G be a finitely generated group with a finite collection of subgroups $\mathbb{P}$. Fixing a finite generating set S for G, we consider the corresponding Cayley graph $\Gamma(G, S)$ equipped with path metric d_S and we denote by $|g|_S = d_S(1, g)$ for the word length.

Denote by $\mathcal{P} = \{gP : g \in G, P \in \mathbb{P}\}$ the collection of peripheral cosets. Let $\hat{G}(\mathcal{P})$ be the coned-off Cayley graph obtained from $\Gamma(G, S)$ as follows. A *cone point* denoted by $c(P)$ is added for each peripheral coset $P \in \mathcal{P}$ and is joined by half edges to each element in P. The union of two half edges at a cone point is called a *peripheral edge*. Denote by $\hat{d}_S$ the induced path metric after coning-off. The pair $(G, \mathbb{P})$ is said to be *relatively hyperbolic* if the coned-off Cayley graph $\hat{G}(\mathcal{P})$ is hyperbolic and *fine*: any edge is contained in finitely many simple circles with uniformly bounded length.

Theorem C. Let G be a hyperbolic group relative to the collection $\mathcal{P}$. If $\mathcal{P}$ satisfies the QR-Assumptions 0,1 and 2, then $(G, \mathcal{P})$ satisfies QR-Assumptions 0,1 and 2.

1.3. **3-manifold groups.** Let M be a non-geometric 3-manifold. The torus decomposition of M yields a nonempty minimal union $\mathcal{T} \subset M$ of disjoint essential tori, unique up to isotopy, such that each component M_v of $M \backslash \mathcal{T}$, called a *piece*, is either Seifert fibered or hyperbolic.

There is an induced graph of groups decomposition $\mathcal{G}$ of $\pi_1(M)$ with underlying graph Γ as follows. For each piece M_v, there is a vertex v of Γ with vertex group $\pi_1(M_v)$. For each torus $T_e \in \mathcal{T}$ contained in the closure of pieces M_v and M_w, there is an edge e of Γ between vertices v and w. The associated edge group is $\pi_1(T_e) \cong \mathbb{Z}^2$ and the edge monomorphisms are the maps induced by inclusion. Note that $\mathcal{G}$ is an admissible graph of groups and $(\pi_1(M), \mathcal{G})$ is an admissible group. When all pieces of M are Seifert fibered spaces then M is called a *graph manifold*. Otherwise, it is called a *mixed manifold*.

As an application of Theorem A and Theorem C we obtain the following result. For the detail discussion, we refer the reader to Section 5.

Theorem D. Let M be a non-geometric 3-manifold. Then $G = \pi_1(M)$ satisfies all three QR-Assumptions and hence ∂G is well-defined.

1.4. **Right-angled Coxeter groups.** A simplicial complex Δ is called *flag* if any complete subgraph of the 1-skeleton of Δ is the 1-skeleton of a simplex of Δ. Let Γ be a finite simplicial graph. The *flag complex* of Γ is the flag complex with 1-skeleton Γ. A simplicial subcomplex B of a simplicial complex Δ is called *full* if every simplex in Δ whose vertices all belong to B is itself in B.

The flag complex of Δ is *planar* if it can be embedded into the 2-dimensional sphere $\mathbb{S}^2$. From now on every time we consider a flag complex it will be as a subspace of the 2-dimensional sphere $\mathbb{S}^2$.

Definition 1.2. Given a finite simplicial graph Γ, the associated *right-angled Coxeter group* W_Γ is generated by the set S of vertices of Γ and has relations $s^2 = 1$ for all s in S and $st = ts$ whenever s and t are adjacent vertices. The graph Γ is the *defining graph* of a right-angled Coxeter group W_Γ and its flag complex $\Delta = \Delta(\Gamma)$ is the *defining nerve* of the group. Therefore, sometimes we also denote the right-angled Coxeter group W_Γ by W_Δ where Δ is the flag complex of Γ.

Let S_1 be a subset of S. The subgroup of W_Γ generated by S_1 is a right-angled Coxeter group W_{Γ_1}, where Γ_1 is the induced subgraph of Γ with vertex set S_1 (i.e. Γ_1 is the union of all edges of Γ with both endpoints in S_1). The subgroup W_{Γ_1} is called a *special subgroup* of W_Γ.

Corollary E (Theorem 6.3)**.** Let Γ be a graph whose flag complex Δ is planar. Then the right-angled Coxeter group W_Γ satisfies all three QR-Assumptions.

2. Preliminaries

In this section, we recall the construction of quasi-redirecting boundary as presented in [QR24]. Please refer to [QR24] for a complete treatment.

Let X and Y be metric spaces and f be a map from X to Y.

(1) We say that f is a (K, A)–*quasi-isometric embedding* if for all $x, y \in X$,

$$\frac{1}{K} d(x, x') - A \leq d(f(x), f(x')) \leq K d(x, x') + A.$$

(2) We say that f is a (K, A)–*quasi-isometry* if it is a (K, A)–quasi-isometric embedding such that $Y = N_A(f(X))$.

2.1. Quasi-redirecting boundary.

Let X be a proper geodesic metric space.

Definition 2.1 (Quasi-Geodesics)**.** A *quasi-geodesic* in a metric space X is a quasi-isometric embedding $\alpha : I \to X$ where $I \subset \mathbb{R}$ is an (possibly infinite) interval. However, in this paper, we always assume α is Lipschitz. And again, we use $\mathfrak{q} = (q, Q)$ to indicate the constants. That is, $\alpha : I \to X$ is a $\mathfrak{q}$–quasi-geodesic if, for all $s, t \in I$, we have

$$\frac{|t-s|}{q} - Q \le d_X\big(\alpha(s), \alpha(t)\big) \le q|s-t|.$$

The assumption that α is Lipschitz is needed so we can apply the Arzelà-Ascoli theorem to a sequence of quasi-geodesics to obtain a limiting quasi-geodesic. However, this assumption can always be achieved by increasing the constants of the quasi-geodesic ([QR24, Lemma 2.3])

2.2. Notation.

Let $\mathfrak{o}$ be a fixed basepoint in X. We use $\mathfrak{q} = (q, Q) \in [1, \infty) \times [0, \infty)$ to indicate a pair of constants. For instance, one can say $\Phi \colon X \to Y$ is a $\mathfrak{q}$–quasi-isometry and α is $\mathfrak{q}$–quasi-geodesic ray or segment.

By a $\mathfrak{q}$–ray we mean a $\mathfrak{q}$–quasi-geodesic ray $\alpha : [0, \infty) \to X$ such that $\alpha(0) = \mathfrak{o}$. For an interval $[s, t] \subset [0, \infty)$, we denote the restriction of α to the time interval $[s, t]$ by $\alpha[s, t]$. However, if points $x, y \in X$ on the image of α are given, we denote the sub-segment of α connecting x to y by $[x, y]_\alpha$. That is, if $\alpha(s) = x$ and $\alpha(t) = y$ for $s \le t$, then $[x, y]_\alpha = \alpha[s, t]$.

Let $\alpha : [s_1, s_2] \to X$ and $\beta : [t_1, t_2] \to X$ be two quasi-geodesics such that $\alpha(s_2) = \beta(t_1)$. In this paper we denote the concatenation of α and β by $\alpha \cup \beta$ by which we mean the following quasi-geodesic:

$$\alpha\cup\beta : [s_1, t_2 - t_1 + s_2] \to X, \quad \alpha\cup\beta(t) = \begin{cases} \alpha(t) & \text{for } t \in [s_1, s_2] \\ \beta(t + t_1 - s_2) & \text{for } t \in [s_2, t_2 - t_1 + s_2] \end{cases}.$$

For $r > 0$, let $B_r^\circ \subset X$ be the open ball of radius r centered at $\mathfrak{o}$, let B_r be the closed ball centered at $\mathfrak{o}$ and let $B_r^c = X - B_r^\circ$.

For a $\mathfrak{q}$–ray α and $r > 0$, we let $t_r \ge 0$ denote the first time when α first intersects B_r^c and $T_r \ge t_r$ be the last time α intersects B_r. We denote $\alpha(t_r)$ by $\alpha_r \in X$. Also, let

$$\alpha|_r := \alpha[0, t_r] \quad \text{and} \quad \alpha|_{\ge r} := \alpha[T_r, \infty)$$

be the restrictions α to the intervals $[0, t_r]$ and $[T_r, \infty)$ respectively. That is, $\alpha|_r$ is the subsegment of α connecting $\mathfrak{o}$ to α_r and $\alpha|_{\ge r}$ is the portion of α that starts at radius r but never returns to B_r.

Lastly, if p is a point on a $\mathfrak{q}$–ray α. We also use $\alpha_{[p,\infty)}$ to denote the tail of α starting from the point p. Note such a point always exists as a quasi-geodesic is always assumed to be a ray without loss of generality. This is because, as discussed in [QRT22, Definition 2.2], one can adjust the quasi-isometric embedding of an interval slightly to make it continuous (see [BH99, Lemma III.1.11]).

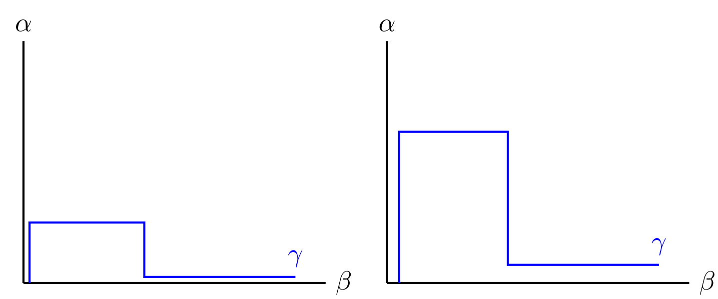
We also use $d(\cdot, \cdot)$ instead of $d_X(\cdot, \cdot)$ when the metric space X is fixed. For $x \in X$, $\|x\|$ denotes $d(\mathfrak{o}, x)$. Now we recall the first of three QR-Assumptions.

QR-Assumption 0. (No dead ends) The space X is a proper, geodesic metric space. Furthermore, there exist a pair of constants $\mathfrak{q}_0$ such that every point $x \in X$ lies on an infinite $\mathfrak{q}_0$–quasi-geodesic ray.

Remark 2.2. QR-Assumption 0 is satisfied by Cayley graphs of all finitely generated groups [QR24, Lemma 2.5].

Definition 2.3. Let X be a geodesic metric space. Let α, β and γ be quasi-geodesic rays in X. We say

(1) γ *eventually coincide with* β if there are times $t_\beta, t_\gamma > 0$ such that, for $t \geq t_\gamma$, we have $\gamma(t) = \beta(t + t_\beta)$.
(2) For $r > 0$, we say γ *quasi-redirects* α *to* β *at radius* r if $\gamma|_r = \alpha|_r$ and β eventually coincides with γ. If γ is a $\mathfrak{q}$–ray, we say α *can be* $\mathfrak{q}$*–redirected to* β *at radius* r or α *can be* $\mathfrak{q}$*–redirected to* β *by* γ *at radius* r. We refer to t_γ as the *landing time*.
(3) We say α is *quasi-redirected* to β, denoted by $\alpha \preceq \beta$, if there is $\mathfrak{q} \in [1, \infty) \times [0, \infty)$ such that α can be $\mathfrak{q}$–redirected to β at radius r.



Definition 2.4. Define $\alpha \simeq \beta$ if and only if $\alpha \preceq \beta$ and $\beta \preceq \alpha$. Then $\simeq$ is an equivalence relation on space of all quasi-geodesic rays in X. Let $P(X)$ denote the set of all equivalence classes of quasi-geodesic rays under $\simeq$. For a quasi-geodesic ray α, let $[\alpha] \in P(X)$ denote the equivalence class containing α. We extend $\preceq$ to $P(X)$ by defining $[\alpha] \preceq [\beta]$ if $\alpha \preceq \beta$. Note that this does not depend on the representative chosen in the given class. The relation $\preceq$ is a partial order on elements of $P(X)$.

Lemma 2.5. [QR24, Lemma 3.2] *Let* α, β, γ *be quasi-geodesic rays. Suppose that* α *can be* (q_1, Q_1)*–quasi-redirected to* β *at a radius* r *and* β *can be* (q_2, Q_2)*–quasi-redirected to* γ *at every radius then* α *can be* (q_3, Q_3)*–quasi-redirected to* γ *at the radius* r *where* $q_3 = \max\{q_1, q_2 + 1\}$ *and* $Q_3 = \max\{Q_1, Q_2\}$.

QR-Assumption 1. (Quasi-geodesic representative) For $\mathfrak{q}_0$ as in QR-Assumption 0, every equivalence class of quasi-geodesics $\mathbf{a} \in P(X)$ contains a $\mathfrak{q}_0$–ray. We fix such a $\mathfrak{q}_0$–ray and denote it by $\underline{a} \in \mathbf{a}$.

QR-Assumption 2. (Uniform redirecting function) For every $\mathbf{a} \in P(X)$, there is a function

$$f_{\mathbf{a}} : [1, \infty) \times [0, \infty) \to [1, \infty) \times [0, \infty),$$

called the redirecting function of the class $\mathbf{a}$, such that if $\mathbf{b} \prec \mathbf{a}$ then any $\mathfrak{q}$–ray $\beta \in \mathbf{b}$ can be $f_{\mathbf{a}}(\mathfrak{q})$–redirected to $\underline{a}$.

Proposition 2.6. [QR24, Proposition 4.3] *Let $X = A \times B$ where A and B are proper metric spaces satisfying QR-Assumption 0, equipped with L^∞–metric. Then $P(X)$ is a point.*

Note that since $P(X)$ is invariant under quasi-isometries, Proposition 2.6 also holds if we equip X with the L^p–metric with $p > 0$.

2.3. Topology on $X \cup P(X)$.

The topology on $X \cup P(X)$ is defined by defining a system of neighbourhoods. Recall that points in $P(X)$ are equivalence classes of quasi-geodesic rays.

$$\mathbb{x} = \Big\{\text{quasi-geodesics rays passing through } x\Big\}.$$

Again recall that $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ to denote elements of $P(X) \cup X$, that is, either a set of quasi-geodesic rays passing through a point $x \in X$ or an equivalence class of quasi-geodesic rays in $P(X)$. For $\mathbf{a} \in P(X)$, define $F_{\mathbf{a}} : [1, \infty) \times [0, \infty) \to [1, \infty) \times [0, \infty)$ by

$$F_{\mathbf{a}}(\mathfrak{q}) = \max\{\mathbf{f_a}(\mathfrak{q}) + (1, 0), (4q + 3Q)\} \qquad \text{for} \qquad \mathfrak{q} \in [1, \infty) \times [0, \infty). \tag{2}$$

Definition 2.7. For $\mathbf{a} \in P(X)$ and $r > 0$, define

$$\mathcal{U}(\mathbf{a}, r) := \Big\{\mathfrak{b} \in P(X) \cup X \text{ such that every } \mathfrak{q}\text{–ray in } \mathfrak{b} \text{ can be } F_{\mathbf{a}}(\mathfrak{q})\text{–redirected to } \underline{a} \text{ at radius } r\Big\}.$$

A system of neighbourhoods. For each $\mathbf{a} \in P(X)$, recall that

$$\mathcal{B}(\mathbf{a}) = \Big\{\mathcal{V} \subset X \cup P(X) s.t. \mathcal{U}(\mathbf{a}, r) \subset \mathcal{V} \quad \text{for some } r > 0\Big\}$$

and for every $x \in X$, define

$$\mathcal{B}(\mathbb{x}) = \Big\{\mathcal{V} \subset X \cup P(X) s.t. B(\mathbb{x}, r) \subset \mathcal{V} \quad \text{for some } r > 0\Big\}.$$

We collect some important facts of QR boundary and poset $P(X)$ from QR24.

Theorem 2.1. [QR24, Theorem B] *Let X, Y be proper geodesic metric spaces.*

(1) Suppose that $\Phi : X \to Y$ is a (k, K) quasi-isometry sending the base point $\mathfrak{o}_X \in X$ to the base point $\mathfrak{o}_Y \in Y$. Then there is a well defined induced map

$$\Phi^* : P(X) \to P(Y) \quad \textit{where} \quad \Phi^*([\alpha]) = [\Phi \circ \alpha].$$

Furthermore, Φ^* preserves the partial order on $P(X)$ and $P(Y)$.

(2) ∂X and $X \cup \partial X$ are QI-invariant as topological spaces.

(3) Sublinearly Morse boundaries are topological subspaces of ∂X.

2.4. Surgery on quasi-geodesics.

We recall a few surgeries related to quasi-geodesics that will be used often in the subsequent arguments.

Lemma 2.8. [QR24, Lemma 2.6] *Let X be a metric space satisfying QR-Assumption 0.*

(1) *(Nearest-point projection surgery) Consider a point $x \in X$ and a (q, Q)–quasi-geodesic segment β connecting a point $z \in X$ to a point $w \in X$. Let y be a closest point in β to x. Then*
$$\gamma = [x, y] \cup [y, z]_\beta$$
is a $(3q, Q)$–quasi-geodesic.

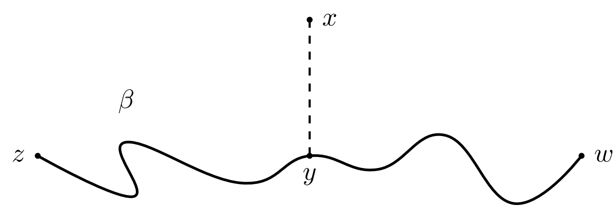


FIGURE 2. The concatenation of the geodesic segment $[x, y]$ and the quasi-geodesic segment $[y, z]_\beta$ is a quasi-geodesic.

(2) *(Quasi-geodesic ray to geodesic ray Surgery) Let β be a geodesic ray and γ be a (q, Q)–ray. For $r > 0$, assume that $d_X(\beta_r, \gamma) \leq r/2$. Then, there exists a $(9q, Q)$–quasi-geodesic γ' where $\gamma'(t) = \beta(t)$ for large values of t and*
$$\gamma|_{r/2} = \gamma'|_{r/2}.$$

(3) *(Segment to quasi-geodesic ray Surgery) Consider a (q, Q)-quasi-geodesic ray $\alpha \colon [0, \infty) \to X$ and a finite (q, Q)–quasi-geodesic segment $\beta \colon [a, b] \to X$. Then there is $s_0 \in [0, \infty)$ such that the following holds: for $s \in [s_0, \infty)$ let $s_\gamma \in [s, \infty)$ and $t_\gamma \in [a, b]$ be such that $[\beta(t_\gamma), \alpha(s_\gamma)]$ is a geodesic segment that realizes the set distance between $\alpha[s, \infty)$ and β. Then*
$$\gamma = \beta[a, t_\gamma] \cup [\beta(t_\gamma), \alpha(s_\gamma)] \cup \alpha[s_\gamma, \infty)$$
is a $(4q, 3Q)$–quasi-geodesic.

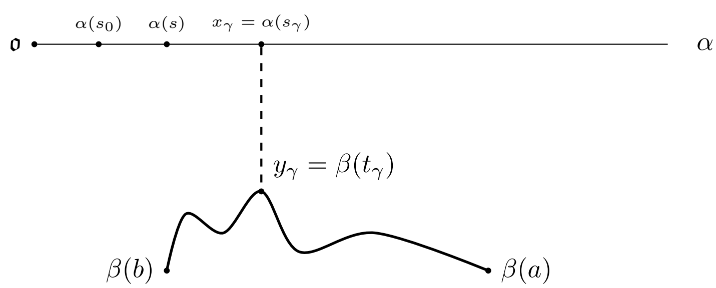


FIGURE 3. Segment-to-geodesic-ray Surgery

(4) *(Fellow travelling surgery) Let X be a metric space satisfying QR-Assumption 0. Let $\mathfrak{q}$-rays α, β and $t_0 > 0$ be such that, for all $t \leq t_0$, we have $d(\alpha(t), \beta(t)) \leq 1$.*

Then there exists a $(q, Q + 1)$-quasi-geodesic ray β' such that
$$\beta'|_{t_0} = \beta|_{t_0} \quad \textit{and} \quad \beta'|_{(t_0+1,\infty)} = \alpha|_{(t_0,\infty)}.$$

Lemma 2.9. *Let α be a (q_1, Q_1)–quasi-geodesic ray and β be a (q_2, Q_2)–quasi-geodesic ray. Suppose there is a sequence of points $\{x_n\}$ on α so that $\|x_n\| \to \infty$ such that the following holds. At every x_n, there exists a (q, Q)–quasi-geodesic ray γ_n where q, Q depends only on q_1, q_2, Q_1, Q_2 such that γ_n and α are identical on the subsegment $[\mathfrak{o}, x_n]_\alpha$ and γ_n is eventually concise with β. Then α is (q, Q)–quasi-redirected to β.*

Proof. Let s_n be the first time in $[0, \infty)$ so that $\alpha(s_n) = x_n$. Let consider the ball $B(\mathfrak{o}, r_n)$ where $r_n := \|x_n\|$. Let t_n be the first time α intersects $X - B(\mathfrak{o}, r_n)$. It follows that $t_n < s_n$. According to the assumption, $(\gamma_n)|_{t_n} = \alpha|_{t_n}$ and γ_n is eventually concise with β. Note that the $t_n \to \infty$ and hence for every $r > 0$, we pick a $t_n > r$. This guarantees that α is quasi-redirected to β at radius r via γ_n. Consequently, $\alpha \preceq \beta$. □

3. QR boundary of relative hyperbolic groups

In this section, we examine the case when X is a Cayley graph of a finitely generated, relatively hyperbolic group pair $(G, \mathcal{P})$ where G is a group and $\mathcal{P}$ is a collection of subgroups. In [QR24] the authors show that if $(G, \mathcal{P})$ is a relatively hyperbolic group where the QR-boundaries of each P is a mono-directional set, i.e. ∂P is a point for each $P \in \mathcal{P}$, then ∂G exists and is homeomorphic to the Bowditch boundary of $(G, \mathcal{P})$. In this section, we drop the assumption that P's are mono-directional. We first show that if ∂P exists for all $P \in \mathcal{P}$, the quasi-redirecting boundary of $(G, \mathcal{P})$ exists. Furthermore, we show in Theorem 3.2 that when it exists, ∂G maps surjectively onto the Bowditch boundary of $(G, \mathcal{P})$. The proof of Theorem 3.2 is largely a modification of the proof of [QR24, Theorem 9.4] but we include it here for the sake of self-containment.

3.1. Relative Hyperbolic groups and redirecting in relative hyperbolic groups.

We first collect the facts regarding the coarse geometry of relatively hyperbolic groups. These results are collected from [QR24, DS05, Hru10] and [Sis12].

Definition 3.1. Fix a finite generating set S once and for all and let $\mathrm{Cay}(G)$ denote the Cayley graph of G with respect to this generating set. We refer to the groups $P \in \mathcal{P}$ as *peripheral* subgroups. Let $\mathcal{A}$ be the set of subgraphs of of $\mathrm{Cay}(G)$ associates to cosets of groups in $\mathcal{P}$. Namely, for $P \in \mathcal{P}$ and $g \in G$, $A_{P,g}$ is the induced subgraph of $\mathrm{Cay}(G)$ with vertex set gP. We form the *coned-off* Cayley graph, denoted $K(G)$ or simply K, by adding a vertex $*p_A$ for each $A \in \mathcal{A}$, and adding edges of length $\frac{1}{2}$ from $*p_A$ to each vertex of A. Since $\mathrm{Cay}(G)$ is a subgraph of K, for any two vertices $v, w \in \mathrm{Cay}(G)$, we have

$$d_K(v, w) \leq d_{\mathrm{Cay}(G)}. \tag{3}$$

Definition 3.2. A graph is *fine* if for each integer n, every edge belongs to only finitely many simple cycles of length n. If the coned-off Cayley graph is hyperbolic and is fine, then G is *relatively hyperbolic* relative to $\mathcal{P}$. A key property of the relative hyperbolic group is the *Bounded Coset Penetration* [Farb98] which we state now. An oriented path $\ell \in K$ is said to *penetrate* $A \in \mathcal{A}$ if it passes through the cone point $*p_A$ of A; its *entering* and *exiting* vertices are the vertices immediately before and after $*p_A$ on ℓ. The path is *without backtracking* if once it penetrates $A \in \mathcal{A}$, it does not penetrate A again. If for each $q \geq 1$ there is a constant $a = a(q)$

such that if $\zeta, \zeta' \subset K$ are $(q,0)$- quasi-geodesics without backtracking in K and with the same pair of endpoints, then

(1) if ζ penetrates some $A \in \mathcal{A}$, but ζ' does not, then the distance between the entering and exiting vertices of ζ in A is at most $a(q)$; and
(2) if ζ and ζ' both penetrate $A \in \mathcal{A}$, then the distance between the entering vertices of ζ and ζ' in A is at most $a(q)$, and similarly for the exiting vertices.

For the rest of this section, let $X = \mathrm{Cay}(G)$ denote the Cayley graph of $(G, \mathcal{P})$.

Definition 3.3. [Sis12, Definition 3.9] Let α be a path in X. For $M, c > 0$, define the $\mathrm{deep}_{M,c}(\alpha)$ to be the set of points $x \in \alpha$ such that there exists a subpath of α containing x with endpoints x_1, x_2 and $A \in \mathcal{A}$ where

$$x_1, x_2 \in N_M(A) \qquad \text{and} \qquad d(x, x_i) \geq c \quad \text{for } i = 1, 2.$$

Thinking of α as a subset of X, define

$$\mathrm{trans}_{M,c}(\alpha) = \alpha - \mathrm{deep}_{M,c}(\alpha)$$

to be the set of (M, c)–transition points of α.

Proposition 3.4. [Sis12, DS05] *Let $X = \mathrm{Cay}(G)$. For every $\mathfrak{q}$ there exist constant $M = M(\mathfrak{q})$, $c = c(\mathfrak{q})$, $D = D(\mathfrak{q})$ and $\rho(\mathfrak{q})$ such that the followings hold. Let $\alpha : [a, b] \to X$ be a $\mathfrak{q}$–quasi-geodesic segment.*

(1) The set $\mathrm{deep}_{M,c}(\alpha)$ is a disjoint union of subpaths each contained in $N_{\rho M}(A)$ for distinct sets $A \in \mathcal{A}$.
(2) For any pair of $\mathfrak{q}$–quasi-geodesic segments α, β with the same endpoints, we have

$$d_{\mathrm{Haus}}\big(\mathrm{trans}_{M,c}(\alpha), \mathrm{trans}_{M,c}(\beta)\big) \leq D.$$

(3) Moreover, for every $A \in \mathcal{A}$ there are times $t, s \in [a, b]$ such that during the interval $[a, s]$ α approaches A at a linear speed, during the interval $[t, b]$ α moves away from A at a linear speed and $\alpha[s, t] \subset N_{\rho M}(A)$.

The same also holds for quasi-geodesic rays.

The statements of (1) and (2) are contained [Sis12, Proposition 5.7]. The statement (2) follows from [DS05, Lemma 4.17].

Definition 3.5. Let α be a $\mathfrak{q}$–ray or $\mathfrak{q}$–segment in X. The *saturation* of α, denoted by $\mathrm{Sat}(\alpha)$, is the union of α and all $A \in \mathcal{A}$ with $N_{M(\mathfrak{q})}(A) \cap \alpha \neq \varnothing$, where $M(\mathfrak{q})$ is as in Proposition 3.4.

The saturation is quasi-convex (see [DS05, Lemma 4.25]).

Lemma 3.6 (Uniform quasi-convexity of saturations)**.** *For every $\mathfrak{q}'$, there exists $\tau(\mathfrak{q}') > 0$ such that for every $L > 1$ and every $\mathfrak{q}$–ray or $\mathfrak{q}$–segment α, $\mathrm{Sat}(\alpha)$ has the property that, for every $\mathfrak{q}'$–segment γ with endpoints $N_L(\mathrm{Sat}(\alpha))$, we have*

$$\gamma \subset N_{\tau(\mathfrak{q}') \cdot L}(\mathrm{Sat}(\alpha)).$$

The quasi-convexity of saturations provides a way to understand the quasi-geodesic rays based on how many and which parabolic sets they travel nearby. The next several definitions and results make this concrete.

Definition 3.7. Let α be a $\mathfrak{q}$–quasi-geodesic segment or $\mathfrak{q}$–ray in X. We say a point $\alpha(t)$ is a $\mathfrak{q}$–*transition point* of α if

$$\alpha(t) \in \mathrm{trans}_{M(\mathfrak{q}),c(\mathfrak{q})}(\alpha),$$

where $M(\mathfrak{q}), c(\mathfrak{q})$ are as Proposition 3.4.

Definition 3.8. Let α be a $\mathfrak{q}$–ray. We say α is a $\mathfrak{q}$–*transient* ray if, there is a sequence of times $t_i \to \infty$ such that $\alpha(t_i)$ is a $\mathfrak{q}$–transition point of α.

Note that if $\mathfrak{q}' \geq \mathfrak{q}$ and α is a $\mathfrak{q}$–ray, then α is also a $\mathfrak{q}'$–ray. But, the set of $\mathfrak{q}$–transition points is not necessarily a subset or a superset of the set of $\mathfrak{q}'$–transition points because to ensure

$$\mathrm{deep}_{M_1,c_1}(\alpha) \subset \mathrm{deep}_{M_2,c_2}(\alpha)$$

we need $c_1 \geq c_2$ and $M_1 \leq M_2$. However, as we shall see, the quality of being a transient ray is independent of the choice of $\mathfrak{q}$. We summarize here that there are exactly two disjoint scenarios of redirecting based on whether a ray is transient or not.

Lemma 3.9. [QR24, Proposition 8.14, Lemma 8.17]. *Let α be a $\mathfrak{q}$–ray, and let M, c and ρ be as in Proposition 3.4. Then either*

- *α is a $\mathfrak{q}$–transient ray, then all quasi-geodesic rays in $\boldsymbol{a} = [\alpha]$ are transient. The class $\boldsymbol{a}$ has a geodesic representative and*
$$f_{\boldsymbol{a}}(q, Q) = (9q, Q).$$
- *Otherwise, α is not transient, then α is eventually contained in $N_{\rho M}(A)$ for some $A \in \mathcal{A}$. Likewise all quasi-geodesic rays in $[\alpha]$ are nona-transient and all quasi-geodesic rays are eventually contained in $N_{\rho(\mathfrak{q})M(\mathfrak{q})}A$ for the same A.*

Furthermore, if α is a $\mathfrak{q}$–transient ray and $\mathfrak{q}' \geq \mathfrak{q}$, then α is also a $\mathfrak{q}'$–transient ray.

We remark without illustration that $K = K(G)$ is a proper hyperbolic space on which G acts properly discontinuously and also the action is a geometrically finite action. Every limit point of K is either a conical limit point or a bounded parabolic point. [Bow12]. In particular, a limit point is a *conical limit point* if the associated geodesic ray is a $(1, 0)$-transient ray.

3.2. **Bowditch boundary.** Now we define the Bowditch boundary for relatively hyperbolic groups. Let ∂K denote the Gromov boundary of K. Let $V(K)$ denote the vertex set of K, let $V_\infty K = \{*p_A, A \in \mathcal{A}\}$ and let $\triangle K = V_\infty(K) \cup \partial K$.

Definition 3.10. For $v, w \in (V(K) \cup \partial K)$, let $[v, w]_K$ denote a geodesic segment (or a geodesic ray) in K connecting v to w. Given any $v \in (V(K) \cup \partial K)$ and a finite set $W \subseteq V(K)$, we write

$$m(v, W) = \Big\{ w \in \triangle K \text{ such that } W \cap [v, w]_K \subseteq \{v\} \quad \text{for every geodesic } [v, w]_K \Big\}.$$

The *Bowditch boundary* $\partial_B G$ of the relative hyperbolic group G is the set $\triangle K$ equipped with a topology generated by the neighborhoods of the form $m(v, W)$.

Every geodesic ray or segment in K can be associated to some quasi-geodesic in $\mathrm{Cay}(G)$. Let ℓ be a path in K, a *lift* of ℓ, denoted $\overline{\ell}$, is a path formed from ℓ by replacing edges incident to vertices in $V_\infty(K)$ with a geodesic in $\mathrm{Cay}(G)$.

Lemma 3.11. *Let ℓ be a geodesic line or segment in K such that $|\ell| \geq 3$, then there exists a geodesic line $\overline{\ell}_0$ in $\mathrm{Cay}(G)$ such that there exists a uniform bound D such that for any such ℓ, the projection of $\overline{\ell}_0$ to K is in a bounded neighborhood of ℓ in K.*

We also recall the *relative thin triangle* property and by [Sis12, Theorem 1.1], the condition holds for geodesic triangles in $\mathrm{Cay}(G)$.

Proposition 3.12. [Sis12, Definition 3.11] *There exists a constant δ_1 such that the following holds. For point $x, y, z \in \mathrm{Cay}(G)$ consider a geodesic triangle (x, y, z) and let w be a $(1, 0)$–transition point along $[x, y]$. Then there exists $w' \in [x, z] \cup [z, y]$ such that*

$$d_{\mathrm{Cay}(G)}(w, w') \leq \delta_1.$$

We first show that $(G, \mathcal{P})$ satisfies the assumptions associated to QR boundaries if the parabolic subgroups do. We need first *shadow* any quasi-geodesic into a parabolic subset A.

Definition 3.13. Let α be a (q, Q)-quasi-geodesic ray emanating from $\mathfrak{o}$, such that α is non transient. By Lemma 3.9, all but a finite segment of α is in a bounded neighborhood of A. Define $\mathrm{Sh}_A(\alpha)$ by composing $\alpha|_{[t_0,\infty)}$ with the closest-point projection to A, and by [QR24, Lemma 2.3] the resulting map can be tamed to be a (q', Q') quasi-geodesic that is also a $2(q + Q)$–Lipschitz and fellow travels α. This tamed (q', Q') quasi-geodesic we call the *shadow* of α in A and we write it as $\mathrm{Sh}_A(\alpha)$.

Theorem 3.1. *Suppose the QR boundaries exist for each subgroup $P \in \mathcal{P}$, then the QR boundary of $(G, \mathcal{P})$ exists.*

Proof. By [QR24, Lemma 2.5], any metric space quasi-isometric to all finitely generated groups satisfies QR-Assumption 0. For QR-Assumption 1, it was shown in Lemma 3.9 that all transient classes have a geodesic ray with a redirecting function

$$f_{\mathbf{a}}(q, Q) = (9q, Q).$$

Consider a quasi-redirecting equivalence class $[\alpha]$ that is non-transient. Then by Lemma 3.9, α is eventually in the associated bounded neighborhood of A for some $A \in \mathcal{A}$. By Definition 3.13, $\mathrm{Sh}(\alpha)$ is a (q, Q)-quasi-geodesic ray in A.

Let the basepoint of ∂A be point in the projection of $\mathfrak{o}$ to A and denote it $\mathfrak{o}_A$. Since ∂A is a quasi-isometry invariant property and thus without loss of generalization we let $\mathfrak{o}_A$ be the basepoint of A via which ∂A is defined. By the Bounded Geodesic Image Theorem, $\mathfrak{o}_A$ is bounded close to the start of $\mathrm{Sh}(\alpha)$. By construction, $\mathrm{Sh}(\alpha)$ and α are bounded distances for all but finite time, and thus $\alpha \sim \mathrm{Sh}(\alpha)$ and there exists a (q'', Q'')-quasi-geodesic ray, denoted α'' that starts at $\mathfrak{o}_A$ whose tail is $\mathrm{Sh}(\alpha)$. By assumptions, there exists a central element emanating from $\mathfrak{o}_A$ which we denote α_0^A and

$$\alpha'' \sim \alpha_0^A.$$

Lastly, we build a central element for $[\alpha]$ based on α_0^A. Indeed, consider the geodesic segments $[\mathfrak{o}, \alpha_0^A(t)], t = 1, 2, 3....$ The limit of the sequence is a geodesic ray we denote α_0. α_0 is in a bounded neighborhood of α_0^A for all but finite time. Thus we see that

$$\alpha \sim \mathrm{Sh}(\alpha) \sim \alpha'' \sim \alpha_0^A \sim \alpha_0. \tag{4}$$

Therefore α_0 is a geodesic representative in the class $[\alpha]$ when α is non-transient. The redirecting function for the non-transient class $f_b fa$ is thus a combination of all the redirecting constants in Equation 4 together with the transitivity lemma. Thus there exists a uniform function $f_{\mathbf{a}}(q, Q)$. ☐

Definition 3.14. Define a map

$$\xi : \partial G \to \partial_B G$$

as follows. Let $\mathbf{a} \in \partial G$ and $\alpha_0 \in \mathbf{a}$ be the central element of $\mathbf{a}$. If α_0 is not transient, then by Lemma 3.9 there exists a set $A \in \mathcal{A}$ such that a tail of α_0 is in a bounded neighborhood of A. In this case we define

$$\xi(\mathbf{a}) := *p_A.$$

Otherwise, α_0 is transient. By the construction and hyperbolicity of K, α_0 is an unbounded unparameterized quasi-geodesic in K and hence converges to a point $\hat{\alpha_0}$ in ∂K. We define

$$\xi(\mathbf{a}) := \hat{\alpha_0}.$$

Lemma 3.15. *The map $\xi : \partial G \to \partial_B G$ is surjective.*

Proof. Let $v \in V_\infty(K)$ be a point in the Bowditch boundary and let A be the associated set in $\mathcal{A}$. Let α be a quasi-geodesic ray that connects $[\mathfrak{o}, \mathfrak{o}_A]$ with a geodesic ray starting at $\mathfrak{o}_A$ and lie entirely in A. By [DS05, Lemma 4.19] α is a bounded constant quasi-geodesic ray in the class of ∂A. Then it follows that $\xi([\alpha]) = v$. Otherwise, let v be a point in ∂K. Since K is hyperbolic, there exists an equivalence class of quasi-geodesic rays associated with v and in fact there exists a geodesic representative in this class (for instance by Arzelà-Ascoli Theorem), which we refer to as α. Since α is a geodesic ray in K, by [Sis13a, Proposition 1.14], there exists a bounded constant quasi-geodesic ray α' in $\mathrm{Cay}(G)$ that is a lift of α. We claim that, for $\mathbf{a} = [\alpha']$, we have

$$\xi(\mathbf{a}) = v.$$

Indeed, the central element α_0 of $\mathbf{a}$ is a geodesic in $\mathrm{Cay}(G)$, and an un parametrized quasi-geodesic in K. Thus it stays in a bounded neighborhood of α and hence converges to v. This finishes the proof. ☐

We now show that ξ and ξ^{-1} are both continuous. First we show that for every $v \in \Delta(K)$ and every finite subset $W \subset V(K)$, $m(v, W)$ is open in ∂G. It suffices to verify this when W has one element as a finite intersection of open sets is open.

Lemma 3.16. *For every $\boldsymbol{b} \in \partial G$ and $p \in V(K)$ there exists $r > 0$ such that*

$$\xi(\mathcal{U}(\boldsymbol{b}, r)) \subset m(\xi(\boldsymbol{b}), p).$$

Therefore, ξ is continuous.

Proof. Let the geodesic ray β_0 be the central element of $\mathbf{b}$. We first assume that $\mathbf{b}$ is transient. Consider β_0 as a subset of K and let $\pi_{\xi(\mathbf{b})}(p)$ be the closest point projection of p to β_0 in K (see Figure 4). Since K is hyperbolic, $\pi_{\xi(\mathbf{b})}(p)$ has a bounded diameter in K. Since $\mathbf{b}$ is transient, β_0 has transition points that are arbitrarily far from $\mathfrak{o}$. Choose $r > 0$ such that, $(\beta_0)_r$ is a $(1, 0)$–transition point of β_0 and

$$d_K(\mathfrak{o}, (\beta_0)_r) \gg d_K(\mathfrak{o}, \pi_{\xi(\mathbf{b})}(p)) + D(9, 0) + 2\delta, \tag{5}$$

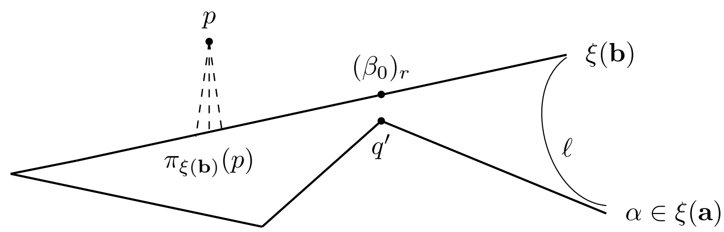


FIGURE 4. A transition point $(\beta_0)_r$ separates the point p and any geodesic line that connects $\xi(\mathbf{b})$ and $\xi(\mathbf{a})$.

where δ is the hyperbolicity constant of K. $D(9,0)$ is as in [QY24, Corollary 8.8] and $d_K(\mathfrak{o}, \pi_{\xi(\mathbf{b})}(p))$ is the maximum distance in K between any point in $\pi_{\xi(\mathbf{b})}(p)$ to $\mathfrak{o}$.

Let $\mathbf{a} \in \mathcal{U}(\mathbf{b}, r)$ and let α_0 be the central element in $\mathbf{a}$. Since $(\beta_0)_r$ is a transition point, there exists points $q \in \alpha_0$ such that

$$d(q, (\beta_0)_r) < D((9,0)),$$

Thus $\|q\| \geq r - D((9,0))$. Since K is hyperbolic, there exists either a geodesic ℓ in K connecting $\xi(\mathbf{a})$ to $\xi(\mathbf{b})$. The line ℓ is an edge in the ideal quadrilateral $((\beta_0)_r, \xi(\mathbf{b}), \xi(\mathbf{a}), q)$ hence it stays in a bounded neighborhood of

$$\beta_0|_{\geq r} \cup \alpha_0|_{\geq r} \cup [(\beta_0)_r, q].$$

Hence, ℓ is far from p in K and hence does not pass through p. Therefore, $\xi(\mathbf{a}) \in m(\xi(\mathbf{b}), p)$.
Case II: Suppose otherwise that $\mathbf{b}$ is not transient. By Lemma 3.9 there exists a unique set $A \in \mathcal{A}$ such that $\xi(\mathbf{b}) = *p_A$. Let β_0 be the central element of $\mathbf{b}$. Let

$$r \gg 2\big(\|\mathfrak{o}_A\| + \|p\|\big).$$

Let $\mathbf{a} \in \mathcal{U}(\mathbf{b}, r)$ and let α_0 be the central element of $\mathbf{a}$. Then α_0 can be $f_{\mathbf{b}}(1,0)$–redirect to β_0 at radius r. Let $\mathfrak{e} \in A$ be the point near where α_0 leaves the M_0–neighborhood of A.

Consider any geodesic segment or ray ℓ in K connecting $\xi(\mathbf{a})$ to $*p_A$. By [Hru10, Proposition 8.13], ℓ enters $N_{\tau(f_{\mathbf{b}}(\mathfrak{q}))}(A)$ at a point that is boundedly close to $\mathfrak{e}$. Since $*p_A$ is the final point in ℓ, $*p_A$ does not appear in interior of ℓ and hence, for any other vertex x in ℓ, we have $\|x\| \geq \|\mathfrak{e}\| - D(1,0)$. This implies $\|x\| \gg \|p\|$ and hence ℓ does not pass through p. Therefore,

$$\mathbf{a} \in m(\xi(\mathbf{b}), p)$$

and hence $\mathcal{U}(\mathbf{b}, r) \subset m(\xi(\mathbf{b}), p)$. □

Now we are ready to conclude:

Theorem 3.2. *Let G be a relatively hyperbolic group with respect to subgroups $P_1, P_2, \ldots P_k$. Assume that ∂A exists for each Cayley graph of the subgroups $P \in \mathcal{P}$, then the quasi-redirecting boundary ∂G exists and ∂G surjects onto $\partial_B G$.*

Proof. Since the map $\xi : \partial X \to \partial_B X$ is onto and ξ is continuous, we conclude that $\xi : \partial G \to \partial_B G$ is a surjective homomorphism. □

Corollary 3.17. *Let G be a relatively hyperbolic group with respect to subgroups $P_1, P_2, ...P_k$. Then the conical limit points of K are embedded as a subset in $P(G)$.*

Proof. Case I of Lemma 3.16 shows that if $\mathbf{b}$ has a transient geodesic ray representative then it maps to exactly one point in ∂K. Therefore there is a 1-1 map between the set of conical limit points of G and the set of transient classes in $P(G)$. □

4. QR boundary of CAT(0) admissible groups

CAT(0) Admissible groups were first introduced by Croke-Kleiner in [CK02]. They are a particular class of graph of groups that includes fundamental groups of 3-dimensional graph manifolds. The QR-boundary of a specific case of CAT(0) admissible group is computed in [QR24]. In this section we follow the the arguments in [QR24, Section 11] closely but adapt and expand them to suit all CAT(0) admissible groups.

Definition 4.1. A *graph of groups* $\mathcal{G} = (\Gamma, \{G_v\}, \{G_e\}, \{\tau_e\})$ consists of the following data:

1. a graph Γ, called the *underlying graph*,
2. a group G_v for each vertex $v \in V\Gamma$, called a *vertex group*,
3. a subgroup $G_e \leq G_{e_-}$ for each edge $e \in E\Gamma$, called an *edge group*,
4. an isomorphism $\tau_e \colon G_e \to G_{\overline{e}}$ for each $e \in E\Gamma$ such that $\tau_e^{-1} = \tau_{\overline{e}}$, called an *edge map*.

The *fundamental group* $\pi_1(\mathcal{G})$ of a graph of groups $\mathcal{G}$ is as defined in [SW79].

Definition 4.2. A graph of groups $\mathcal{G}$ is *admissible* if

1. $\mathcal{G}$ is a finite graph with at least one edge.
2. Each vertex group G_v has center $Z(G_v) \cong \mathbb{Z}$, $H_v \colon = G_v/Z(G_v)$ is a non-elementary hyperbolic group, and every edge subgroup G_e is isomorphic to $\mathbb{Z}^2$.
3. Let e_1 and e_2 be distinct directed edges entering a vertex v, and for $i = 1, 2$, let $K_i \subset G_v$ be the image of the edge homomorphism $G_{e_i} \to G_v$. Then for every $g \in G_v$, gK_1g^{-1} is not commensurable with K_2, and for every $g \in G_v - K_i$, gK_ig^{-1} is not commensurable with K_i.
4. For every edge group G_e, if $\alpha_i \colon G_e \to G_{v_i}$ is the edge monomorphism, then the subgroup generated by $\alpha_1^{-1}(Z(G_{v_1}))$ and $\alpha_2^{-1}(Z(G_{v_1}))$ has finite index in G_e.

Definition 4.3. A group G is *admissible* if it is the fundamental group of an admissible graph of groups. We say that an admissible group G is a *CAT(0) admissible group* if there is a complete proper CAT(0) space X such that $G \curvearrowright X$ properly discontinuous, cocompactly. Such action $G \curvearrowright X$ is called a *CKA action* and the space X is called a *CAT(0) admissible space* of G.

Below are some examples of CAT(0) admissible groups.

Example 4.4.

1. (Tori complexes) Let $n \geq 3$ be an integer. Let $T_1, T_2, \ldots, T_n$ be a family of flat two-dimensional tori. For each i, we choose a pair of simple closed geodesics a_i and b_i such that length(b_i) = length(a_{i+1}), identifying b_i and a_{i+1} and denote the resulting space by X. The space X is a graph of spaces

with $n-1$ vertex spaces $V_i := T_i \cup T_{i+1}/\{b_i = a_{i+1}\}$ (with $i \in \{1, \ldots, n-1\}$) and $n-2$ edge spaces $E_i := V_i \cap V_{i+1}$.

The fundamental group $G = \pi_1(X)$ has a graph of groups structure where each vertex group is the fundamental group of the product of a figure eight and S^1. Vertex groups are isomorphic to $F_2 \times \mathbb{Z}$ and edge groups are isomorphic to $\pi_1(E_i) \cong \mathbb{Z}^2$. The generators $[a_i], [b_i]$ of the edge group $\pi_1(E_i)$ each map to a generator of either a $\mathbb{Z}$ or F_2 factor of $F_2 \times \mathbb{Z}$. It is clear that with this graph of groups structure, $\pi_1(X)$ is an admissible group.

(2) (Graph manifolds) Let M be a nongeometric graph manifold that admits a nonpositively curved metric. Lift this metric to the universal cover $\tilde{M}$ of M, and we denote this metric by d. Then the action $\pi_1(M) \curvearrowright (\tilde{M}, d)$ is a CKA action.

(3) One may build CAT(0) admissible groups algebraically from any finite number of hyperbolic CAT(0) groups. The following example is for $n = 2$ but the same principle works for any $n \geq 2$. Let H_1 and H_2 be two torsion-free hyperbolic groups that act geometrically on $CAT(0)$ spaces X_1 and X_2 respectively. Then $G_i = H_i \times \langle t_i \rangle$ (with $i = 1, 2$) acts geometrically on the $CAT(0)$ space $Y_i = X_i \times \mathbb{R}$. Any primitive hyperbolic element h_i in H_i gives rise to a totally geodesic torus T_i in the quotient space Y_i/G_i with basis $([h_i], [t_i])$. We re-scale Y_i so that the translation length of h_i is equal to that of t_i for each i. Let $f \colon T_1 \to T_2$ be a *flip* isometry respecting these lengths, that is, an orientation-reversing isometry mapping $[h_1]$ to $[t_2]$ and $[t_1]$ to $[h_2]$. Let M be the space obtained by gluing Y_1 to Y_2 by the isometry f. There is a metric on the space M that gives rise to a locally $CAT(0)$ space (see e.g. [BH99, Proposition II.11.6]). By the Cartan-Hadamard Theorem, the universal cover $\widetilde{M}$ with the induced length metric from M is a CAT(0) space. Let G be the fundamental group of M. Then the action $G \curvearrowright \widetilde{M}$ is geometric, and G is an example of a Croke-Kleiner admissible group.

4.1. **Vertex and edge spaces in CAT(0) admissible spaces.** Let G be an admissible group that acts properly discontinuous, cocompactly, and by isometries on a complete proper CAT(0) space X. Let $G \curvearrowright T$ be the action of G on the associated Bass-Serre tree T of the graph of group $\mathcal{G}$ (we refer the reader to [CK02, Section 2.5] for a brief discussion).

Let $V(T)$ and $E(T)$ be the vertex and edge sets of T. For each $\sigma \in V(T) \cup E(T)$, we let $G_\sigma \leq G$ be the stabilizer of σ. We review facts from [CK02, Section 3.2] that will be used thoroughly in this paper and refer the reader to [CK02] for further explanation. From the given actions $G \curvearrowright X$ and $G \curvearrowright T$ we have

(1) for every vertex $v \in V(T)$, the set $X_v := \cap_{g \in Z(G_v)} \operatorname{Minset}(g)$ splits as metric product
$$X_v = H_v \times \mathbb{R}$$
where $Z(G_v)$ acts by translation on the $\mathbb{R}$–factor and the quotient $Q_v := G_v/Z(G_v)$ acts geometrically on the CAT(0) space H_v.

(2) for every edge $e \in E(T)$, the minimal set $X_e := \cap_{g \in G_e} \operatorname{Minset}(g)$ splits as
$$\overline{X_e} \times \mathbb{R}^2 \subset X_v$$
where $\overline{X_e}$ is a compact CAT(0) space and $G_e = \mathbb{Z}^2$ acts co-compactly on the Euclidean plane $\mathbb{R}^2$.

Definition 4.5. For every vertex $v \in V(T)$, edge $e \in E(T)$, the spaces X_v and X_e are called *vertex space* and *edge spaces* of X.

Remark 4.6. For each vertex space X_v, since the quotient $Q_v := G_v/Z(G_v)$ is a non-elementary hyperbolic group and it acts geometrically on the CAT(0) space H_v, it follows that H_v is a hyperbolic space.

In the sequel, it will be useful to to make the following specific choices.

Definition 4.7. There exists a G–equivariant coarse L–Lipschitz map $\mathfrak{i}\colon X \to T^0$ such that $x \in X_{\mathfrak{i}(x)}$ for all $x \in X$. The map $\mathfrak{i}$ is called an *indexed map*. We refer the reader to Section 3.3 in [CK02] for existence of such a map $\mathfrak{i}$.

4.2. **Strips and walls in CAT(0) admissible spaces.** [CK02, Section 4.2] We note that the assignments $v \to X_v$ and $e \to X_e$ are G–equivariant in the sense that $gX_v = X_{gv}$ and $gX_e = X_{ge}$ for any $g \in G$.

Definition 4.8 (Walls and strips)**.** We first choose, in a G–equivariant way, a plane $F_e \subset X_e$ which we will call *wall* for each edge $e \in E(T)$.

For every pair of adjacent edges e, e', we choose, again equivariantly, a minimal geodesic from F_e to $F_{e'}$; by the convexity of $X_v = H_v \times \mathbb{R}$ where $v := e \cap e'$, this geodesic determines a *strip* in the CAT(0) admissible space X:

$$\mathcal{S}_{ee'} := h_{ee'} \times \mathbb{R}$$

(possibly of width zero) for some geodesic segment $h_{ee'} \subset H_v$.

Remark 4.9.

(1) Note that lines $\mathcal{S}_{ee'} \cap F_e$ and $\mathcal{S}_{ee'} \cap F_{e'}$ are axes of $Z(G_v)$. Hence if $e, e', e'' \in E(T)$ be three consecutive edges then the angle between the geodesics $\mathcal{S}_{ee'} \cap F_{e'}$ and $\mathcal{S}_{e'e''} \cap F_e'$ is bounded away from zero.

(2) If $\langle f_1 \rangle = Z(G_{v_1}), \langle f_2 \rangle = Z(G_{v_2})$ then $\langle f_1, f_2 \rangle$ generates a finite index subgroup of G_e. We remark that the intersection of two strips $\mathcal{S}_{e_1 e}$ and $\mathcal{S}_{e_2 e}$ is a point. Indeed, we have

$$\mathcal{S}_{e_1 e} \cap \mathcal{S}_{e_2 e} = (\mathcal{S}_{e_1 e} \cap F_e) \cap (\mathcal{S}_{e_2 e} \cap F_e)$$

As two lines $\mathcal{S}_{e_1 e} \cap F_e$ and $\mathcal{S}_{e_2 e} \cap F_e$ in the wall F_e are axes of $\langle f_{v_1} \rangle = Z(G_{v_1})$, $\langle f_{v_1} \rangle = Z(G_{v_2})$ respectively and $\langle f_1, f_2 \rangle$ generates a finite index subgroup of G_e, it follows that these two lines are non-parallel, and hence their intersection must be a single point.

Lemma 4.10. *For every $\mathfrak{q} \in [1, \infty] \times [0, \infty)$ and $\rho > 0$, there is $\mathfrak{q}' \in [1, \infty] \times [0, \infty)$ such that the following holds. Let F_i be a wall, $R \geq (1 + \rho) \cdot r > 0$ be a pair of radii and α and β be two $\mathfrak{q}$–rays. Assume $\alpha_r \in F_i$ and that $\beta|_{\geq R}$ starts at a point in F_i. Then, α can be $\mathfrak{q}'$-redirected to β at radius r.*

Proof. Since $\rho > 0$ there exists an annulus $A = B(\mathfrak{o}, R) - B(\mathfrak{o}, r)$ such that $\alpha|r$ and $\beta|_{\geq R}$ are in different connect components of $X - A$. In polar coordinates, the point α_r can be denoted (θ_1, r). Consider the geodesic segment ℓ_1 connecting (θ_1, r) and the point $p := (\theta_1, r + \frac{1}{2}(R - r)) = (\theta_1, (R + r)/2)$. Likewise let $\beta|_R$ be a point with coordinates (θ_2, R) and let

$$\ell_2 := [(\theta_2, R), (\theta_2, R - \frac{1}{2}(R - r)],$$

where we use q to denote the point $(\theta_2, R - \frac{1}{2}(R-r) = (\theta_2, (R+r)/2)$. Lastly, consider the arc (as part of the circle of radius $r + \frac{1}{2}(R-r)$) that connects p and q along the shorter half of the circle and denote it $C(p,q)$. The concatenation

$$\alpha|_r \cup \ell_1 \cup C(p,q) \cup \ell_2 \cup \beta|_{\geq R}$$

redirects $\alpha|_r$ to the tail of β. Since $\beta|_{\geq R}$ starts at β_R, By Surgery Lemma 2.8 (1), $\alpha|_r \cup \ell_1$ is a (3q, Q)-segment. Now consider two points x, y on $\alpha|_r$ and $C(p,q)$, respectively. Then the quasi-geodesic constants are bounded above by $(D, 0)$ where

$$\begin{aligned} D = D(\mathfrak{q}, \rho) &:= \frac{\ell(\alpha|_r) + \frac{1}{2}(R-r) + r(1+\frac{\rho}{2})}{\frac{\rho}{2}r} \\ &\leq \frac{qr + Q + r\frac{1}{2}\rho + r(1+\frac{\rho}{2})}{\frac{\rho}{2}r} \\ &\leq \frac{2(q+Q+1+\rho)}{\rho} \end{aligned}$$

Lastly apply Surgery Lemma 2.8 (3) to connect $\alpha|_r \cup \ell_1 \cup C(p,q)$ with $\beta|_{\geq R}$ as the segment ℓ_2 realizes the set distance and we get that the concatenation is a $(4D, 3Q)$ quasi-geodesic ray that redirects α_r to the tail of β. Since the construction holds for all r and $\mathfrak{q}' := (4D, 3Q)$ depends only on $\mathfrak{q}$ and ρ, we have that α $\mathfrak{q}'$-redirects to β at r. □

That is, we can transition from $\alpha|_r$ to $\beta|_{\geq R}$ as long as there is buffer between them that have a product structure and a thickness that is a linear function of r.

4.3. **Types of quasi-geodesics.** Let $\mathfrak{i}\colon X \to T$ be the index map given by Definition 4.7 and fix a wall F in X. We also assume that the basepoint $\mathfrak{o} \in F$ and $F \subset X_{v_0}$ where $v_0 := \mathfrak{i}(\mathfrak{o})$.

Recall that X_{v_0} splits as a metric product $H_{v_0} \times \mathbb{R}$. In the rest of this paper, we fix a geodesic ray ζ based at $\mathfrak{o}$ follows the line $\mathbb{R}$ in the $\mathbb{R}$ factor of X_{v_0}, and call it the *main flat ray*.

We remark that the choice of ζ is arbitrary since any quasi-geodesic ray in X_{v_0} is the same equivalent class with ζ by Proposition 2.6.

Definition 4.11. Let α be a arbitrary $\mathfrak{q}$–ray in the CKA space X emanating from $\mathfrak{o}$. There is a unique geodesic segment/ray γ in T associated with $\mathfrak{i}(\alpha)$ defined as follows.

Let $v_1 \in \text{Link}(v_0)$ be the vertex where $\mathfrak{i}(\alpha)$ enters immediately after $\mathfrak{i}(\alpha)$ leaves v_0 in the sense that $\mathfrak{i}(\alpha)$ never visit v_0 again. Similarly, we define $v_2 \in \text{Link}(v_1), v_3 \in \text{Link}(v_2)$, etc.

(1) Note that it is possible that $\mathfrak{i}(\alpha)$ contains some v_i infinitely many times. In this situation, we call the $\mathfrak{q}$–ray α is of *Type I*.
(2) Otherwise, $\mathfrak{i}(\alpha)$ is a multiset that contains an ordered, infinite sequence of vertices where each v_i appears a finitely many times. The radii of v_i in $\mathfrak{i}(\alpha)$ tends to infinity monotonically. Since T is a tree there is exactly one geodesic ray whose vertex set is contained in $\mathfrak{i}(\alpha)$. Denote this geodesic ray γ. Relabel again such that γ traverses vertices $v_0, v_1, v_2, ...$etc. In this case, we will call the $\mathfrak{q}$–ray α is of *Type II*.

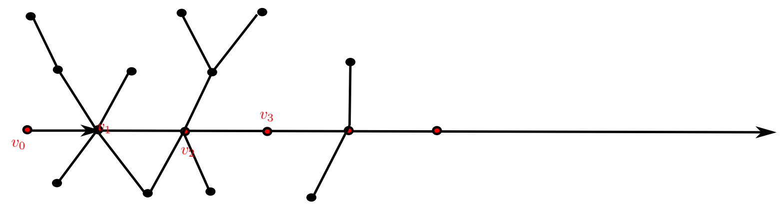


FIGURE 5. The figure illustrates a portion of vertices $\mathfrak{i}(\alpha)$ visits. With respect to $\mathfrak{i}(\alpha)$, there is the unique geodesic ray $\gamma_\alpha := [v_0, v_1] \cdot [v_1, v_2] \cdot [v_2, v_3] \cdots$ associated to α.

Since γ is unique and associated with α, it also makes sense to write γ_α. We call γ_α and the associated ordered, infinite sequence of vertices $v_0, v_1, v_2, ..., etc$ the *simplified itinerary* associated with α.

We define $e_i := [v_{i-1}, v_i]$ and let $v_0 := \mathfrak{o}$. The geodesic in $X_{v_0} = H_{v_0} \times \mathbb{R}$ that realizes the distance $d(\mathfrak{o}, F_{e_1})$ form a strip which we denote by $\mathcal{S}_{e_0e_1} = h_{e_0e_1} \times \mathbb{R}$ where the geodesic $h_{e_0e_1} \subset H_{v_0}$ is the projection to H_{v_0} of the intersection of this minimal geodesic with X_{v_0}.

Lastly, for the rest of this paper, we adopt the notations below:

(1) The intersection point of any two adjacent strips by
$$p_i := \mathcal{S}_{e_{i-1}e_i} \cap \mathcal{S}_{e_ie_{i+1}}$$
(2) For each $i \geq 2$, denote the two *singular geodesics* in the wall F_{e_i} by
$$\mathfrak{f}_i^- := F_{e_i} \cap \mathcal{S}_{e_{i-1}e_i} \quad \text{and} \quad \mathfrak{f}_i^+ = F_{e_i} \cap \mathcal{S}_{e_ie_{i+1}}$$

Remark 4.12. These two singular geodesic rays $\mathfrak{f}_i^-$ and $\mathfrak{f}_i^+$ in the wall F_{e_i} will form an angle denoted by θ_i. This angle is in $(0, \pi)$. Up to group action, there are only finitely many angles shown up.

Definition 4.13. Associated to the geodesic ray γ_α, we define
$$s_\alpha := [p_0, p_1] \cdot [p_1, p_2] \cdots$$
to be the concatenation of geodesic segments $[p_i, p_{i+1}]$. The path s_γ will be called the *special ray* with respect to the simplified itinerary γ_α. There exists $\mu > 1$ such that all special rays are (μ, μ)–quasi-geodesics [NY23, Proposition 3.8], where μ depends only on X and are independent of the specific itinerary addressed.

4.4. **Backward spiral paths.** In this section, we are going to show that every $\mathfrak{q}$–ray α can be quasi-redirected to the main flat ray ζ at every radius $r > 0$ via a quasi-geodesic γ_r with uniform quasi-geodesic constants see Proposition 4.16.

Definition 4.14. For each $i \geq 1$, an *L–path* in a wall F_{e_i} of X is a concatenation of two geodesics l and l' in the wall F_{e_i} such that l is parallel to the singular geodesic $\mathfrak{f}_i^-$ and l' is parallel to the singular geodesic $\mathfrak{f}_i^+$.

An *extended L–path* in X is a concatenation of an L–path in F_{e_i} with a geodesic segment in and strip that either on one side or the other (spliced with l, or l', respectively).

Lemma 4.15. *There exists a uniform constant $R > 0$ such that the following holds. Let x and y be two points in walls F_e and F'_e of a vertex space $X_v = H_v \times \mathbb{R}$ respectively. Then any path in X_v connecting x to y must come within* *the R–neighborhood of two singular geodesics $\mathfrak{f}_e := F_e \cap \mathcal{S}_{ee'}$ and $\mathfrak{f}_{e'} := F_{e'} \cap \mathcal{S}_{ee'}$ respectively.*

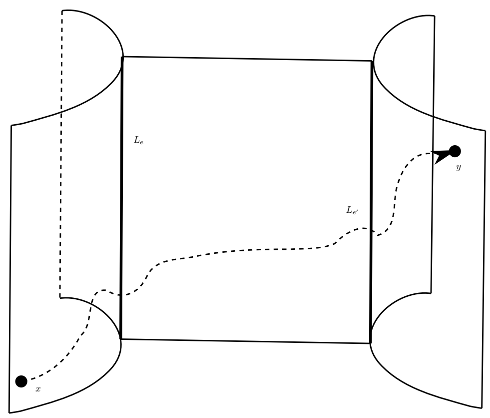


FIGURE 6. Any path from any point $x \in F_e$ to any point $y \in F_e'$ need not intersect $\mathfrak{f}_e$ and $\mathfrak{f}_{e'}$. However, this path must come within an R–neighborhood of $\mathfrak{f}_e$ and $\mathfrak{f}_{e'}$.

Proposition 4.16. *Let α be a $\mathfrak{q}$–ray in X. Then α can be $\mathfrak{q}'$–redirected to ζ where $\mathfrak{q}'$ depends only on $\mathfrak{q}$. In particular, $\alpha \preceq \zeta$.*

Proof. If α does not intersect any wall then α necessarily lies in the same block as the basepoint $\mathfrak{o}$. By Proposition 2.6, α and ξ redirect to each other. Otherwise, α intersects a finite of infinite set of walls. According to Lemma 4.15, for each such wall F_{e_n}, the path α must come within $N_R(\mathfrak{f}_n^-)$. Let $x_n \in \alpha$ be the first point α enters $N_R(\mathfrak{f}_n^-)$. Without lost of generality, we can assume that $x_n \in \mathfrak{f}_n^-$.

In the following, we are going to construct a $\mathfrak{q}'$–ray γ so that $[\mathfrak{o}, x_n]_\alpha$ is a subpath of γ and γ is eventually concise with ζ.

Since each wall is isometric to $\mathbb{E}^2$, every L–path is a $(\lambda, 0)$–quasi-geodesic where λ is a constant depending on the angle between the two singular geodesics. Since X is cocompact, there are only finitely many angles shown up, and thus λ can be made to be a uniform constant, that is every L–path in X is $(\lambda, 0)$–quasi-geodesic.

(1) At x_n, we choose a point y_n in F_n so that $[x_n, y_n]$ is parallel to $\mathfrak{f}_n^+$ and $d(y_n, x_n) = 1$. Since x_n is a closest point in $[\mathfrak{o}, x_n]_\alpha$ to any point $x \in [x_n, y_n]$, it follows from Surgery Lemma 2.8(1) that

$$\mathcal{L}_{n+1} := \alpha|_{[\mathfrak{o}, x_n]} \cup [x_n, y_n]$$

is a $(3q, Q)$–quasi-geodesic. Next, let

$$\rho > 36q^2\lambda^2 \tag{6}$$

and at y_n we attach to it an extended L–path

$$\mathcal{L}_n := \zeta_n \cdot \eta_{n-1}$$

where η_{n-1} is a geodesic in the $\mathcal{S}_n$ and ζ_n is an L–path in F_n so that

$$d((\mathcal{L}_n)_+, (\mathcal{L}_n)_-) \geq \rho\, d((\mathcal{L}_{n+1})_+, (\mathcal{L}_{n+1})_-)$$

Here for a path γ, we mean γ_- and γ_+ to be the initial and terminal points of γ respectively.

(2) Next at the terminal point $(\mathcal{L}_n)_+$ of $\mathcal{L}_n$ we attach to it an extended L–path

$$\mathcal{L}_{n-1} := \zeta_{n-1} \cdot \eta_{n-2}$$

where η_{n-2} is a geodesic in the $\mathcal{S}_{n-1}$ and ζ_{n-1} is an L–path in W_{n-1} so that

$$d((\mathcal{L}_{n-1})_+, (\mathcal{L}_{n-1})_-) \geq \rho\, d((\mathcal{L}_n)_+, (\mathcal{L}_n)_-)$$

(3) We continue this pattern to define extended L–paths:

$$\mathcal{L}_{n-2}, \mathcal{L}_{n-3}, \ldots, \mathcal{L}_i, \ldots, \mathcal{L}_1$$

(4) At the terminal point $(\mathcal{L}_1)_+$ of $\mathcal{L}_1 = \zeta_1 \cdot \eta_0$ which belongs to the singular geodesic $\mathfrak{f}_0^+$, we then don't attach an L–path in the wall F_0, instead, we just need to attach to it a geodesic ray, denoted by $\mathcal{L}_0$, perpendicular to the singular geodsic $\mathfrak{f}_0^+$.

Let $\mathcal{L}$ be the concatenation $\mathcal{L} := \mathcal{L}_{n+1} \cdot \mathcal{L}_n \cdots \mathcal{L}_0$, we refer to $\mathcal{L}$ as a *backward spiral path* in X with slope ρ. It remains to show that $\mathcal{L}$ is a $\mathfrak{q}'$–ray where $\mathfrak{q}'$ depends only on q and Q. To simplify notations, we relabel $\mathcal{L}_i$ by γ_{n+2-i} with $i \in \{0, 1, \ldots, n+1\}$, and hence $\mathcal{L} = \gamma := \gamma_1 \cdot \gamma_2 \cdots \gamma_{n+2}$. Let $0 = a_0 < a_1 < a_2 < \ldots < a_{n+1}$ so that $\gamma_i = \gamma|_{[a_{i-1}, a_i]}$ with $i \in \{1, 2, \ldots n+1\}$ and $\gamma_{n+2} = \mathcal{L}|_{[a_{n+1}, a_{n+2})}$ with $a_{n+2} := \infty$.

By Surgery Lemma 2.8(1), we obtain that every extended L–path is a $(6\lambda, 0)$–quasi-geodesic. It follows that each γ_i is a $(6\lambda, 0)$–quasi-geodesic with $i \geq 2$ except γ_1 is a $(3q, Q)$–quasi-geodesic. Thus all $\gamma_1, \gamma_2, \ldots, \gamma_{n+2}$ are $(6q, Q)$–quasi-geodesic. Again, using Surgery Lemma 2.8(1), we have that the concatenation $\gamma_i \cdot \gamma_{i+1}$ is a $(18q\lambda, Q)$–quasi-geodesic.

Now, let t_1 and t_2 be distinct points in $[a_0, a_{n+2}) = [0, \infty)$. Since each γ_i is a $(6q\lambda, Q)$–quasi-geodesic (in fact all are $(6\lambda, 0)$–quasi-geodesic except γ_1 is $(6q\lambda, Q)$–quasi-geodesic), we have

$$d\left(\gamma\left(t_1\right), \gamma\left(t_2\right)\right) \leq 6q\lambda\lambda \left|t_2 - t_1\right| + Q$$

For the rest of the proof, we only need to work on the lower bound of $d(\gamma(t_1), \gamma(t_2))$ in terms of $|t_2 - t_1|$. Since $\gamma_i \cdot \gamma_{i+1}$ is a $(18q\lambda, Q)$–quasi-geodesic for every i, we only need to consider the case where $t_1 \in [a_k, a_{k+1}]$ and $t_2 \in [a_j, a_{j+1}]$ with $j \geq k+2$. By the triangle inequality,

$$d\left(\gamma\left(t_2\right), \gamma\left(t_1\right)\right) \geq d\left(\gamma\left(t_2\right), \gamma\left(a_{j-1}\right)\right) - d\left(\gamma\left(a_{j-1}\right), \gamma\left(t_1\right)\right) \tag{7}$$

To simplify notation let us denote

$$|\gamma_i| := d\left(\gamma\left(a_{i-1}\right), \gamma\left(a_i\right)\right)$$

for $i = 1, 2, 3..., n+1$. For $i = n+2$ we denote $|\gamma_{n+2}| := \infty$ as γ_{n+2} is a geodesic ray.

The slope condition then says

$$\sum_{i=1}^{j-1} |\gamma_i| \leq \frac{1}{\rho} |\gamma_j| \tag{8}$$

From the triangle inequality and the fact $|a_{k+1} - t_1| \leq |a_{k+1} - a_k|$ we have

$$\begin{aligned} d\left(\gamma\left(t_1\right), \gamma\left(a_{j-1}\right)\right) &\leq d\left(\gamma\left(t_1\right), \gamma\left(a_{k+1}\right)\right) + \sum_{i=k+2}^{j-1} |\gamma_i| \\ &\leq 6q\left|a_{k+1} - a_k\right| + Q + \sum_{i=k+2}^{j-1} |\gamma_i| \\ &\leq 6q\lambda(6q\lambda|\gamma_{k+1}|) + 6q\lambda Q) + Q + \sum_{i=k+2}^{j-1} |\gamma_i| \\ &\leq 36q^2\lambda^2\left(\sum_{i=k+1}^{j-1} |\gamma_i|\right) + 36q^2\lambda^2 Q + Q \end{aligned}$$

From the construction of γ, we have

$$d(\gamma(t_2), \gamma(a_{j-1})) \geq d(\gamma(a_j), \gamma(a_{j-1})) \geq |\gamma_j|$$

Then by applying inequality (8), we have

$$\begin{aligned} d(\gamma(t_1), \gamma(a_{j-1})) &\leq 36q^2\lambda^2\left(\sum_{i=k+1}^{j-1} |\gamma_i|\right) + 36q^2\lambda^2 Q + Q \\ &\leq 36q^2\lambda^2\frac{1}{\rho}|\gamma_j| + 36q^2\lambda^2 Q + Q \\ &\leq 36q^2\lambda^2\frac{1}{\rho}|\gamma_j| + 36q^2\lambda^2 Q + Q \leq 36q^2\lambda^2\frac{1}{\rho}d(\gamma(t_2), \gamma(a_{j-1})) + 36q^2\lambda^2 Q + Q \end{aligned}$$

Substituting this into inequality (7) and then use the fact that $\gamma_j * \gamma_{j+1}$ is a $(18q\lambda, Q)$–quasi-geodesic we obtain

$$\begin{aligned} d\left(\gamma\left(t_2\right), \gamma\left(t_1\right)\right) &\geq (1 - 36q^2\lambda^2\frac{1}{\rho})d(\gamma(t_2), \gamma(a_{j-1})) - 36q^2\lambda^2 Q - Q \\ &\geq (1 - 36q^2\lambda^2\frac{1}{\rho})(\frac{1}{18q\lambda}|t_2 - a_{j-1}| - Q) - 36q^2 Q\lambda^2 - Q \\ &= (1 - 36q^2\lambda^2\frac{1}{\rho})\frac{1}{18q}|t_2 - a_{j-1}| - Q(1 - 36q^2\lambda^2\frac{1}{\rho}) - 36q^2\lambda^2 Q - Q \end{aligned}$$

From inequality (8) and the fact γ_j is a $(6\lambda, 0)$–quasi-geodesic with $j \geq 2$ and γ_1 is $(6q\lambda, Q)$–quasi-geodesic, we obtain

$$\begin{aligned} \frac{1}{\rho}\left(6q\lambda\left|a_j - a_{j-1}\right| + Q\right) \geq \frac{1}{\rho}|\gamma_j| \geq \sum_{i=1}^{j-1}|\gamma_i| &\geq \sum_{i=1}^{j-1} \frac{1}{6q\lambda}\left|a_i - a_{i-1}\right| - Q \\ &\geq \frac{1}{6q\lambda}\left|a_{j-1} - t_1\right| - Q \end{aligned}$$

Hence

$$|a_{j-1} - t_1| \leq 6q\lambda(Q + Q\frac{1}{\rho} + 6q\lambda\frac{1}{\rho}|a_j - a_{j-1}|)$$

and we can conclude

$$\begin{aligned}|t_2 - t_1| &= |t_2 - a_j| + |a_j - a_{j-1}| + |a_{j-1} - t_1| \\ &\leq |t_2 - a_j| + 6q\lambda(Q + Q\frac{1}{\rho}) + (1 + 36q^2 \frac{\rho}{\rho+1})|a_j - a_{j-1}| \\ &\leq |t_2 - a_{j-1}| + 6q\lambda(Q + Q\frac{1}{\rho}) + (1 + 36q^2\lambda^2 \frac{\rho}{\rho+1})|t_2 - a_{j-1}| \\ &\leq (2 + 36q^2\lambda^2 \frac{\rho}{\rho+1}))|t_2 - a_{j-1}| + 6q\lambda(Q + Q\frac{1}{\rho})\end{aligned}$$

and hence

$$|t_2 - a_{j-1}| \geq \frac{1}{2 + 36q^2 \frac{\rho}{\rho+1}}|t_2 - t_1| - \frac{6q(Q + Q\frac{1}{\rho})}{2 + 36q^2 \frac{\rho}{\rho+1}}$$

Therefore

$$\begin{aligned}d(\gamma(t_2), \gamma(t_1)) &\geq (1 - 36q^2\lambda^2 \frac{1}{\rho}) d(\gamma(t_2), \gamma(a_{j-1})) - 36q^2\lambda^2 Q - Q \\ &\geq (1 - 36q^2\lambda^2\frac{1}{\rho})\frac{1}{18q\lambda}(\frac{1}{2 + 36q^2\lambda^2\frac{\rho}{\rho+1}})|t_2 - t_1| - (1 - 36q^2\frac{1}{\rho})\frac{1}{18q\lambda}\frac{6q\lambda(Q + Q\frac{1}{\rho})}{2 + 36q^2\lambda^2\frac{\rho}{\rho+1}} \\ &- Q(1 - 36q^2\lambda^2\frac{1}{\rho}) - 36q^2\lambda^2 Q - Q \\ &\geq \frac{1}{q'}|t_2 - t_1| - Q'\end{aligned}$$

where

$$q' := (1 - 36q^2\lambda^2\frac{1}{\rho})\frac{1}{18q}(\frac{1}{2 + 36q^2\lambda^2\frac{\rho}{\rho+1}})$$

and

$$Q' := (1 - 36q^2\lambda^2\frac{1}{\rho})\frac{1}{18q}\frac{6q(Q + Q\frac{1}{\rho})}{2 + 36q^2\lambda^2\frac{\rho}{\rho+1}} + Q(1 - 36q^2\lambda^2\frac{1}{\rho}) + 36q^2\lambda^2 Q + Q$$

Since we choose ρ sufficiently large, it implies that constants q' and Q' are possitive. The claim is proved.

Since for every $n > 0$ we have shown that α can be quasi-redirected to ζ at x_n via a combinatorial spiral path γ that is a (q', Q')–quasi-geodesic. As $\|x\|_n \to \infty$, it follows from Lemma 2.9 that α is (q', Q')–quasi-redirected to ζ. □

4.5. **Forward spiral path.** In Section 4.4 we constructed backward spiral paths that redirects any q-ray (Type I or Type II) to ζ. The Proof can be adapted to show that if α is a type I, then ζ can also be redirected analogously to α. In this section we address redirecting when α is of type II.

Definition 4.17. (Sub-exponential Excursion.) Let α be a ray of type II. Let $\alpha(t_i)$ be the first time α intersects F_{e_i} where F_{e_i} associated with v_i in the simplified itinerary of α. We say α has *sub-exponential excursion* with respect to the distance in T if

$$\lim_{i\to\infty} \frac{\log|t_i - t_{i-1}|}{i} = 0$$

Now we construct similar quasi-geodesic paths which we call *forward spiral paths*. Let γ be the geodesic ray in the Bass-Serre tree T associated to α. Let s_γ be the special ray in X associated to γ. Recall that $p_i = \mathfrak{f}_i^- \cap \mathfrak{f}_i^+$, $p_{i+1} = \mathfrak{f}_{i+1}^- \cap \mathfrak{f}_{i+1}^+$ where $\mathfrak{f}_i^+$ and $\mathfrak{f}_{i+1}^-$ are the two singular geodesics of the strip $\mathcal{S}_{e_i e_{i+1}}$. Also recall that $p_0 := \mathfrak{o}$ and the special path s_γ is the concatenation

$$[p_0, p_1] \cdot [p_1, p_2] \cdots$$

Let $\ell_i^+ \subset \mathfrak{f}_i^+$ be the geodesic ray in $\mathfrak{f}_i^+$ based at p_i and $\ell_{i+1}^- \subset \mathfrak{f}_{i+1}^-$ be the geodesic ray in $\mathfrak{f}_{i+1}^-$ based at p_{i+1} so that the projection point of p_{i+1} into $\mathfrak{f}_i^+$ will belong to ℓ_i^+ (see Figure 7).

Assume that the excursion of α is sub-exponential. Note that $d(\alpha(s_i), \alpha(t_{i-1}))$ is bigger than $d(p_i, p_{i+1})$. Pick a constant $0 < \rho_0 < 1/4$. Then there exists $C = C(\rho_0) > 0$ such that

$$w_i := d(p_i, p_{i+1}) \leq C(1 + \rho_0)^i$$

for every i. For every $r > C$ we define

$$\kappa_i = r(1 + \rho_0)^i$$

which is greater than w_i.

On each ℓ_i^+, choose z_i so that $d(z_i, p_i) = \kappa_i$.

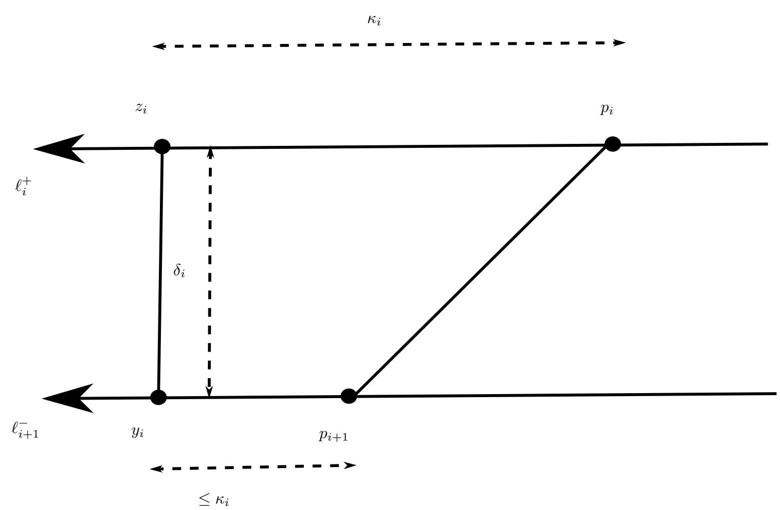


FIGURE 7. The figure illustrates how we choose geodesic rays ℓ_i^+ and ℓ_{i+1}^- on the strip. Our choice of constant $\kappa_i > w_i = d(p_i, p_{i+1})$ ensures that the projection point y_i of z_i into $\mathfrak{f}_{i+1}^-$ will lie in ℓ_{i+1}^- and $d(y_i, p_{i+1}) \leq \kappa_i$.

Let us denote the width of the strip $\mathcal{S}_{e_i e_{i+1}}$ by δ_i. Let y_i be the projection point of z_i into $\mathfrak{f}_{i+1}^-$. We note that since $w_i < \kappa_i$, it follows that $y_i \in \ell_{i+1}^-$, $d(y_i, z_i) = \delta_i$, and $d(y_i, p_{i+1}) < \kappa_i$.

Let $L_{r,k}$ be the concatenation

$$L_{r,k} := \zeta|_{[0,r]} \cdot [\zeta(r), z_1] \cdot [z_1, y_1] \cdot [y_1, z_2] \cdots [y_{k-1}, z_{k-1}]$$

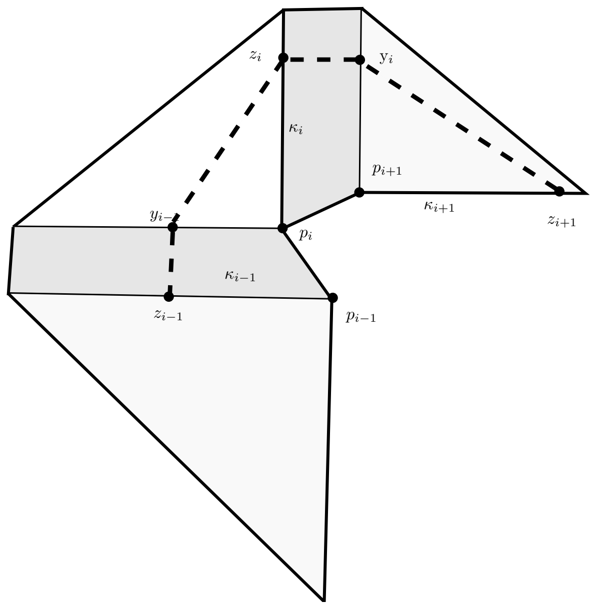


FIGURE 8. The figure illustrates a portion of $L_{r,k}$ which is a concatenation of dashed segments. The sum of all dashed segments is bounded above by an exponential function $(1+\rho_0)^k$ up to some multiplicative constant.

The *forward spiral path* $\mathcal{L}_{r,k}$ is the path obtained by attaching to z_{k-1} the geodesic ray in $\mathfrak{f}^+_{k-1}$ based at z_{k-1} which does not contain p_{k-1}.

We have

$$\sum_{i=1}^{k-1} w_i < r\sum_{i=1}^{k-1}(1+\rho_0)^i = r\frac{1+\rho_0}{\rho_0}\big((1+\rho_0)^{k-1}-1)\big)$$

and

$$\sum_{i=1}^{k} \kappa_i = \sum_{i=1}^{k} r(1+\rho_0)^i < r\frac{1+\rho_0}{\rho_0}\big((1+\rho_0)^{k}-1)\big)$$

By our construction, we have $d(z_i, y_i) = \delta_i \le w_i$ and $d(y_i, z_{i+1}) \le d(y_i, p_{i+1}) + d(p_{i+1}, z_{i+1}) \le \kappa_i + \kappa_{i+1}$. For $i < j$, let us denote the subpath of $L_{r,k}$ from y_{i-1} to z_j by $L_{r,k}|_{[y_{i-1},z_j]}$. We have

$$\begin{aligned}\text{Length}\big(L_{r,k}|_{[y_{i-1},z_j]}\big) &\le d({}_{y_{i-1}}, z_i) + d(z_i, y_i) + \cdots + d(z_{j-1}, y_{j-1}) + d(y_{j-1}, z_j)\\ &\le 2\sum_{m=i-1}^{j}\kappa_m + 2\sum_{m=i-1}^{j-1} w_m\\ &\le 4r\sum_{m=i-1}^{j}(1+\rho_0)^m \le \frac{4r}{\rho_0}(1+\rho_0)^{j+1}\end{aligned}$$

By [NY23, Proposition 3.8], the subpath $[p_i, p_{i+1}]\cdots[p_{j-1}, p_j]$ of the special ray s_γ is a (μ,μ)–quasi-geodesic. By Lemma 2.8, the concatenation $\sigma := [y_{i-1}, p_i]\,\cdot$

$[p_i, p_{i+1}] \cdots [p_{j-1}, p_j] \cdot [p_j, z_j]$ is a $(9\mu, 9\mu)$–quasi-geodesic. It implies that

$$d(y_{i-1}, z_j) \geq \frac{\text{Length}(\sigma)}{9\mu} \geq \frac{d(p_j, z_j)}{9\mu} \geq \frac{\kappa_j}{9\mu} = \frac{r(1+\rho_0)^j}{9\mu}$$

We thus can control the upper bound of the ratio:

$$\frac{\text{Length}\big(L_{r,k}|_{[y_{i-1}, z_j]}\big)}{d(y_{i-1}, z_j)} \leq \frac{36\mu}{\rho_0}(1+\rho_0)$$

Similar argument shows that there is an uniform constant $\Delta = \Delta(\mu, \rho_0)$ such that for any points x, y in $L_{r,k}$, we have

$$\frac{\text{Length}\big(L_{r,k}|_{[x,y]}\big)}{d(x,y)} \leq \Delta$$

In other words, $L_{r,k}$ is a (Δ, Δ)–quasi-geodesic. Applying Lemma 2.8(1), we have that $\mathcal{L}_{r,k}$ is a $(3\Delta, \Delta)$–quasi-geodesic. The forward spiral path $\mathcal{L}_{r,k}$ has the following property: Let t_i be the first time $\mathcal{L}_{r,k}$ visists the wall F_{e_i}. By our construction, we have

$$t_{k+1} - t_k = d(p_k, y_{k-1}) + d(p_k, z_k) + d(z_k, y_k) = \kappa_{k-1} + \kappa_k + \delta_k < 2r(1+\rho_0)^k + \delta_k$$

Routine computation yields to a constant $\rho = \rho(\rho_0)$ which tends to 0 when $\rho_0 \to 0$ such that

$$2r(1+\rho_0)^k < \rho \,\text{Length}(L_{r,k}) = \rho t_k$$

for k sufficiently large. We then have

$$t_{k+1} - t_k < \rho t_k + \delta_k$$

for sufficiently large k. We summary the above discussion in the next proposition.

Proposition 4.18. *Let α be a $\mathfrak{q}$–ray of Type II. Given any $\rho_0 > 0$, let $0 < \rho < \rho_0$. There exists a quasi-geodesic ray $\mathcal{L}$ with quasi-geodesic constants depending only on $\rho, \mathfrak{q}$ such that the following holds. If t_i is the first time $\mathcal{L}$ visits the wall F_{e_i} then*

$$t_{k+1} - t_k < \rho t_k + \delta_k$$

for sufficiently large k.

Proposition 4.19. *Let α be an arbitrary $\mathfrak{q}$–ray of Type II in X. If the excursion of α is not sub-exponential then $\alpha \sim \zeta$.*

Proof. Since α is not sub-exponential excursion, then there exists a constant $\rho_0 \in (0, 1/4)$ so that for every $r > 0$ then there exists $k \in \mathbb{Z}_+$ satisfying

$$t_k - t_{k-1} \geq r(1+\rho_0)^k$$

Let $k \in \mathbb{Z}_+$ be the first integer so that

$$\begin{cases} t_k - t_{k-1} & \geq r(1+\rho_0)^k \\ t_i - t_{i-1} & < r(1+\rho_0)^i \quad \forall 1 \leq i \leq k-1 \end{cases}$$

We define

$$\kappa_i := r\rho_0(1+\rho_0)^i \quad \text{for each} \quad 1 \leq i \leq k$$

We have

$$\kappa_k = r\rho_0(1+\rho_0)^k < \rho_0(t_k - t_{k-1}) < \frac{t_k - t_{k-1}}{4}$$

since $0 < \rho_0 < 1/4$, and hence it implies that $4\kappa_k < t_k - t_{k-1}$.

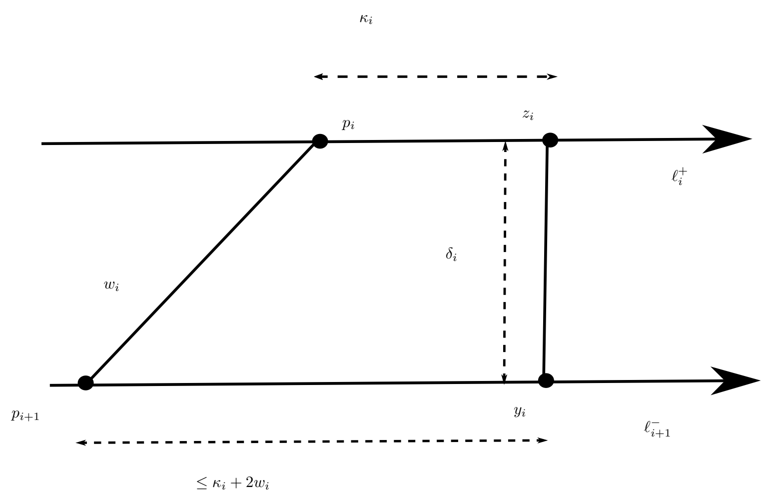


FIGURE 9.

Let $\ell_i^+ \subset \mathfrak{f}_i^+$ be the geodesic ray in L_i^+ based at p_i and $\ell_{i+1}^- \subset \mathfrak{f}_{i+1}^-$ be the geodesic ray in $\mathfrak{f}_{i+1}^-$ based at p_{i+1} so that the projection point of p_i into $\mathfrak{f}_{i+1}^-$ will belong to ℓ_{i+1}^- (see Figure 9).

Let $L_{r,k}$ be the concatenation

$$L_{r,k} := \zeta|_{[0,r]} \cdot [\zeta(r), z_1] \cdot [z_1, y_1] \cdot [y_1, z_2] \cdots [y_{k-1}, z_{k-1}]$$

Let $\mathcal{L}_{r,k}$ be the path obtained by attaching to z_{k-1} the geodesic ray in $\mathfrak{f}_{k-1}^+$ based at z_{k-1} which does not contain p_{k-1}.

Using similar arguments as forward spiral paths, we can verify that $\mathcal{L}_{r,k}$ is a $(3\nu, 3\nu)$–quasi-geodesic for some constant $\nu = \nu(\rho_0, \mathfrak{q})$.

As $4\kappa_k < t_k - t_{k-1}$, applying Lemma 4.10, $\mathcal{L}_{r,k}$ can be quasi-redirected to α at radius r, and hence ζ can be quasi-redirected to α at radius r because $\zeta|_r = \mathcal{L}_{r,k}|_r$. Since this is true for every $r > 0$, it follows that $\zeta \preceq \alpha$. By Proposition 4.16, we have $\alpha \preceq \zeta$. Therefore $\zeta \sim \alpha$. □

Lemma 4.20. *Let α be a $\mathfrak{q}$–ray of Type II. Then $[\alpha] \neq \boldsymbol{z}$ if and only if the excursion of α is sub-exponential.*

Proof. According to Proposition 4.19, $\alpha \sim \zeta$ if the excursion of α is not sub-exponential. Thus, to complete the proof, we only need to show that if the excursion of α is sub-exponential then ζ can not be quasi-redirected to α.

By way of contradiction, suppose that at every radius r, there is always a uniform quasi-geodesic γ that quasi-redirects ζ to α at the radius r.

Let T_k be the first time γ visits F_{e_k} and denote

$$\ell_k := d(p_k, \gamma(T_k))$$

Since γ is a $\mathfrak{q}$–ray, there exists a constant $\rho_0 = \rho_0(q, Q) > 0$ such that

$$T_0 = r \quad \text{and} \quad T_{k+1} - T_k \geq \rho_0 \ell_k \tag{9}$$

Another way to travel from $\mathfrak{o} = p_0$ to $\gamma(T_k)$ is to go along the special path $[p_0, p_1], [p_1, p_2], \ldots, [p_{k-1}, p_k]$ which is a (μ, μ)–quasi-geodesic (μ is mentioned in Definition 4.13) and then go up or down a distance of ℓ_k to reach $\gamma(T_k)$.

Again since γ is a $\mathfrak{q}$–ray we have that

$$\ell_k + \sum_{i=0}^{k-1} w_i \geq \rho_0 T_k \tag{10}$$

Define

$$\rho_1 = \rho_0^2/2$$

and pick an arbitrary $0 < \rho < \rho_1$.

Since the excursion of α is sub-exponential, it implies that there exists a constant $C = C(\rho)$ such that for every $i \geq 0$ then

$$d(\alpha(t_i), \alpha(t_{i-1})) \leq C(1+\rho)^i$$

and hence

$$\sum_{i=0}^{k} w_i \leq C \sum_{i=0}^{k} (1+\rho)^i \leq \frac{C}{\rho}(\rho+1)^k$$

as we know that $w_i \leq d(\alpha(t_i), \alpha(t_{i-1}))$.

Claim 3:

$$\forall r > 2C/(\rho\rho_0) \Longrightarrow T_{k+1} \geq r(1+\rho_1)^{k+1} \quad \text{and} \quad \ell_k \geq \frac{r\rho_0}{2}(1+\rho_1)^{k+1} \tag{11}$$

Indeed, we prove the above claim by induction. The base case is obvious, so we assume the claim is true for all $i \leq k$. We have

$$\begin{aligned} T_{k+1} &\geq T_k + \rho_0 \ell_k \\ &\geq r(1+\rho_1)^k + \frac{r\rho_0^2}{2}(1+\rho_1)^k \geq r(1+\rho_1)^k(1+\frac{\rho_0^2}{2}) \\ &\geq r(1+\rho_1)^{k+1} \end{aligned}$$

Using this and (10), we have

$$\begin{aligned} \ell_{k+1} &\geq \rho_0 T_{k+1} - \sum_{i=0}^{k} w_i \geq \rho_0 T_{k+1} - \frac{C}{\rho_0}(1+\rho_0)^{k+1} \\ &\geq r\rho_0(1+\rho_1)^{k+1} - \frac{C}{\rho_0}(1+\rho_1)^{k+1} \\ &= (1+\rho_1)^{k+1}(r\rho_0 - \frac{C}{\rho}) \geq \frac{r\rho_0}{2}(1+\rho_1)^{k+1} \end{aligned}$$

On the other hand, we have

$$\sum_{i=0}^{k} d(\alpha(t_i), \alpha(t_{i-1})) \leq \frac{C}{\rho}(1+\rho)^k < r(1+\rho_1)^k < \frac{C}{\rho}(1+\rho_1)^k < T_k$$

for r sufficiently large. In other words, γ arrives in F_{e_k} long after α has left F_{e_k} and the distance between γ and α goes to infinity. In particular, it is impossible for γ to eventually coincide with α.

In conclusion, we have shown that for every $\mathfrak{q} = (q, Q)$, there exists a sufficiently large constant $r > 0$ such that there is no $\mathfrak{q}$–ray γ with $\gamma|_r = \zeta|_r$ and γ is eventually equal to α. Therefore ζ can not be quasi-redirected to α. □

Proposition 4.21. *Let α be a $\mathfrak{q}$-ray that is of Type II and is sub-exponential. Let α_0 in X be a geodesic ray whose simplified itinerary is the sequence γ_α. Then $\alpha \sim \alpha_0$.*

Proof. Consider the geodesic segments $[\mathfrak{o}, p_i]$, by Arzela-Ascoli Theorem, the sequence $\{[\mathfrak{o}, p_i]\}$ has a limit that is a geodesic ray which we denote as α_0. By way of contradiction, suppose α is not sub-exponential. By Lemma 4.20, $\alpha \sim \mathbf{z}$ and for every $r > 0$ and every $C > 0$, there exists a bounded-constants quasi-geodesic that is a forward spiral path that redirects to α eventually in a wall. In particular, in a infinite sequence of walls, these forward spiral paths intersects F_{e_i} at a point whose distances to the associated p_i is greater than $C\|p_i\|$. Therefore, for each $C > 0$, there exists an infinite sequence of times where

$$d(p_k, \alpha) \leq \rho \|p_k\|$$

Therefore for s large enough, the segment $[x, y]$ that realizes the distance between $\alpha(s, \infty)$ and $\alpha_0|_{p_k}$, where $x \in \alpha_0|_{p_k}$ has

$$\|x\| \geq \frac{9}{10} \|p_k\|.$$

Therefore, by Surgery Lemma 2.8 (3), there exists a $(4q, 3Q)$-quasi-geodesic ray that redirects $\alpha_0|_x$ to α where $\|x\| \to \infty$ as $k \to \infty$. Thus $\alpha_0 \preceq \alpha$. On the other hand, by Surgery Lemma 2.8(2), we obtain that $\alpha \preceq \alpha_0$ with redirecting constant $(9q, Q)$. □

Proposition 4.22. *Let α and α' be two $\mathfrak{q}$–rays of Type II in X with different simplified itineraries and with sub-exponential excursions. Then α can not be quasi-redirected to α' and vice versa.*

Proof. By way of contradiction, suppose that $[\alpha] = [\alpha']$. In particular, we have $\alpha' \preceq \alpha$. We claim that $\zeta \preceq \alpha$. Indeed, let $r > 0$ be an arbitrary constant. Let γ be an arbitrary forward spiral path given by Proposition 4.18 such that $\gamma|_r = \zeta|_r$.

Let t_k be the first time $\gamma(t_k) \in F_{e_k}$ and denote

$$\ell_k := d(\gamma(t_k), p_k)$$

Now choose $R \gg \ell_k$ we consider a quasi-geodesic β' quasi-redirecting α' to α at radius R. Such a β' exists since $\alpha' \preceq \alpha$. Then β' arrives at and leaves F_{e_k} much later than γ. Hence, by Lemma 4.10, we can redirect γ to β', that is, construct a quasi-geodesic ray γ' where $\gamma[0, t_k] = \gamma'[0, t_k]$ and γ' is eventually equal to β'. Since β' is eventually equal to α it implies that γ' quasi-redirects ζ to α at radius r. This can be done for every r with uniform constants. Hence $\zeta \preceq \alpha$. This would contradict to Proposition 4.19(2). □

Now we have enough ingredients to claim the existence of the QR-boundary of X.

Theorem 4.23 (Theorem A)**.** *The quasi-redirecting boundary ∂X exists and it is non-Hausdorff.*

Proof. By [QR24, Lemma 2.3], all finitely generated groups satisfy QR Assumption 0. Here we check QR-Assumptions 1 and 2. That is, for every $\mathbf{a} \in P(X)$, there is a geodesic representative, and there is a function

$$f_{\mathbf{a}} : [1, \infty) \times [0, \infty) \to [1, \infty) \times [0, \infty),$$

any $\mathfrak{q}$–ray $\alpha \in \mathbf{a}$ can be $f_{\mathbf{a}}(\mathfrak{q})$–redirected to the representative of $\mathbf{a}$. If α is of Type I or of Type II but it does not have sub-exponential excursion, then by Proposition 4.16 $\alpha \preceq \zeta$ with constants $\mathfrak{q}'(\mathfrak{q})$. If otherwise, Proposition 4.19 show that $\zeta \preceq \alpha$ with constant $(3\nu(\rho_0, \mathfrak{q}), 3\nu(\rho_0, \mathfrak{q}))$. Thus ζ is a suitable geodesic representative of $[\alpha]$ and $f_{[\alpha]} = \mathfrak{q}'(\mathfrak{q})$. Otherwise, α is of type II and sub-exponential, then Proposition 4.21 shows that α_0 is a geodesic representative of $[\alpha]$ and the redirecting function is $f_{[\alpha]} = (9q, Q)$. Thus X satisfies all three QR-Assumption $0, 1, 2$, and ∂X is well-defined and QI-invariant.

To see that ∂X is not Hausdorff, we first argue that $\preceq$ on ∂G is not symmetric, then by [QR24, Theorem 7.3], ∂G is not Hausdorff. To see this, let α be a $\mathfrak{q}$–ray α with sub-exponential excursion. By Proposition 4.16, we have $\alpha \preceq \zeta$ and by Proposition 4.19, $\zeta \not\preceq \alpha$. Therefore, QR relation $\preceq$ on ∂G is not symmetric. □

5. QR boundaries of 3-manifold groups

It is important to note that Theorem A applies to non-positively curved graph manifolds. However, it is worth mentioning that there exist many graph manifolds such that their fundamental groups are not CAT(0) groups [KL96]. As a result, Theorem A cannot be applied directly to graph manifolds.

Fortunately, it has been demonstrated by Kapovich-Leeb [KL98] that graph manifolds are Hadamard spaces in the large-scale sense, meaning that they are quasi-isometric to CAT(0) graph manifolds. Hence, as the quasi-redirecting boundary is a quasi-isometric invariant, Theorem A can be applied to graph manifolds without the CAT(0) assumption.

Proposition 5.1. *Let M be a graph manifold and let $G = \pi_1(M)$. Then the following properties hold.*

1. *G satisfies all three QR-Assumptions. Thus ∂G is a quasi-invariant topological space.*
2. *The boundary ∂G is non-Hausdorff.*

Proof. We equip M with a Riemannian metric. By [KL98, Theorem 1.1], there exists a nonpositively curved graph manifold N and a bilipschitz homeomorphism $\phi \colon \widetilde{M} \to \widetilde{N}$ such that ϕ preserves their geometric decompositions. Here the metrics on $\widetilde{M}$ and $\widetilde{N}$ are the induced metrics from M and N respectively. Since $\pi_1(N)$ is an admissible group and the action $\pi_1(N) \curvearrowright \widetilde{N}$ is a geometric action, we apply Theorem A to $\pi_1(N) \curvearrowright \widetilde{N}$ to obtain the existence of the quasi-redirecting boundary of $\pi_1(N)$. Since $\pi_1(M)$ and $\pi_1(N)$ are quasi-isometric and QR-boundary is a quasi-isometric invariant, it implies the existence of the quasi-redirecting boundary of $\pi_1(M)$. □

Proposition 5.2. *Let M be a mixed 3-manifold. Then the quasi-redirecting boundary of $\pi_1(M)$ exists.*

Proof. Let $M_1, \ldots, M_k$ be the maximal graph manifold components and Seifert fibered pieces of the torus decomposition of M. Let $S_1, \ldots, S_\ell$ be the tori in the boundary of M that bound a hyperbolic piece, and let $T_1, \ldots, T_m$ be the tori in the torus decomposition of M that separate two hyperbolic components. According to [Dah03] (see also [BW13]), $\pi_1(M)$ is hyperbolic relative to

$$\mathbb{P} = \{\pi_1(M_p)\}_{p=1}^{k} \cup \{\pi_1(S_q)\}_{q=1}^{\ell} \cup \{\pi_1(T_r)\}_{r=1}^{m}.$$

We note that the quasi-redirecting boundaries of $\pi_1(S_q)$, $\pi_1(T_r)$ exist since they are isomorphic to $\mathbb{Z}^2$. Proposition 5.1 implies the existence of the quasi-redirecting boundary of $\pi_1(M_p)$. Thus, we apply Theorem C to conclude that the quasi-redirecting boundary of $\pi_1(M)$ exists. □

Proof of Theorem D. The proof is a combination of Proposition 5.1 and Proposition 5.2. □

6. QR boundary of certain right-angled Coxeter groups

Given a graph Γ, define Γ^4 as the graph whose vertices are induced 4–cycles of Γ. Two vertices in Γ^4 are adjacent if and only if the corresponding induced 4-cycles in Γ have two nonadjacent vertices in common.

Definition 6.1 (Constructed from squares)**.** A graph Γ is $\mathcal{CFS}$ if Γ is the join $\Omega * K$ where K is a (possibly empty) clique and Ω is a non-empty subgraph such that Ω^4 has a connected component T such that every vertex of Ω is contained in a 4–cycle that is a vertex of T. If Γ is $\mathcal{CFS}$, then we will say that the right-angled Coxeter group W_Γ is $\mathcal{CFS}$.

Standing Assumptions. The planar flag complex $\Delta \subset \mathbb{S}^2$:

(1) is connected with no separating vertices and no separating edges (W_Δ is one-ended);
(2) contains at least one induced 4-cycle (W_Δ is not hyperbolic);
(3) is not a 4-cycle and not a cone of a 4-cycle (G_Δ is not virtually $\mathbb{Z}^2$).

Proposition 6.2. *Let $\Delta \subset \mathbb{S}^2$ be a flag complex satisfying Standing Assumptions. Assume that either $\Delta = \mathbb{S}^2$ or the boundary of each region in $\mathbb{S}^2 - \Delta$ is a 4–cycle. Then the quasi-redirecting boundary of the right-angled Coxeter groups W_Γ exists.*

Proof. It is shown in [NT19, Theorem 1.1] and [HNT19] that there are mutually exclusive cases as bellow:

(1): If Δ is a suspension of some n-cycle ($n \geq 4$) or some broken line (i.e a finite disjoint union of vertices and finite trees with vertex degrees 1 or 2), then G contains a finite index subgroup G' which is isomorphic to $\pi_1(M)$ with M is a Seifert manifold. In this case, there is a finite cover $M' \to M$ such that $M' = F \times S^1$ where F is a hyperbolic surface with a nonempty boundary, and thus $\partial(\pi_1(M'))$ consists only one point by Proposition 2.6. Since G is quasi-isometric to $\pi_1(M')$, it follows from Theorem 2.1 that ∂G consists only one point.

(2): If the 1-skeleton of Δ is $\mathcal{CFS}$ and does not satisfy (1) then G contains a finite index subgroup G' which is isomorphic to $\pi_1(M)$ with M is a graph manifold. If the 1-skeleton of Δ contains a separating induced 4-cycle and is not $\mathcal{CFS}$, then M is a mixed manifold. In these two cases, it follows from Theorem D that the quasi-redirecting boundary of $\pi_1(M)$ exists, and so does G.

(3): If the 1-skeleton of Δ has no separating induced 4-cycle and is not $\mathcal{CFS}$, then G contains a finite index subgroup G' which is isomorphic to $\pi_1(M)$ with M is a hyperbolic 3-manifold with tori boundary. In this case, $\pi_1(M)$ is hyperbolic relative to a finite collection of $\mathbb{Z}^2$ which have trivial QR-boundaries, and Theorem C implies the existence of the quasi-redirecting boundary of $\pi_1(M)$, and so does G. □

Theorem 6.3. *Let Γ be a graph whose flag complex Δ is planar. Then the quasi-redirecting boundary of the right-angled Coxeter group W_Γ exists.*

Proof. According to [HNT19, Theorem 1.2], there is a collection $\mathbb{J}$ of $\mathcal{CFS}$ subgraphs of Γ such that the right-angled Coxeter group G_Γ is relatively hyperbolic with respect to the collection $\mathbb{P} = \{ G_J \mid J \in \mathbb{J} \}$. By Proposition 6.2, the quasi-redirecting of each peripheral subgroup $G_J \in \mathbb{P}$. We now apply Theorem C to obtain the conclusion. □

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