# GEOMETRICITY OF OUTER AUTOMORPHISMS IS ALGORITHMIC

EDGAR A. BERING IV, YULAN QING, AND DERRICK R. WIGGLESWORTH

Abstract. An outer automorphism of a free group is geometric if it can be represented by a homeomorphism of a compact surface. Bestvina and Handel gave an algorithmic characterization of geometricity for irreducible automorphisms, using relative train tracks. Using advances in train-track theory, in conjunction with the Guirardel core of tree actions and Nielsen-Thurston theory for surfaces, we give an algorithm that can decide if a general outer automorphism is geometric. The algorithm is constructive and produces a realizing surface homeomorphism if one exists.

## 1. Introduction

A free group $\mathbb{F}$ of rank $n$ can be realized as the fundamental group of a surface $\Sigma$ with boundary, and homeomorphisms of $\Sigma$ induce outer automorphisms on $\mathbb{F}_n$. Any outer automorphism $\phi \in \mathrm{Out}(\mathbb{F})$ that arises from this construction is *geometric* (see Section 2.4). We describe an algorithm to decide whether or not an outer automorphism is geometric. Our algorithm is constructive, in the sense that if a realization exists there is a procedure to compute the surfaces together with the homeomorphism.

**Theorem A.** *Let $\phi \in \mathrm{Out}(\mathbb{F})$ be an outer automorphism. There exists an algorithm that decides if $\phi$ is geometric and a procedure to compute a realization of $\phi$ if one exists.*

**Geometric free-by-cyclic groups.** A finitely-generated, free-by-infinite-cyclic group is a group $G$ which admits an epimorphism onto $\mathbb{Z}$ such that the kernel is a finitely generated non-cyclic free group. For brevity we refer to such groups as free-by-cyclic groups. If $G$ is a free-by-cyclic group, then the epimorphism $G \to \mathbb{Z}$ splits, and we can thus write $G$ as a semidirect product

$$\mathbb{F} \rtimes_\phi \mathbb{Z}$$

where $\mathbb{F}$ is a non-cyclic free group and $\phi \colon \mathbb{F} \to \mathbb{F}$ is an isomorphism. If $\phi$ is geometric and is realized by a surface homeomorphism $h \colon \Sigma \to \Sigma$, then $G$ is the fundamental group of the mapping torus of $h$, a 3–manifold with non-trivial torus boundary. Not every free-by-cyclic group is the fundamental group of a 3-manifold. However, it is a consequence of Stallings' fibering theorem [18, Chapter 11] that if a free-by-cyclic group $G$ is the fundamental group of a 3-manifold then it is a mapping torus of some surface homeomorphism realizing the monodromy. As a consequence, our algorithm can detect 3-manifold groups among all free-by-cyclic groups.

**Corollary B.** *There exists an algorithm to decide whether a given free-by-cyclic group is a 3-manifold group.*

**History.** An (outer) automorphism of $\mathbb{F}$ is *fully irreducible* if no conjugacy class of a nontrivial proper free factor is periodic. An (outer) automorphism of $\mathbb{F}$ is *irreducible* if it fixes no conjugacy class of a nontrivial proper free factor. Bestvina and Handel characterized irreducible automorphisms that are geometric [4]. Given an outer automorphism $\phi$, Bestvina and Handel's characterization finds a periodic conjugacy class $[c]$. This conjugacy class represents a boundary curve of the surface, and a surface where $\phi$ can be realized as a homeomorphism can be obtained from a suitable graph by attaching an annulus along a cycle representing $[c]$. This idea is one of the building blocks of our algorithm.

Using the core of tree actions, Guirardel gave a limiting characterization of a geometric fully irreducible. The projectivization of the space of actions of $\mathbb{F}$ on very-small $\mathbb{R}$-trees is known as the compactification of Culler-Vogtmann outer space, $\overline{CV}$ [3, 7, 9]. For a fully irreducible automorphism $\phi$ acting on $\overline{CV}$ there are unique forward and backward limit points, with representatives $T^+$ and $T^-$. Guirardel proved that if both $T^+$ and $T^-$ are dual to measured foliations on surfaces, then $\phi$ is realized by a pseudo-Anosov homeomorphism of a punctured surface [15, Corollary 9.3]. Moreover, the universal cover of the realizing surface, equipped with the singular Euclidean metric coming from the transverse foliations for the pseudo-Anosov is the core of the two tree actions. The core (though not of limit points) also plays a role in our algorithm.

There has been relatively little progress in understanding the geometricity of elements of $\mathrm{Out}(\mathbb{F})$ beyond the irreducible setting. Ye treated the case of polynomially growing outer automorphisms with an analysis of generalized Dehn twists [27]. For any given polynomially growing automorphism $\phi \in \mathrm{Out}(\mathbb{F})$, Ye provides an explicit power $t(n)$ and an algorithm to decide whether $\phi^{t(n)}$ is geometric or not.

**Key tools and technical advances.** Our algorithm proceeds in two stages. First, we pass to what is known as a *rotationless power* to avoid the complications of finite order behavior. Algorithm 1 uses Feighn and Handel's algorithmic CT representatives to decide if a rotationless outer automorphism is geometric. Once this is done, Algorithm 2 uses the Guirardel core and Krstic, Lustig, and Vogtmann's equivariant Whitehead algorithm to decide if a given root of a geometric rotationless outer automorphism is again geometric. To combine these tools we requires certain extensions of ideas in the literature that we highlight here, in hopes they have broader application.

*Surface data of CT representatives.* CTs are graph maps representatives of outer automorphisms designed by Feighn and Handel to satisfy the properties that have proven most useful for investigating elements of $\mathrm{Out}(\mathbb{F})$ [11, 12]. Moreover, given an outer automorphism, a CT representative can be produced algorithmically [13]. Algorithm 1 uses such a representative as a starting point. A wealth of data can be derived from a CT representative, including surfaces and pseudo-Anosov homeomorphisms that represent surface-like exponentially growing behavior of the outer automorphim [17, Chapter I.2]. We add to this literature with Corollary 3.11, which roughly says that all of the exponentially growing surface data of a CT is a property of the outer automorphism alone and does not depend on the choice of representative. This extends work of Handel and Mosher, as detailed in Section 3.

Using this surface data, the final step of Algorithm 1 is to verify that the candidate boundary curves can all be glued together into a surface. This is accomplished using the Whitehead algorithm.

*Cores and spines.* Guirardel generalized the idea of intersection of curves on surfaces to the *intersection complex* of a pair of minimal group actions on a tree [15]. In the context of $\mathrm{Out}(\mathbb{F})$, suppose $\phi$ is a geometric outer automorphism, and $T$ is the universal cover of a spine for a surface $\Sigma$ where $\phi$ is realized as a homeomorphism. Twisting the action by $\phi$ gives two $\mathbb{F}$ actions on trees, and Guirardel proved that the intersection core $\mathrm{Core}(T, T\phi)$ embeds in the universal cover $\Sigma$. We extend this characterization to what we call *partially geometric* outer automorphisms. An outer automorphism is partially geometric if it can be realized as a homotopy equivalence of a surface $h\colon \Sigma \to \Sigma$ equipped with a subsurface $Q \subseteq \Sigma$. Subject to some technical conditions, we prove a relative version of Guirardel's result in Proposition 6.5. Roughly, given a spine $K$ for $Q$, where $K$ is a subgraph of a spine for $\Sigma$, the portion of the intersection core $\mathrm{Core}(T, T\phi)$ projecting to $K$ embeds in some copy of the universal cover of $Q$ inside the universal cover of $\Sigma$. That is, this "surface detection" is a local property.

Behrstock, Bestvina, and Clay give an algorithm for computing a fundamental domain for $\mathrm{Core}(T, T\phi)$ for any outer automorphism $\phi$, so our local condition is algorithmic [1]. This is used by our general case algorithm, Algorithm 2, to verify that an outer automorphism with geometric rotationless power is compatible with the surface data of that rotationless power. As in the rotationless case, the final verification is completed with the Whitehead algorithm. However, due to finite-order behavior we require the equivariant Whitehead algorithm of Krstic, Lustig, and Vogtmann [19].

**Organization of the paper.** This paper draws on a breadth of $\mathrm{Out}(\mathbb{F})$ theory; we endeavor to recall all relevant definitions with a common notation. This is done in Section 2. The first part of the paper is devoted to developing the necessary tools for the rotationless case: Section 3 introduces the geometric models of EG strata and derives new invariance results; these results are used in Section 4 to give an algorithm for handling the rotationless case. The second part of the paper develops the notion of partially geometric outer automorphisms and connects detecting this notion to Guirardel's core (Sections 5 and 6). The general algorithm is given in Section 8 after developing one more tool for treating some finite order behavior in Section 7. Illustrative examples of some of the varied behavior of different cases in the algorithm are provided in Section 9.

**Acknowledgements.** The authors are grateful to the Fields Institute for hospitality during the 2018 thematic program on "Teichmüller Theory and its Connections to Geometry, Topology and Dynamics", where this work began. E. A. Bering and Y. Qing thank the American Institute of Mathematics for hospitality where this work was completed. E. A. Bering was also partially supported Azraeli foundation.

## 2. Preliminaries

Let $\mathbb{F}$ denote the free group of rank $\mathfrak{r} \geq 3$. For an element or subgroup of $\mathbb{F}$, we use $[\cdot]$ to denote its conjugacy class. Let $\mathrm{Out}(\mathbb{F}) = \mathrm{Aut}(\mathbb{F})/\mathrm{Inn}(\mathbb{F})$ denote the group of outer automorphisms of $\mathbb{F}$.

2.1. **Graphs, paths, and splittings.** Fix an identification $\mathbb{F} = \pi_1(\mathfrak{R}, *)$ for the $\mathfrak{r}$-petaled rose $\mathfrak{R}$. A *marked graph* $G$ is a graph where each vertex has degree at least three and a homotopy equivalence $m : \mathfrak{R} \to G$ called a *marking*, which identifies $\mathbb{F}$ with $\pi_1(G, m(*))$. The outer automorphism group of $\mathbb{F}$ acts on the set of marked graphs on the right by twisting the marking: $(G, m) \mapsto (G, m \circ \phi)$. A homotopy equivalence $f : G \to G$ determines an outer automorphism $\phi \in \mathrm{Out}(\mathbb{F})$ by $\phi = [\bar{m} \circ f \circ m]$ where $\bar{m}$ is a homotopy inverse to $m$. If $f$ sends vertices to vertices and is an immersion on each edge, then we say $f$ is a *topological representative* of $\phi$.

A *path* in a marked graph is an isometric immersion $\sigma : I \to G$ of an interval or a constant map, the latter is called a trivial path. Any map $\sigma : I \to G$ is homotopic relative to endpoints to a unique path $[\sigma]$, called its *tightening.* A *circuit* is an immersion $\sigma : S^1 \to G$, and similarly any map $\sigma : S^1 \to G$ has a tightening. Any path or circuit has a decomposition into edges $E_1 \ldots E_{\ell(\sigma)}$ where $\ell(\sigma)$ is the *length* of this decomposition, and each $E_i$ is an isometry to a given edge. For any path or circuit $\overline{\sigma}$ denotes $\sigma$ with reversed orientation. A finite graph is a *core graph* if every edge is contained in some circuit, and every finite graph deformation retracts onto a unique core subgraph.

2.2. **Lines and laminations.** For a free group $\mathbb{F}$ the boundary pairs

$$\tilde{\mathcal{B}}(\mathbb{F}) = (\partial\mathbb{F} \times \partial\mathbb{F} \setminus \Delta)/(\mathbb{Z}/2\mathbb{Z})$$

is the set of unordered pairs of distinct boundary points of $\partial\mathbb{F}$, given the topology induced by the usual topology on $\partial\mathbb{F}$. The space of *abstract lines* in $\mathbb{F}$ is the quotient $\mathcal{B}(\mathbb{F}) = \tilde{\mathcal{B}}(\mathbb{F})/\mathbb{F}$ of boundary pairs by the action of $\mathbb{F}$, equipped with the quotient topology. For a line $\ell \in \mathcal{B}(\mathbb{F})$ a *lift* is any point $\tilde{\ell} \in \tilde{\mathcal{B}}(\mathbb{F})$ that projects to $\ell$. Every conjugacy class $[w] \in \mathbb{F}$ determines a well-defined *axis* characterized as the image of a boundary pair fixed by a representative $w$. For a finite rank subgroup $K \leq \mathbb{F}$ there is a natural inclusion $\tilde{B}(K) \subseteq \tilde{B}(\mathbb{F})$ which induces an inclusion $\mathcal{B}(K) \subseteq \mathcal{B}(\mathbb{F})$. The image of the latter map depends only on the conjugacy class $[K]$.

A closed subset $\Lambda \subseteq \mathcal{B}(\mathbb{F})$ is called a *lamination* of $\mathbb{F}$. A line $\ell \in \Lambda$ is a *leaf* of the lamination and $\ell$ is a *generic leaf* if the closure of $\ell$ equals $\Lambda$. A lamination *fills* a subgroup $K \leq \mathbb{F}$ if it is not carried by any proper free factor system of $K$. It is a consequence of the Shenitzer and Swarup theorems on cyclic splittings of free groups that if a lamination is carried by the vertex group of a cyclic splitting then it is not filling [24, 25].

Associated to each $\phi \in \mathrm{Out}(\mathbb{F})$ is a finite $\phi$-invariant finite set of laminations $\mathcal{L}(\phi)$, called the set of *attracting laminations*, and a bijection $\mathcal{L}(\phi) \to \mathcal{L}(\phi^{-1})$, a pair of laminations $\Lambda^+, \Lambda^-$ related by this bijection are a *dual lamination pair* for $\phi$ and the set of dual lamination pairs is denoted $\mathcal{L}^{\pm}(\phi)$.

Given a finite graph $G$ the *space of lines* in $G$, denoted $\mathcal{B}(G)$ is the set of all isometric immersions $\ell \colon \mathbb{R} \to G$. The space of lines is topologized by the *weak topology* which has a basic open set $V(G, \alpha)$ for each path $\alpha$ consisting of lines that have $\alpha$ as a subpath. The marking $m$ of a marked graph identifies $\mathcal{B}(G)$ and $\mathcal{B}(\mathbb{F})$ via homeomorphism, induced by the identification of $\partial\mathbb{F}$ with the ends of $\tilde{G}$. An outer automorphism $\phi$ induces a self-homeomorphism

$$\phi_{\#} \colon \mathcal{B}(\mathbb{F}) \to \mathcal{B}(\mathbb{F})$$

and if $f$ is a topological realization of $\phi$ the composition

$$\mathcal{B}(\mathbb{F}) \cong \mathcal{B}(G) \xrightarrow{f_\#} \mathcal{B}(G) \cong \mathcal{B}(\mathbb{F})$$

is equal to $\phi_\#$. For a marked graph $G$, we say a line or lamination in $\mathcal{B}(G)$ *realizes* the corresponding line or lamination in $\mathcal{B}(\mathbb{F})$.

2.3. **Free factor systems and supports.** A *free factor system* of $\mathbb{F}$ is a finite collection of proper free factors of $\mathbb{F}$ such that there exists a free factor decomposition

$$\mathbb{F} = A_1 * \cdots * A_k * B.$$

A free factor system $\mathcal{F}$ is the collection

$$\mathcal{F} = \{[A_1], \ldots, [A_k]\}, \text{ where } k \geq 0.$$

We say that a free factor system *carries* a conjugacy class $[K]$ if there is some $[A] \in \mathcal{F}$ such that $K \leq A$. Free factor systems are partially ordered by extending the carrying relationship $\mathcal{F}_1 \sqsubset \mathcal{F}_2$ if $\mathcal{F}_2$ carries each $[A] \in \mathcal{F}_1$.

A subgroup $K$ *carries* a set of lines $W$ if $W \subset \mathcal{B}(K)$, and a free factor system $\mathcal{F}$ *carries* $W$ if each element of $W$ is carried by some $[A] \in \mathcal{F}$. For a set $\mathcal{C}$ of conjugacy classes of subgroups, conjugacy classes of elements and lines we define the *free factor support* $\mathcal{F}_{supp}(\mathcal{C})$ is the $\sqsubset$-minimal free factor system carrying each element of $\mathcal{C}$. Such a support exists and is unique [17, Fact I.1.10].

2.4. Out($\mathbb{F}$) **and surface homeomorphisms.** When $\Sigma$ is a compact surface the mapping class group Map($\Sigma$) is the set of isotopy classes of homeomorphisms of $\Sigma$. We explicitly include homeomorphisms that permute the boundary and orientation-reversing homeomorphisms, in order to fully capture the behavior exhibited by geometric outer automorphisms.

Similar to a marked graph, a marked surface is a compact surface $\Sigma$ and a homotopy equivalence $m\colon \mathfrak{R} \to \Sigma$ called a *marking*, identifying $\mathbb{F}$ and $\pi_1(\Sigma, m(*))$. As with graphs we will suppress the marking unless necessary.

For algorithmic purposes, for each rank $n$ we fix a finite list $S_{n,1}, \ldots, S_{n,k}$ of *standard surfaces* such that $\mathfrak{R} \subset S_{n,i}$ and $S_{n,i}$ deformation retracts onto $\mathfrak{R}$; one for each homeomorphism type of surface with fundamental group $\mathbb{F}_n$ (including non-orientable surfaces). Observe that if $(\Sigma, m)$ is a marked surface homeomorphic to some $S_{n,i}$ then there is an outer automorphism $\phi$ such that $(S_{n,i}, \phi)$ is marked-homeomorphic to $(\Sigma, m)$.

An outer automorphism $\phi$ is *geometric* if there exists a marked surface $\Sigma$ and a homeomorphism $g : \Sigma \to \Sigma$ with $g_*$ a representative of $\phi$. It is important to note $\Sigma$ need not be orientable and $g$ may not restrict to the identity on $\partial\Sigma$. In addition to this standard notion, our algorithm will need a partial notion in intermediate steps. A connected subsurface $Q \subseteq \Sigma$ is *essential* if it has infinite fundamental group and is $\pi_1$ injective. A general subsurface $Q \subseteq \Sigma$ is essential if each connected component is essential and no component of the complement $\Sigma \setminus Q$ is an annulus.

**Definition 2.1.** An outer automorphism $\phi$ is *partially geometric* on a marked surface $\Sigma$ with respect to a (possibly disconnected) essential subsurface $Q \subset \Sigma$ if $\phi$ is realized by a homotopy equivalence $g\colon \Sigma \to \Sigma$ such that:

- The decomposition of $\Sigma$ into $Q$ and $\Sigma \setminus Q$ is $g$ invariant,
- $g$ restricted to $Q$ is a homeomorphism.

Any homotopy equivalence with these properties is a *geometric witness* for $\phi$.

*Remark* 2.2. Note that the notion of a partially geometric outer automorphism is stronger than that of an outer automorphism having geometric strata, because the complementary subsurface is preserved up to homotopy.

As we are working with un-oriented compact, connected surfaces and not requiring that our homeomorphisms restrict to the identity on the boundary, we quote the following two classical surface theorems in full with references to the specific formulations used.

**Theorem 2.3** (Dehn-Nielsen-Baer Theorem)**.** *Let $f\colon \Sigma \to \Sigma'$ be a homotopy equivalence between compact, connected surfaces with $\chi(\Sigma) = \chi(\Sigma') < 0$. Assume that $f$ restricts to a homeomorphism $\partial\Sigma = \partial\Sigma'$. Then $f$ is homotopic (relative to $\partial\Sigma$) to a homeomorphism $g\colon \Sigma \simeq \Sigma'$.*

*Proof.* Fujiwara [14, §3] records the non-orientable case of Maclachlan and Harvey's generalization of the Dehn-Nielsen-Baer theorem for so-called type-preserving outer automorphisms of a Fuchsian group [21, Theorem 1]. In the setting of homotopy equivalences of compact surfaces with negative Euler characteristic, type-preserving reduces to preserving the set of boundary conjugacy classes. Thus, the hypothesis on $f$ is exactly that it induces a type-preserving isomorphism of $\pi_1(\Sigma) \to \pi_1(\Sigma')$. In turn this implies the two surfaces are homeomorphic and $f$ is homotopic to a homeomorphism by Fujiwara's formulation of the Dehn-Nielsen-Baer theorem. □

In this article we will need to apply the Dehn-Nielsen-Baer theorem to restrictions of maps to disconnected subsurfaces. To do so we need the following standard result.

**Lemma 2.4.** *Suppose $Q, R$ are homeomorphic connected subsurfaces of a compact surface $\Sigma$. Let $\phi\colon \Sigma \to \Sigma$ be a homotopy equivalence preserving the decomposition of $\Sigma$ into $\Sigma \setminus (Q \cup R)$ and $Q \cup R$, and that $\phi(Q) = R$ . For any homeomorphism $h\colon R \to Q$, the composition $h \circ \phi\colon Q \to Q$ is homotopic to a homeomorphism if and only if the restriction $\phi\colon Q \to R$ is homotopic to a homeomorphism.*

*Proof.* One direction is clear. Suppose that $h \circ \phi\colon Q \to Q$ is homotopic to a homeomorphism and let $H$ be the homotopy. The composition $h^{-1}H$ is the desired homotopy. □

Given a surface $\Sigma$, let $\mathrm{Map}(\Sigma)$ denote the equivalence classes of orientation-preserving homeomorphisms on $\Sigma$, up to isotopy. Let $[g] \in \mathrm{Map}(\Sigma)$ be a mapping class represented by a homeomorphism $g$. A *reducing system* for $g$ is a collection of disjoint, simple closed curves $\mathcal{C}$ such that, $g(\mathcal{C}) = \mathcal{C}$. A mapping class is *reducible* if it has a representative with a reducing system and *irreducible* otherwise. The reduction of $[g]$ along $\mathcal{C}$ is the image of $[g]$ under the natural homomorphism $\mathrm{Map}(\Sigma) \to \mathrm{Map}(\Sigma_{\mathcal{C}})$, where $\Sigma_{\mathcal{C}}$ is the complement of a system of disjoint open neighborhoods of $\mathcal{C}$. The *canonical reduction system* for $g$ is the intersection of all inclusionwise maximal reduction systems. If $\mathcal{C}$ is the canonical reduction system, $\mathcal{C}$ has a power such that $\bar{g}$ fixes each component and each restriction is finite-order or irreducible.

**Theorem 2.5** (Thurston Normal Form [10, 26])**.** *If $[g] \in \mathrm{Map}(\Sigma)$ is a mapping class of a compact, connected surface $\Sigma$, then either $[g]$ is irreducible or there is a representative $g$ and a canonical reduction system $\mathcal{C}$ fixed by $g$.*

It is a standard consequence of this normal form that after passing to a sufficient power any mapping class can be factored as a product of Dehn twists about the reducing curves and the images of irreducible mapping classes on the complement under inclusion; see 4.2 for the precise formulation.

2.5. **Automorphisms and lifts.** In this section we introduce all the notations and facts needed for actions and dynamics on $\mathbb{F}$ and $\partial\mathbb{F}$. As in the previous sections $\mathfrak{R}$ is a rose with base-point $(*)$ and $G$ is a marked rose. Let $\phi \in \mathrm{Out}(\mathbb{F})$ and let $f\colon G \to G$ be a topological representative of $\phi$. Let $b = m(*) \in G$ be the base-point in $G$. The set of paths $\sigma$ from $b$ to $f(b)$ determines a bijection between the automorphism lifts of $\phi$ and lifts of $f$ to $\widetilde{G}$. This bijection can be seen without reference to a base-point as we now detail.

Let $\Phi$ be an automorphism of $\mathbb{F}$ that represents $\phi$. Each $u \in \mathbb{F}$ acts on $\widetilde{G}$ by the covering transformation $\tau_u$. Additionally, there are a pair of points

$$\{u^+_\infty, u^-_\infty\} \subset \partial\mathbb{F}$$

that are respectively the limits of positive and negative powers of $u$. The marking identifies $\partial\mathbb{F}$ with $\partial\widetilde{G}$. The line in $\widetilde{G}$ whose ends converge to $u^+_\infty$ and $u^-_\infty$ is called the *axis* of $\tau_u$. The bijection between lifts and automorphisms pairs $\tilde{f}$ to $\Phi$ when

$$\tilde{f}\tau_u = \tau_{\Phi(u)}\tilde{f} \text{ for all } u \in \mathbb{F}.$$

An automorphism $\Phi$ of $\mathbb{F}$ induces a homeomorphism $\hat{\Phi}$ on $\partial\mathbb{F}$. The action of the group $\mathrm{Aut}(\mathbb{F})$ on $\mathbb{F}$ has a continuous extension to an action on the Gromov compactification $\mathbb{F} \cup \partial\mathbb{F}$: Given $\Phi \in \mathrm{Aut}(\mathbb{F})$, let $\hat{\Phi} : \partial\mathbb{F} \to \partial\mathbb{F}$ denote its continuous extension, and let $\mathrm{Fix}(\hat{\Phi}) \subset \partial\mathbb{F}$ be the set of fixed points of $\hat{\Phi}$. Let $\mathrm{Fix}(\Phi) < \mathbb{F}$ denote the subgroup of elements fixed by $\Phi$.

Let $\mathrm{Fix}_+(\hat{\Phi})$ denote the set of attractors in $\mathrm{Fix}(\hat{\Phi})$, a discrete subset consisting of points $\xi \in \mathrm{Fix}(\hat{\Phi})$ such that for some neighborhood $U \subset \partial\mathbb{F}$ of $\xi$ we have $\hat{\Phi}(U) \subset U$ and the sequence $\hat{\Phi}^n(\eta)$ converges to $\xi$ for each $\eta \in U$. Let

$$\mathrm{Fix}_-(\hat{\Phi}) := \mathrm{Fix}_+(\hat{\Phi}^{-1})$$

denote the set of repellers in $\mathrm{Fix}(\hat{\Phi})$. This gives a decomposition of the fixed set of $\hat{\Phi}$:

$$\mathrm{Fix}(\hat{\Phi}) = \partial\,\mathrm{Fix}(\Phi) \cup \mathrm{Fix}_-(\hat{\Phi}) \cup \mathrm{Fix}_+(\hat{\Phi}). \tag{2.1}$$

In the sequel we are interested primarily in the *nonrepelling fixed points*,

$$\mathrm{Fix}_N(\hat{\Phi}) = \mathrm{Fix}(\hat{\Phi}) \setminus \mathrm{Fix}_-(\hat{\Phi}).$$

We also denote the periodic point set

$$\mathrm{Per}(\hat{\Phi}) := \bigcup_{k\geq 1} \mathrm{Fix}(\hat{\Phi}^k)$$

and its subsets $\mathrm{Per}_+(\hat{\Phi}), \mathrm{Per}_-(\hat{\Phi}), \mathrm{Per}_N(\hat{\Phi})$ defined by similar unions.

2.6. **Principal lifts and rotationless outer automorphisms.**

**Definition 2.6.** A representative $\Phi \in \mathrm{Aut}(\mathbb{F})$ of an outer automorphism $\phi \in \mathrm{Out}(\mathbb{F})$ is *a principal lift* if either:

- $\mathrm{Fix}_N(\hat{\Phi})$ contains at least three points.

- $\mathrm{Fix}_N(\hat{\Phi})$ is a two point set that is neither the set of endpoints of an axis nor the set of endpoints of a lift of a generic leaf of an element of $\mathcal{L}(\phi)$.

The corresponding lift $\tilde{f} : \tilde{G} \to \tilde{G}$ of a topological representative of $\phi$ is also called a *principal lift*. The set of principal lifts for $\phi$ is denoted $P(\phi)$.

Note that there is a map $P(\phi) \to P(\phi^k)$ for $k \geq 1$. Two automorphisms $\Phi_1, \Phi_2$ are *isogredient* if there is a $c \in \mathbb{F}$ such that for the inner automorphism $i_c$, $\Phi_2 = i_c \Phi_1 i_c^{-1}$. An outer automorphism $\phi$ has finitely many iso-gredience classes of principal lifts [12, Remark 3.9].

**Definition 2.7.** An outer automorphism $\phi$ is *(forward) rotationless* if for all $k \geq 1$ the map $P(\phi) \to P(\phi^k)$ is a bijection and $\mathrm{Fix}_N(\Phi) = \mathrm{Per}_N(\Phi)$ for all $\Phi \in P(\phi)$.

Rotationless automorphisms are without periodic behavior: if $\phi$ is rotationless then any $\phi$-periodic conjugacy class, lamination, or free factor system is in fact $\phi$ invariant [12, Lemma 3.30].

2.7. **Principal lifts and the Nielsen approach to** $\mathrm{Map}(\Sigma)$**.** Suppose now $\phi \in \mathrm{Map}(\Sigma)$ for a compact surface $\Sigma$. There exists a homeomorphism $g \colon \Sigma \to \Sigma$ representing $\phi$ that preserves both the stable and unstable geodesic laminations [5, Lemma 6.1]. If in addition $g$ preserves each individual principal region of the stable and unstable geodesic laminations, its boundary leaves, and their orientations then we say that $g$ is *rotationless*; A rotationless power of $g$ exists because there are only finitely many principal regions and boundary leaves. We say that $\phi$ is rotationless if it has a rotationless representative. If $g$ is rotationless and if $\widetilde{g} \colon \widetilde{\Sigma} \to \widetilde{\Sigma}$ is a lift of $g$, we say that $\widetilde{g}$ is an *s-principal lift* ($s$ stands for stable) if there exists a principal region $R^s$of the stable geodesic laminations such that $\widetilde{g}$ preserves $\widetilde{R}$ and preserves some (every) boundary leaf of $\widetilde{R}$.

**Observation 2.8.** *As shown by Handel and Mosher the principal lifts in the* $\mathrm{Out}(\mathbb{F})$ *and* $\mathrm{Map}(\Sigma)$ *sense agree* [17, Proposition I.2.12].

For geometric outer automorphims, the boundary curves of the surface can be detected by the dynamics of principal lifts.

**Proposition 2.9.** *Suppose $g$ is a pseudo-Anosov diffeomorphism of a compact surface $\Sigma$. Then the following are equivalent:*

(1) *$c \in \pi_1(\Sigma)$ is a boundary conjugacy class.*
(2) *There is a principal lift of $\tilde{g}$ such that the endpoints of $c$ are a non-repelling fixed point for the action of $\tilde{g}$ on $\partial\mathbb{H}^2$.*
(3) *The endpoints of $c$ are non-repelling fixed points for the action of $\tilde{g}_*$ on $\partial\pi_1(\Sigma)$.*

*Proof.* The equivalence of (2) and (3) is standard [23], since there's a continuous embedding of $\partial\pi_1(\Sigma) \in \partial\mathbb{H}^2$ that respects the action. The equivalence of (1) and (2) is established with care by Handel and Mosher [17, Proposition 2.12]. □

2.8. **Train tracks, splittings, and CTs.** Our analysis of individual outer automorphisms requires the consequences of particularly refined topological representatives, known as *completely split improved relative train-track maps (CTs)*, introduced by Feighn and Handel [12]. In lieu of the precise definition, the statement of which requires significant structure that we do not use directly, we introduce the parts of the definition needed for this paper, and indicate whenever a particular quoted lemma for topological representatives applies only to CTs.

A *filtration* of a topological representative $f : G \to G$ of an outer automorphism $\phi$ is a choice of an $f$-invariant chain of subgraphs

$$\emptyset = G_0 \subset G_1 \subset \cdots \subset G_R = G.$$

Associated to a filtration of a marked graph $G$ is a nested free factor system

$$\mathcal{F}_i = [\pi_1(G_i)]$$

that comes from the fundamental group of each connected component of $G_i$. The action of $f$ on the edges of $G$ determines a square matrix known as the *transition matrix*. The set $H_i = \overline{G_i \setminus G_{i-1}}$ is the $i^{th}$ stratum of $G$, and the submatrix of $M$ corresponding to the rows and columns indexed by $H_i$ is denoted $M_i$. For the topological representatives in this article, $M_i$ will be either a zero matrix, a $1 \times 1$ identity matrix, or an irreducible matrix with Perron-Frobenius eigenvalue $\lambda_i > 1$. We refer respectively to the stratum $H_i$ as a *zero*, *non-exponentially growing (NEG)*, or *exponentially growing* stratum. Any stratum that is not a zero stratum is *irreducible*.

If $f : G \to G$ is a topological representative and $\sigma$ a path or circuit in $G$ define $f_\#(\sigma) = [f(\sigma)]$. A decomposition of $\sigma = \sigma_1 \cdot \sigma_2 \cdots \sigma_n$ into subpaths is a *splitting for $f$* if

$$f^k_\#(\sigma) = f^k_\#(\sigma_1) \cdot f^k_\#(\sigma_2) \cdots f^k_\#(\sigma_n) \text{ for all } k \geq 1;$$

we reserve $\cdot$ to denote splittings and use adjacency for plain concatenation. A path or circuit $\sigma$ is called a *periodic Nielsen path* if $f^k_\#(\sigma) = \sigma$ for some $k \geq 1$, if $k = 1$ then $\sigma$ is a *Nielsen path.* A Nielsen path is *indivisible* if it is not a concatenation of non-trivial Nielsen paths. In general a closed path $\sigma$ is *root-free* if $\sigma \neq \gamma^k$ for a closed path $\gamma$. If $w$ is a closed, root-free Nielsen path and $E$ is an edge such that $f(E) = Ew^k$ then we call $E$ a *linear edge* and $w$ the axis of $E$. If $E$ and $E'$ have a common axis $w$ where $k \neq k'$ and $k, k' > 0$, then any path of the form $Ew^*\overline{E}'$ is an *exceptional path.*

For an EG stratum $H_r$ a nontrivial path $\sigma$ in $G_{r-1}$ with endpoints in $H_r \cap G_{r-1}$ is a *connecting path* for $H_r$. A path $\sigma$ contained in a zero stratum is *taken* if there is an edge $E$ in an irreducible stratum and $k \geq 1$ such that $\sigma$ is a maximal subpath of $f^k_\#(E)$ contained in that zero stratum. A non-trivial path or circuit $\sigma$ is *completely split* if there is a splitting $\sigma = \sigma_1 \cdot \sigma_2 \cdots \sigma_k$ where each $\sigma_i$ is one of the following: a single edge in an irreducible stratum, an indivisible Nielsen path, an exceptional path, or a maximal connecting taken path in a zero stratum. Each piece $\sigma_i$ is referred to as a *splitting unit* of $\sigma$.

**Definition 2.10** ([12, Definition 4.7])**.** A filtered topological representative

$$f \colon G \to G$$

of a rotationless outer automorphism $\phi$ with filtration

$$\emptyset = G_0 \subset G_1 \subset \cdots \subset G_R = G$$

is a *completely split improved relative train-track map (CT)* if it is a *relative train track map* satisfying

- **(Completely Split):** For every edge $E$ in an irreducible stratum $f(E)$ is completely split; and for every taken connecting path $\sigma$ in a zero stratum $f_\#(\sigma)$ is completely split.

**(NEG Nielsen Paths):** If the highest edges in an indivisible Nielsen path $\sigma$ belong to an NEG stratum then there is a linear edge $E$ and a closed root-free Nielsen path $w$ such that $f(E) = Ew^d$ for $d \neq 0$ and $\sigma = Ew\overline{E}$.

**(Other Good Stuff):** The requirements of the cited definition due to Feighn and Handel, which will not be directly appealed to in this paper, but are necessary for the quoted consequences.

For a rotationless outer automorphism $\phi$, the EG strata of the CT are intimately related to the set of attracting laminations. Specifically, there is a bijection between $\mathcal{L}(\phi)$ and EG strata of a CT [17, Fact I.1.55]: given any lamination $\Lambda \in \mathcal{L}(\phi)$, we can associate it with the stratum $i$ where

$$\mathcal{F}_{supp}(\Lambda) \not\sqsubset \mathcal{F}_{i-1} \text{ and } \mathcal{F}_{supp}(\Lambda) \sqsubset \mathcal{F}_i.$$

In this case we say $\Lambda$ is the lamination associated to the stratum $H_i$.

While the definition provides complete splittings only for edges and taken connecting paths, iteration suffices to obtain complete splittings of any path.

**Lemma 2.11** ([12, Lemma 4.25])**.** *If $f : G \to G$ is a CT and $\sigma \subset G$ is a path with endpoints at vertices then $f^k_\#(\sigma)$ is completely split for all sufficiently large $k$.*

In order to analyze certain CTs we make routine use of the "moving up through the filtration" lemma, which describes how the strata change as we move from filtration element that is a core graph to the next filtration element that is a core graph.

**Lemma 2.12** ([11, Lemma 8.3])**.** *Suppose $f : G \to G$ is a CT. Fix a filtration level $r$ such that $G_r$ is core, and let $s > r$ be the smallest integer such that $G_s$ is a core graph. Then there are two possible cases:*

$H_s$ **is NEG:** *In this case either $s = r+1$ or $s = r+2$ and $H_{r+1}$ is also NEG.*

$H_s$ **is EG:** *In this case there exists $r \leq u < s$ such that*

- *For each $r < j \leq u$ the stratum $H_j$ is a single non-fixed edge with terminal vertex in $G_r$.*
- *For each $u < j < s$, $H_j$ is a zero stratum enveloped by $H_s$.*

2.9. **Computing with CTs.** Feighn and Handel introduced an algorithm that produces a CT for any rotationless outer automorphism, which additionally can start from any prescribed filtration as input. This algorithm is instrumental in our algorithm.

**Theorem 2.13** ([13, Theorem 1.1])**.** *There is an algorithm, refered to as* ComputeCT*, that takes as input a rotationless $\phi \in \mathrm{Out}(\mathbb{F})$ and a nested sequence $\mathcal{F}_0 \sqsubset \mathcal{F}_1 \sqsubset \cdots \sqsubset \mathcal{F}_k$ of $\phi$-invariant free factor systems and produces a filtered topological representative $f : G \to G$ such that every non-empty $\mathcal{F}_i = \mathcal{F}(G_r)$ for a core filtration element $G_r$ and $f$ is a CT.*

In addition to an algorithm for computing a CT, Feighn and Handel demonstrate that many properties of a CT are computable from the CT. Specifically, there are algorithms, referred to as ZeroStrata?, NonlinearNEG?, and NongeometricEG?, that take as input a CT $f$ and decide respectively if $f$ has zero, nonlinear NEG, or nongeometric EG strata [13].

2.10. **The (equivariant) Whitehead algorithm.** In its most general form, the classical Whitehead algorithm is a procedure for understanding orbit-equivalence of tuples of conjugacy classes under the action of $\mathrm{Out}(\mathbb{F})$. We will make repeated use of this formulation, as recorded in Lyndon and Schupp.

**Theorem 2.14** ([20, Proposition 4.21])**.** *Let $\mathcal{C}$ denote the set of conjugacy classes of $\mathbb{F}$ with its natural $\mathrm{Out}(\mathbb{F})$ action. Let $\mathcal{C}^n$ denote the product of $n$ copies of $\mathcal{C}$ with diagonal $\mathrm{Out}(\mathbb{F})$ action. Then there is an algorithm which takes as input $c, c' \in \mathcal{C}^n$ and either produces $\phi \in \mathrm{Out}(\mathbb{F})$ such that $\phi(c) = c'$ or `No` if no such automorphism exists.*

Krstic, Lustig, and Vogtmann, in their solution to the conjugacy problem for roots of Dehn twists in $\mathrm{Out}(\mathbb{F})$, gave an equivariant generalization of the Whitehead algorithm that respects finite subgroups of $\mathrm{Out}(\mathbb{F})$, which we will also need.

**Theorem 2.15** ([19, Theorem 1.1])**.** *Let $\mathcal{C}$ denote the set of conjugacy classes of $\mathbb{F}$ with its natural $\mathrm{Out}(\mathbb{F})$ action. Let $\mathcal{C}^n$ denote the product of $n$ copies of $\mathcal{C}$ with diagonal $\mathrm{Out}(\mathbb{F})$ action. Let $G$ be a finite group. Then there is an algorithm which takes as input $c, c' \in \mathcal{C}^n$ and homomorphisms $\alpha, \alpha' \colon G \to \mathrm{Out}(\mathbb{F})$ and either produces $\phi \in \mathrm{Out}(\mathbb{F})$ such that $\phi(c) = c'$ and $(\alpha(g))^\phi = \alpha'(g)$ for all $g \in G$ or `No` if no such automorphism exists.*

2.11. **Guirardel's core.** Given two simplicial $T, T'$ with $G$-actions, Guirardel [15] constructed a *core*, denoted $\mathrm{Core}(T, T')$, that captures the compatibility of the two actions. Guirardel defines the core in a greater generality than what we require, and we recall a specialized definition.

**Definition 2.16.** Suppose $T$ and $T'$ are simplicial trees with $\mathbb{F}$ action. There exists a unique smallest non-empty, closed, connected, $\mathbb{F}$-invariant subset $C \subset T \times T'$ with convex fibers. We denote this set $\mathrm{Core}(T, T')$ and call it the *core of* $T, T'$.

The decision to use the connected subset is usually called the *augmented core*, but for our applications the connectivity is technically convenient, and we pray the reader will accept our slight abuse of terminology. Guirardel proved that a core always exists, and gave an explicit characterization of it in terms of the dynamics of the $\mathbb{F}$ action on each tree [15].

We will use a different construction of the Guirardel core found in different generalities in separate work of the first two authors and in Behrstock, Bestvina, and Clay [1, 2, 6]. Given a simplicial $\mathbb{F}$–tree $T$, let $\mathfrak{v}$ be a fixed vertex of $T$. Fix orientations on all edges of $T$. For a given edge $e \subseteq T$, the complement of the interior partitions $T$ into two connected components, which we call $\delta^+(e)$ and $\delta^-(e)$ according to the orientation of $e$. This induces a partition of the loxodromic elements $\mathbb{F}_{hyp} = \partial^+(e) \cup \partial^-(e)$ by the rule

$$\partial^\pm(e) = \{g \in \mathbb{F} | g^n \mathfrak{v} \in \delta^+(e) \text{ for large } n\}$$

Note that the sets $\partial^\pm(e)$ are independent of the choice of $\mathfrak{v}$.

**Definition 2.17.** Consider an edge $a$ in $T$ and an edge $b$ in $T'$. We say $a \times b$ is an *intersection square* if all of the following four intersections, as subsets of $\mathbb{F}$, are nonempty:

$$\partial^+(a) \cap \partial^+(b) \neq \emptyset \qquad \partial^+(a) \cap \partial^-(b) \neq \emptyset$$
$$\partial^-(a) \cap \partial^+(b) \neq \emptyset \qquad \partial^-(a) \cap \partial^-(b) \neq \emptyset$$

We call this intersection condition the *4-sets condition.*

A square $a \times b \subset \mathrm{Core}(T, T')$ if and only if it is an intersection square [2, Lemma 3.4.1; 6, Lemma 3.4].

## 3. Geometric models of geometric EG strata

One of the major ingredients in our algorithm is the relationship between the irreducible pieces of Thurston normal form and the EG strata of *any* CT representing a geometric automorphism $\phi$. Before we make the connection with Thurston normal form, we recall the definition of a geometric model for an EG stratum, some key facts established by Handel and Mosher about such models, and use them to derive invariants of $\phi$.

### 3.1. Defining the models.

**Definition 3.1** ([17, Definition I.2.1])**.** A *weak geometric model* for the EG stratum $H_r$ of a CT $f : G \to G$ is the following collection of objects:

(1) A compact, connected surface $S$ with nonempty boundary, enumerated as $\partial S = \partial_0 S \cup \cdots \cup \partial_m S$. The component $\partial_0 S$ is called the *upper boundary* of $S$ and $\partial_1 S, \ldots, \partial_m S$ are called the *lower boundaries*.
(2) For each lower boundary $\partial_i S, i = 1, ..., m$, a homotopically nontrivial closed edge path $\alpha_i : \partial_i S \to G_{r-1}$.
(3) The 2-complex $Y$ that is the quotient of the disjoint union $S \sqcup G_{r-1}$ by the attaching map $\sqcup_i \alpha_i$ on the lower boundaries, with quotient map

$$j \colon S \sqcup G_{r-1} \to Y.$$

(4) An embedding $G_r \hookrightarrow Y$ which extends the embedding $G_{r-1} \hookrightarrow Y$ such that $G_r \cap \partial_0 \Sigma$ is a single point denoted $p_r$, and there is a closed indivisible Nielsen path $\rho_r$ of height $r$ in $G_r$ and based at $p_r$, such that the loop $\partial_0 \Sigma$ based at $p_r$ and the path $\rho_r$ are homotopic rel base point in $Y$. The existence of this embedding implies the existence of a deformation retraction $d : Y \to G_r$ such that $d|_{\partial_0 S}$ is a parameterization of $\rho_r$.
(5) A homotopy equivalence $h : Y \to Y$ and a homeomorphism

$$g \colon (S, \partial_0 S) \to (S, \partial_0 S)$$

whose mapping class $[g] \in \mathrm{Map}(S)$ is infinite order and irreducible, subject to the following compatibility conditions:
   (a) The maps $(f|G_r) \circ d$ and $d \circ h$ are homotopic.
   (b) The maps $j \circ g$ and $h \circ j$ are homotopic.

Following Handel and Mosher, we will make use of Thurston's theorem that $[g] \in \mathrm{Map}(\Sigma)$ is infinite order and irreducible if and only if there is a pseudo-Anosov representative, and when working with geometric models take $g$ to be pseudo-Anosov. The full data of a weak geometric model is quite notationally heavy, for brevity we will say $Y$ is a weak geometric model for the stratum $H_r$, and use the other notation introduced by the definition in a standard fashion. When working with several geometric models we will consistently decorate $Y$ and the other notation the same way, e.g. $S'$ is the surface associated to the weak geometric model $Y'$.

**Definition 3.2.** An EG stratum $H_r$ of a CT $f : G \to G$ is *geometric* if there is a weak geometric model $Y$ for $H_r$.

We recall that geometric EG strata are characterized by Nielsen paths.

**Fact 3.3** ([17, Fact I.2.3]). *For an EG stratum $H_r$ of a CT $f : G \to G$, the following are equivalent:*

- *$H_r$ is geometric.*
- *There exists a closed, height $r$ indivisible Nielsen path $\rho_r$.*
- *There exists a closed, height $r$ indivisible Nielsen path $\rho_r$ which crosses each edge of $H_r$ exactly twice.*

*Furthermore each component of $G_{r-1}$ is non-contractible.*

Combining this fact with a lemma of Feighn and Handel gives two useful structural results about geometric EG strata.

**Lemma 3.4** ([12, Lemma 4.24]). *Suppose $f : G \to G$ is a CT and $H_r$ is a geometric EG stratum. If $E$ is an edge of $H_r$ then each maximal subpath of $f(E)$ in $G_{r-1}$ is a Nielsen path.*

**Lemma 3.5** ([12, Lemma 4.24]). *Suppose $f : G \to G$ is a CT and $H_r$ is a geometric EG stratum. Then $H_r$ does not envelop a zero stratum.*

A weak geometric model $Y$ for a stratum $H_r$ only captures a CT $f : G \to G$ up to height $r$ of the filtration, but encapsulates most of the technical notation and results. To connect the weak geometric model to the full automorphism we recall the definition of a geometric model for a stratum.

**Definition 3.6** ([17, Definition I.2.4]). A *geometric model* $X$ for a geometric EG stratum $H_r$ of a CT $f : G \to G$ is the quotient of $Y \sqcup G$ obtained by identifying the embedded copies of $G_r$ in each, where $Y$ is some weak geometric model for $H_r$, a deformation retraction $d : X \to G$ extending $d : Y \to G$ and a homotopy equivalence $h : X \to X$ extending $h : Y \to Y$ such that $d \circ h$ is homotopic to $f \circ d$.

The extra information contained in a geometric model but not a weak geometric model needed for this work is encapsulated in the *peripheral splitting* associated to $X$. This is a $\mathcal{Z}$-splitting of $\mathbb{F}$ obtained from a decomposition of the geometric model $X$ as a graph of spaces [17, Definition I.2.10]. Vertex spaces are the surface $S$ and components of the "complementary subgraph" $(G - H_r) \cup \partial_0 S$; edge spaces are annuli (coming from boundary components of $S$) and intervals (coming from attaching points of the $G - G_r$ to interior points of $H_r$). This splitting need not be minimal. Using this splitting, one can see that for a geometric model $X$ of an EG stratum of $f : G \to G$, the composition $dj : S \to G$ is $\pi_1$-injective [17, Lemma I.2.5].

## 3.2. Geometric models as invariants.

We need a few facts about the invariance of geometric models for EG strata that are suggested by, but not proved in, Chapter I.2 of Handel and Mosher's monograph [17]. In that chapter, Handel and Mosher prove the following structure proposition for the surface laminations of a geometric stratum.

**Proposition 3.7** ([17, Proposition I.2.15]). *Given any finite type surface $S$ and any pseudo-Anosov homeomorphism $g : S \to S$ with stable and unstable lamination pair $\Lambda^s, \Lambda^u \subset B(\pi_1 S)$, we have:*

- *$\Lambda^s/\Lambda^u$ is an attracting/repelling lamination pair for $g_* \in \mathrm{Out}(\pi_1 S)$.*
- *$\Lambda^s$ and $\Lambda^u$ each fill $\pi_1 S$.*

*Furthermore, suppose that $f : G \to G$ is a CT representative for some $\phi \in \mathrm{Out}(\mathbb{F})$, that $H_r \subset G$ is an EG-geometric stratum of $f$ corresponding to a geometric lamination pair $\Lambda^{\pm}$ of $\phi$, and that the given surface $S$ is the surface associated to a weak geometric model for $f$ and $H_r$. If in addition $g$ is the associated pseudo-Anosov homeomorphism of the weak geometric model then we have:*

- *The map $\widehat{dj} : \mathcal{B}(\pi_1\Sigma) \to \mathcal{B}(\mathbb{F})$ takes $\Lambda^u, \Lambda^s$ homeomorphically to $\Lambda^+, \Lambda^-$ respectively.*
- *Every leaf of $\Lambda^+$ is dense in $\Lambda^-$, and similarly for $\Lambda^-$.*
- *All leaves of $\Lambda^+$ and $\Lambda^-$ are generic.*
- *$\mathcal{F}_{supp}(\Lambda^+) = \mathcal{F}_{supp}(\Lambda^-) = \mathcal{F}_{supp}([\pi_1 S])$.*

From this structural result on laminations, they conclude that geometricity is a property of a lamination pair $\Lambda^{\pm} \in \mathcal{L}^{\pm}(\phi)$ associated to an outer automorphism, independent of the geometric model or CT used.

**Proposition 3.8** ([17, Proposition I.2.18])**.** *If $\Lambda^{\pm} \in \mathcal{L}^{\pm}(\phi)$ is an invariant lamination pair associated to $\phi \in \mathrm{Out}(\mathbb{F})$ and there is some rotationless iterate $\phi^k$ and CT $f : G \to G$ for $\phi^k$ such that $\Lambda^{\pm}$ is associated to a geometric EG stratum of $f$, then for any CT $f' : G' \to G'$ of any rotationless iterate $\phi^m$, the EG stratum of $f'$ associated to $\Lambda^{\pm}$ is geometric.*

In this section we pick up where Handel and Mosher leave off, and expand upon Proposition 3.8 to conclude that "for a weak geometric model of a geometric EG stratum of a rotationless automorphism, the homeomorphism type of the surface, the mapping class of the pseudo-Anosov, and the conjugacy class of the inclusion of the surface fundamental group under $dj$ are invariants of $\phi$"; they do not depend on the CT representative.

**Lemma 3.9.** *Suppose $\phi \in \mathrm{Out}(\mathbb{F})$ is a rotationless outer automorphism and $\Lambda^{\pm} \in \mathcal{L}^{\pm}(\phi)$ is a geometric lamination pair. Suppose $f : G \to G$ and $f' : G' \to G'$ are two CT representatives for $\phi$. Let $H_r$ be the EG stratum of $f$ corresponding to $\Lambda$ with weak geometric model $Y$, and $H'_s$ be the EG stratum of $f'$ corresponding to $\Lambda$ with weak geometric model $Y'$. Let $[K] \leq \mathbb{F}$ be the conjugacy class of the fundamental group of the surface $S$ associated to $Y$; that is, $[K] = [(d \circ j)_*(\pi_1 S)]$. Similarly, let $[K']$ be the conjugacy class of the fundamental group of the surface $S'$ associated to $Y'$. Then $[K] = [K']$.*

*Proof.* By Proposition 3.7, both $[K]$ and $[K']$ carry $\Lambda^{\pm}$. Thus, for a representative $K \in [K]$ there is a subset $\widetilde{\Lambda}^{\pm}_K$ of $\widetilde{\mathcal{B}}(K) \subseteq \widetilde{B}(\mathbb{F})$ that is a lift of $\Lambda^{\pm}$. Let $K' \in [K']$ be the representative such that $\widetilde{\Lambda}^{\pm}_K \subseteq \widetilde{B}(K') \subseteq \widetilde{B}(\mathbb{F})$. This choice exists since there a representative of $[K']$ carries some lift of $\Lambda^{\pm}$ and all such lifts are $\mathbb{F}$ conjugate.

We first show that the intersection $\Gamma = K \cap K'$ has finite index in $K$ (and $K'$). Indeed, for a contradiction suppose $[K : \Gamma] = \infty$ and consider the cover $S_\Gamma$ of $S$ corresponding to $\Gamma$. Intersections of finitely generated subgroups of free groups are finitely generated, so $S_\Gamma$ is an infinite sheeted cover with compact core. The laminations $\Lambda^{\pm}$ are carried by both $K$ and $K'$, and therefore by $\Gamma$ as well. Thus $\Lambda^{\pm}$ lift to $S_\Gamma$ and are therefore not filling. However, Proposition 3.7 specifies that that $\Lambda^{\pm}$ are precisely the images of the stable and unstable laminations of the pseudo-Anosov model $g \colon S \to S$ under the map $\mathcal{B}(S) \hookrightarrow \mathcal{B}(\mathbb{F})$. This is a contradiction: the stable and unstable laminations of any pseudo-Anosov homeomorphism are filling. Thus $\Gamma$ has finite index in $K$. By a symmetric argument $[K' : \Gamma] < \infty$.

It remains to show that $K' \leq K$; equality will follow by symmetry. Let $T$ be the $\mathbb{F}$-minimal subtree of the Bass-Serre tree coming from the peripheral splitting associated to the full geometric model $X$. The subgroup $K$ is the stabilizer of a vertex $v$ in $T$, so it suffices to show that the $K'$-minimal subtree of $T$ is $v$. Let $T_{K'}$ be the minimal subtree for $K'$. If $T_{K'}$ had an infinite orbit of vertices, then $\mathrm{Stab}(v) \cap K' = K \cap K' = \Gamma$ would have infinite index in $K'$, which we have ruled out. So $T_{K'}$ has finitely many vertices and contains $v$. Since $T_{K'}$ is finite, the fixator $\mathrm{Fix}(T_{K'}) \leq K'$ is finite-index in $K'$ and contained in each edge stabilizer of $T_{K'}$. Recall now that $T$ is a subtree of the Bass-Serre tree of a $\mathcal{Z}$-splitting, hence all edge groups are trivial or cyclic. However, the group $K'$ is the fundamental group of a surface carrying a pseudo-Anosov homeomorphism, so is a free group of rank at least 2. Since $\mathrm{Fix}(T_{K'})$ is finite index in $K'$ it is also free of rank at least 2, and contained in each edge stabilizer of $T_{K'}$. The only way for this to be possible is if $T_{K'}$ contains no edges, that is $T_{K'} = v$. Hence $K' \leq K$. As equality follows by symmetry this completes the proof. ☐

**Lemma 3.10.** *Suppose $f : G \to G$ is a CT representative of a rotationless $\phi \in \mathrm{Out}(\mathbb{F})$ and $\Lambda^{\pm} \in \mathcal{L}^{\pm}(\phi)$ is a geometric lamination pair. Let $H_r$ be the geometric EG stratum of $f$ associated to $\Lambda^{\pm}$ and $Y$ a geometric model for $H_r$. Let $[K] \leq \mathbb{F}$ be the fundamental group of the surface $S$ associated to $Y$; that is, $[K] = [(d \circ j)_*(\pi_1 S)]$. For any representative $K$ the set of $K$-conjugacy classes representing $\partial S$ is independent of the CT and depends only on $\phi$.*

*Proof.* Let $g$ be the pseudo-Anosov homeomorphism of $S$ coming from the geometric model. Denote the stable and unstable laminations of $g$ by $\Lambda^s$ and $\Lambda^u$. By Proposition 3.7 $\widehat{dj}$ is a homeomorphism from $\Lambda^u$ to $\Lambda^+$ and from $\Lambda^s$ to $\Lambda^-$. We will identify the boundary classes of $K$ using principal lifts, so some care with basepoints is required.

Fix the basepoint $b \in H_r \subseteq G$, as well as a basepoint $* \in S$ such that $dj(*) = b$ in the image of the geometric model attaching maps. Fix basepoints $\tilde{b} \in \tilde{G}$, and $\tilde{*} \in \tilde{S}$, and a lift $\widetilde{dj} : \tilde{S} \to \tilde{G}$ covering $dj$ at these base-points. Let $K \in [K]$ be the representative image of $\pi_1(S)$ picked by these base0point choices.

For each principal lift $\Psi$ of $\phi$ that fixes some element of $[K]$ there is an iso-gredient principal lift $\Phi$ fixing $K$. Since $[K]$ carries $\Lambda^{\pm}$ by Proposition 3.7, there is a lift of the laminations $\Lambda^{\pm}$ to $\tilde{\Lambda}^{\pm} \subseteq \tilde{\mathcal{B}}(K)$. A principal lift $\Phi$ satisfies $|\tilde{\Lambda}^+ \cap \mathrm{Fix}_N(\Phi)| \geq 3$ if and only if it fixes $K$.

Let $\Phi$ be a principal lift of $\phi$ fixing $K$, and select a realization $\tilde{f} : \tilde{G} \to \tilde{G}$ so that $\tilde{f}(b) \in \widetilde{dj}(\tilde{S})$. This is possible since $\Phi$ fixes $K$. Let $\tilde{g}$ be a lift of $g$ such that $\widetilde{dj}(\tilde{g}(\tilde{*})) = \tilde{f}(b)$. By the homotopy lifting property of covers and item 5a in the definition of geometric model, the action of $\widetilde{g}$ on the lifts of $\Lambda^s$ and $\Lambda^u$ is conjugate via $\widetilde{dj}$ to the action of $\widetilde{f}$ on $\tilde{\Lambda}^{\pm}$. Thus $\tilde{g}$ is a principal lift of $g$.

In fact, every principal lift of $g$ arises this way. Indeed, if $\tilde{g}$ is a principal lift of $g$, then, again by the homotopy lifting property and item 5a in the definition of geometric model, the lift of $f$ such that $\tilde{f}(b) = \widetilde{dj}(\tilde{g}(*))$ is a realization of a principal lift of $\phi$.

By Lemma 3.9 the conjugacy class $[K]$ is an invariant of $\phi$. For each representative $K$ the boundary conjugacy classes of $S$ are determined by the dynamics of the principal lifts of $g$ on $\widetilde{\mathcal{B}}(K)$, see Proposition 2.9. Since each principal lift of $g$ fixing $K$ arises as a principal lift of $\phi$ fixing $K$, these dynamics are invariants of $\phi$. This completes the proof. □

**Corollary 3.11.** *Suppose $\Lambda^{\pm} \in \mathcal{L}(\phi)$ is a geometric lamination pair. Let $f : G \to G$ be a CT representing $\phi$ and suppose $H_r$ is the geometric EG stratum of $f$ corresponding to $\Lambda^{\pm}$. The surface $S$ associated to a weak geometric model $Y$ for $H_r$ and the mapping class of $g : S \to S$ are both invariants of the outer automorphism $\phi$ and do not depend on $Y$ or $f$.*

*Proof.* Suppose $f' : G \to G$ is another CT for $\phi$ and $H'_s$ is the geometric EG stratum of $f'$ corresponding to $\Lambda^{\pm}$. Let $S'$ be the surface associated to a geometric model $Y'$ of $H'_s$. We will show that there is a homeomorphism $\eta : S \to S'$ such that $[\eta g \eta^{-1}] = [g'] \in \mathrm{Map}(S')$.

Pick base-points in the strata $H_r$ and the surface $S$ to fix a representative $K = dj_*(\pi_1(S), *) \leq \mathbb{F}$. By Lemma 3.9, $[K] = [d'j'_*(\pi_1(S'))]$, so there is a choice of basepoint in $H'_s$ and on $S'$ such that $d'j'_*(\pi_1(S', *') = K$.

Let

$$F_* = (d'{j'}_*^{-1}) \circ dj_* : \pi_1(S, *) \to \pi_1(S', *)$$

be the induced isomorphism (this is indeed well defined as both $dj_*$ and $d'j'_*$ are isomorphisms onto their images). Since $\pi_1(S, *)$ is free, there is a homotopy equivalence $\Theta$ inducing this isomorphism.

By Lemma 3.10, $\Theta_*$ takes the boundary classes of $S$ to the boundary classes of $S'$, so by the Dehn-Nielsen-Baer theorem (Theorem 2.3) $\Theta$ is homotopic rel boundary to a homeomorphism $\eta : S \to S'$.

Finally, by Proposition 3.7, $\widehat{\eta} = \widehat{\Theta}$ induces homeomorphisms from $\Lambda^s$ to $\Lambda^{s\prime}$ and $\Lambda^u$ to $\Lambda^{u\prime}$, and $g$ and $g'$ have the same dilatation, thus

$$[\eta g \eta^{-1}] = [g'] \in \mathrm{Map}(S)$$

by the recognition theorem for pseudo-Anosov mapping classes [22, Theorem 1]. □

## 4. Deciding geometricity for rotationless automorphisms

In this section we introduce Algorithm 1, which decides if a rotationless outer automorphism $\phi$ is geometric, and verify its correctness. Our description of Algorithm 1 begins with an investigation of the properties of CTs representing a geometric automorphism $\phi$.

This investigation begins with a particular CT coming from the surface on which $\phi$ is realized. We then examine the extent to which these properties are shared by all CTs representing $\phi$. In this way, we provide a list of computable necessary conditions for any CT representing a rotationless automorphism to be geometric. These conditions are split into dynamical constraints on the strata of a representative CT (Section 4.1) and algebraic constraints on the fixed subgroups (Section 4.2).

Feighn and Handel [13] provide an algorithm to compute a CT representative of a rotationless outer automorphism. We arrive at Algorithm 1 by computing a surface and a realizing homeomorphism from a CT satisfying the necessary conditions, demonstrating their sufficiency.

4.1. **CT representatives of a geometric automorphism.** The following notation will be used throughout the remainder of this subsection and the next.

**Notation 4.1.** *Fix a rotationless element $\phi \in \mathrm{Out}(\mathbb{F})$ which we assume to be geometric. Let $\Sigma$ be a (possibly non-orientable) connected compact surface whose fundamental group is identified with $\mathbb{F}$ ($\pi_1\Sigma \simeq \mathbb{F}$). Let $g\colon \Sigma \to \Sigma$ be a homeomorphism inducing $\phi$ ($[g_*] = \phi$). We assume further that $g$ is in Thurston normal form for its homotopy class, so that it has no periodic behavior as detailed in Lemma 4.2.*

**Lemma 4.2.** *The Thurston normal form of $g$ is without periodic behavior. That is, let $\{c_1, \ldots, c_m\}$ be the canonical reduction system of $g$ and choose representatives $c_i$ with pairwise disjoint closed neighborhoods $R_1, \ldots, R_m$. Let $R_{m+1}, \ldots, R_{m+n}$ denote the closures of the connected components of the complement of the reduction system $\Sigma - \cup_{i=1}^m R_i$. Let $\eta_i\colon \mathrm{Homeo}(R_i, \partial R_i) \to \mathrm{Homeo}(\Sigma)$ denote the homomorphism induced by the inclusion $R_i \hookrightarrow \Sigma$. Then $g(R_i) = R_i$ for all $i$ and moreover,*

$$g = \prod_{i=1}^{m+n} \eta_i(g_i)$$

*where $[g_i] \in \mathrm{Map}(R_i)$ is a power of a Dehn twist for $i \in \{1, \ldots, m\}$ and $[g_i] \in \mathrm{Map}(R_i)$ is either pseudo-Anosov or the identity for $i \in \{m+1, \ldots, m+n\}$.*

*Proof.* This follows immediately from the fact that if $\phi \in \mathrm{Out}(\mathbb{F})$ is rotationless and $c$ is a periodic conjugacy class, then $c$ is in fact fixed by $\phi$. Were it necessary to pass to a power of $g$ to obtain the above form, there would be a conjugacy class which is periodic but not fixed. □

We now choose a nested sequence $\emptyset = \mathcal{F}_0 \leq \mathcal{F}_1 \leq \ldots \mathcal{F}_M = \mathbb{F}$ of $\phi$-invariant free factor systems determined by (the fundamental groups of) an appropriately chosen maximal nested sequence $Q_1 \subset Q_2 \subset \ldots \subset Q_M$ of $g$-invariant subsurfaces of $\Sigma$. One way to ensure such a sequence of subsurfaces determines a nested sequence of free factor system is as follows. Using the notation in Lemma 4.2, assume after reordering that $R_1, \ldots, R_b$ are annular neighborhoods of the boundary components of $\Sigma$. Start by defining $Q_i = R_1 \cup \ldots \cup R_i$ for $i \in \{1, \ldots, b-1\}$; note that we have omitted the last boundary component, as carried by $R_b$. Then "work towards" the final boundary component as follows: define a partial order on the remaining invariant subsurfaces $R_i$ for $i \in \{b+1, m+n\}$ by $R_i \leq R_j$ if $\mathcal{F}_{supp}(\pi_1 R_i, \pi_1 Q_{b-1}) \sqsubset \mathcal{F}_{supp}(\pi_1 R_j, \pi_1 Q_{b-1})$. Then for $i \in \{b+1, \ldots, m+n\}$ inductively define $Q_i = Q_{i-1} \cup R$ where $R$ is any subsurface which is minimal with respect to this partial order. Finally, define $\mathcal{F}_i = \mathcal{F}_{supp}(\pi_1 Q_i)$ as the smallest free factor system carrying the fundamental group(s) of the not necessarily connected subsurface.

**Notation 4.3.** *Let $f\colon G \to G$ be a CT representing $\phi$ with filtration*

$$\emptyset = G_0 \subset \ldots \subset G_N = G$$

*and realizing the (not necessarily maximal) nested sequence of $\phi$-invariant free factor systems defined above.*

The first step in the construction of a CT involves completing the prescribed nested sequence of $\phi$-invariant free factor systems to a maximal one. We will distinguish those core subgraphs $G_r$ whose fundamental groups are elements of the prescribed filtration (that is $\mathcal{F}_{supp}([\pi_1(G_r)]) \in \{\mathcal{F}_i\}$) from those core subgraphs whose fundamental groups are not in the prescribed filtration; will say subgraphs in the former category and their strata *determine subsurfaces*, while subgraphs in the latter category *do not determine subsurfaces.*

4.1.1. *Strata of the preferred CT.* In the previous section we proved that the surface data of a geometric EG stratum of a CT is an invariant. We now use $f : G \to G$ to show that these invariants agree with the pieces of the Thurston normal form of $g$.

**Observation 4.4.** *Suppose the restriction of $g\colon \Sigma \to \Sigma$ to a subsurface $R$ is pseudo-Anosov, and that $Q_i$ is the subsurface of $\Sigma$ such that $Q_i = Q_{i-1} \cup R$ as defined above, with associated $\phi$-invariant free factor system $\mathcal{F}_i$. Then there are no $\phi$-invariant free factors properly containing $\mathcal{F}_{i-1}$ and properly contained in $\mathcal{F}_i$.*

**Lemma 4.5.** *If $H_r$ is an EG stratum of $f\colon G \to G$, then $H_r$ is geometric and the subsurface $R$ of $\Sigma$ such that $Q_r = Q_{r-1} \cup R$ can be attached to $G_{r-1}$ to produce a weak geometric model for $H_r$.*

*Proof.* Since $H_r$ is an EG stratum, $G_r$ is necessarily a core graph; let $\mathcal{F}_{supp}([\pi_1 G_r])$ be the associated $\phi$-invariant free factor system. Again because $H_r$ is EG there are conjugacy classes in $\mathcal{F}_{supp}([\pi_1 G_r])$ that grow exponentially under iteration by $\phi$ and are not contained in $\mathcal{F}_{supp}([\pi_1 G_{r-1}])$. In particular, this implies that the $\phi$-invariant free factor system $\mathcal{F}_{supp}([\pi_1 G_r])$ determines a subsurface $Q_i$ of $\Sigma$. Let $R$ be the subsurface of $\Sigma$ such that $Q_i = Q_{i-1} \cup R$. Note that the presence of exponentially growing conjugacy classes in $R$ and the maximality of the filtration imply that $g|_R$ is pseudo-Anosov. There is a simple closed curve $c$ in $\partial R$ which is not homotopic into $Q_{i-1}$ and is fixed by $g$ (it's the "upper boundary component"). Since $g|_R$ is pseudo-Anosov, Observation 4.4 implies that the core graph of $G_{r-1}$ has fundamental group equal to $\mathcal{F}_{i-1}$. Abusing notation and thinking of $c$ as an element of $\mathbb{F}$, it is clear that $[c]$ is not carried by $\mathcal{F}_{i-1}$ and therefore not carried by $G_{r-1}$. Thus there is a conjugacy class in $G$ of height $r$ that is $\phi$-invariant, so is represented by a height $r$ closed indivisible Nielsen path $\rho_r$. This is equivalent to $H_r$ being geometric by Fact 3.3.

Now suppose $S \xrightarrow{j} X \xrightarrow{d} G$ is a geometric model for $H_r$. The argument is similar to Lemma 3.9. Both $\pi_1(R)$ and $\pi_1(S)$ carry the laminations of the stratum $\Lambda^{\pm}$, so the intersection $K = \pi_1(R) \cap \pi_1(S)$ is finite index in both groups. Indeed, in both surfaces $\Lambda^{\pm}$ are realized as attracting laminations for pseudo-Anosov homeomorphisms, so each is filling and carried by $K$; this is impossible if $K$ is infinite index.

Let $X$ be a geometric model of $H_r$ and $T$ be the minimal tree associated to the corresponding $\mathcal{Z}$-splitting. As in Lemma 3.9 we conclude that $\pi_1(R) \le \pi_1(S)$. While we cannot appeal to symmetry exactly, to see that $\pi_1(S) \le \pi_1(R)$, consider the Bass-Serre tree $T_R$ coming from the $\mathcal{Z}$-splitting of $\Sigma$ induced by $\partial R$. The minimal tree $T_S$ for the action of $\pi_1(S)$ must be finite since the intersection $K$ is finite-index in $\pi_1(S)$; since $\pi_1(S)$ is free of rank at least 2 this implies it is a single vertex.

Finally the upper boundary $\partial_0 S$ represents $[c]$ by the definition of a geometric model so upper boundaries of $S$ and $R$ are identified. Since $\pi_1(S) \cong \pi_1(R)$ to see that the lower boundaries are identified we appeal to Lemma 3.10. The boundary classes are an invariant of the lamination pair $\Lambda^{\pm}$, independent of the CT or geometric model. Moreover, as the surface and free group notions of principal lift coincide [12], the lamination pair $\Lambda^{\pm}$ determines the boundary classes of $R$. Thus, for an appropriate choice of basepoints we obtain an isomorphism $\Theta_* = \iota_*^{-1} \circ dj_* : \pi_1(S) \to \pi_1(R)$ that takes boundary classes to boundary classes, respecting the distinction of upper and lower classes. As in the proof of Corollary 3.11 it follows from the Dehn-Nielsen-Baer theorem that there is a homeomorphism $\eta : S \to R$ conjugating the geometric model homeomorphism to the restriction of the Thurston normal form to $R$. The homeomorphism $\eta$ can be used to attach $R$ to $G_r$ to produce a new weak geometric model. □

Geometricity also imposes a strict constraint on NEG strata of $f$, capturing the growth dichotomy for conjugacy classes under iteration by a surface homeomorphism.

**Lemma 4.6.** *If $H_r$ is a non-fixed NEG stratum, then it is linear.*

*Proof.* Suppose for a contradiction that there exists a nonlinear NEG stratum and let $H_r$ be the lowest such. The stratum $H_r$ consists of a unique edge $E$ such that $f(E) = Ew$ for some completely split conjugacy class $w$. We first show that if $w$ contains an EG edge $E'$ as a splitting unit, there will be a conjugacy class $[\sigma]$ in $\mathbb{F}$ whose asymptotic growth rate is super-exponential and this behavior does not occur for geometric automorphisms.

To produce such a conjugacy class, we simply need to construct a completely split conjugacy class containing $E$ as a splitting unit. To do so, consider the smallest integer $s \geq r$ $r$ such that $G_s$ is a core graph.

We first argue that $H_s$ cannot be an EG stratum. If $H_s$ were EG, then it would be geometric by Lemma 4.5. Thus, by Lemma 3.4, for each edge $E''$ of $H_s$, the maximal subpaths of $f(E'')$ in $G_{s-1}$ are Nielsen paths. Thus, the EG case of Lemma 2.12 implies the existence of a Nielsen path $w$ of height $r$ whose first edge is $E$. This is a contradiction to (NEG Nielsen Paths), since $E$ is non-linear. Thus, $H_s$ cannot be EG.

So we conclude that $H_s$ is NEG and the relation between $H_s$ and $G_r$ is described in the NEG case of Lemma 2.12. Since $G_s$ is core, there exists a closed loop $\sigma$ in $G_s$ crossing $E$ either once or twice according to whether or not $E$ is separating in $G_s$. Iterating $f$, we may assume that $\sigma$ is completely split, and we claim $E$ is a splitting unit in the complete splitting of $\sigma$. Indeed, the edge $E$ is not contained in a zero stratum, so it is not a taken path. It also cannot appear in an indivisible Nielsen path or exceptional path in the complete splitting of $\sigma$: Gupta and Wigglesworth characterize the structure of indivisible Nielsen paths and exceptional paths that cross non-fixed irreducible strata [16, Lemma 7.4], and their characterization rules out the possibility of $E$ appearing in either type of splitting unit.

Therefore, the asymptotic growth of iterates of $E$ gives a lower bound for the asymptotic growth of $\sigma$. Our assumption that $f(E)$ contains an EG edge as a splitting unit implies that $\ell(f^n(E)) \geq n\lambda^n$ (here $\lambda$ is the exponential growth rate of the EG edge $E'$). Hence $\ell(f^n(\sigma) \geq n\lambda^n$ but this cannot occur for geometric automorphisms. Thus $w$ contains no EG edges as splitting units.

The minimality of $H_r$ implies that, absent EG splitting units, $w$ must contain a linear edge $E'$ in its complete splitting. A similar argument then implies the existence of a conjugacy class that grows quadratically under iteration by $\phi$ (and hence by $g$), something that cannot happen for conjugacy classes under iteration by a surface homeomorphism. □

4.1.2. *Strata of an arbitrary CT.* We now consider an arbitrary CT representing the geometric automorphism $\phi$.

**Notation 4.7.** *For the remainder of this section, we let $f'\colon G' \to G'$ be* any *CT representing the geometric automorphism $\phi$. In particular, $f'$ need not have anything to do with a surface $\Sigma$ on which $\phi$ can be realized.*

**Lemma 4.8.** *If $f'\colon G' \to G'$ is as above, then every EG stratum of $f'$ is geometric and every nonfixed NEG stratum is linear.*

*Proof.* The first statement is implied by Lemma 4.5 together with Proposition 3.8. For the second, we simply note that the proof of Lemma 4.6 relies only on the asymptotic growth rates of conjugacy classes in $\mathbb{F}$, and this depends only on $\phi$ and not $f$. □

As a consequence of Lemma 4.8 we obtain two more necessary conditions for a rotationless automorphism to be geometric:

**Corollary 4.9.** *There are no elements $\Lambda_1, \Lambda_2 \in \mathcal{L}(\phi)$ such that $\Lambda_1 \supsetneq \Lambda_2$. Any CT $f'\colon G' \to G'$ representing $\phi$ does not have a zero stratum.*

*Proof.* Since $\phi$ is geometric, by Lemma 4.8 every EG stratum $H'_r$ is geometric.

For every $E$ edge in an EG stratum $H'_r$ Lemma 3.4 implies each maximal subpath of $f'(E)$ in $G'_{r-1}$ itself a Nielsen path. In particular, there are no EG edges of height $< r$ in the complete splitting of $f'(E)$, so the lamination $\Lambda$ associated to $H'_r$ cannot contain a sub-lamination.

Every zero stratum is enveloped by some EG stratum, but Lemma 3.5 states that geometric EG strata do not envelop zero strata. □

In fact, the pseudo-Anosov pieces used for geometric models are an invariant of $\phi$ independent of the CT.

**Lemma 4.10.** *With the conventions established by Notation 4.1, Notation 4.3, and Notation 4.7 above, there is a bijection*

$$\{\textit{EG strata of } f'\} \longleftrightarrow \{R \subseteq \Sigma \mid g(R) = R \textit{ and } g|_R \textit{ is pA}\}.$$

*Moreover, a subsurface on the right-hand side of the bijection can be used as the surface part of a geometric model for the corresponding stratum.*

*Proof.* For the CT $f\colon G \to G$ obtained using a filtration coming from the surface $\Sigma$; this is a consequence of Lemma 4.5. For any other CT $f'$ the requisite the bijection comes from the bijection between EG strata of $f'$ and attracting laminations $\mathcal{L}(\phi)$. That this bijection provides surface pieces of geometric models follows from Corollary 3.11. □

In summary, we conclude that for any CT $f'$ representing a rotationless geometric automorphism $\phi$, every stratum of $f'$ must be either fixed, linearly growing, or geometric.

4.2. **The fixed subgroups.** Beyond the "stratum constraints" from the previous section, the fixed subgroups of a rotationless geometric automorphism reflect the fixed subsurfaces of the Thurston normal form. The precise formulation is the notion of a $\partial$-realizable set of conjugacy classes, Definition 4.11. We first show that this property is algorithmic, and then apply it to the fixed subgroups of a rotationless automorphism.

We continue to use Notation 4.1, Notation 4.3, and Notation 4.7 in this subsection.

4.2.1. *$\partial$-realizable sets in a free group.*

**Definition 4.11.** Let $\mathbb{F}$ be a finite rank free group and $\mathcal{C}$ a finite set of conjugacy classes of $\mathbb{F}$. We say that $\mathcal{C}$ is *$\partial$-realizable* if there exists a surface $\Sigma$ and an identification $\mathbb{F} \simeq \pi_1(\Sigma)$ such that the free homotopy class corresponding to every element of $\mathcal{C}$ is the class of a boundary component of $\Sigma$. We do not allow a class $[c] \in \mathcal{C}$ to be a proper power of a boundary component. A finite multiset $\mathcal{C}$ is $\partial$-realizable if either $\mathcal{C}$ is in fact a $\partial$-realizable set, or if $\mathbb{F} = \langle c \rangle$ and $\mathcal{C}$ is $[c]$ with multiplicity at most 2.

Suppose that $\mathcal{C}$ is $\partial$-realizable and let $\Sigma$ be a surface witnessing this fact. If $[\partial\Sigma]$ is the collection of conjugacy classes of elements of $\pi_1(\Sigma) \simeq \mathbb{F}$ determined by the boundary components of $\Sigma$, then there are two possibilities: either $\mathcal{C} = [\partial\Sigma]$ or else $\mathcal{C} \subsetneq [\partial\Sigma]$. In the latter case, $\mathcal{C}$ will be part of a basis for $\mathbb{F}$, so Whitehead's algorithm (Theorem 2.14) will determine if such an $\mathcal{C}$ is $\partial$-realizable. The following corollary shows that the same holds when $\mathcal{C} = [\partial\Sigma]$ again using Whitehead's algorithm.

**Corollary 4.12.** *There is an algorithm ($\partial$-REALIZABLE?) that determines whether a multiset $\mathcal{C}$ of conjugacy classes is $\partial$-realizable in $\mathbb{F}$.*

*Proof.* By definition a multiset is $\partial$-realizable if either it is a $\partial$-realizable set (not multiset) or if $\mathbb{F} \cong \mathbb{Z}$ and $\mathcal{C}$ is a generator of $\mathbb{F}$ with multiplicity at most 2. This second condition is readily computed, so it remains to give an algorithm for the case that $\mathcal{C}$ is a set.

Observe that a *set* of conjugacy classes $\mathcal{C}$ is $\partial$-realizable if and only if $\mathcal{C}$ is in the $\mathrm{Out}(\mathbb{F})$ orbit of the set of conjugacy classes $[\partial S_{n,i}]$ represented by the boundary of some standard surface $S_{n,i}$. Thus, $\partial$-realizability is computable with Whitehead's algorithm (Theorem 2.14): For each standard surface $S_{n,i}$ of the relevant rank, use Theorem 2.14 to determine if any $|\mathcal{C}|$-subset of $[\partial S_{n,i}]$ and $\mathcal{C}$ are in the same $\mathrm{Out}(\mathbb{F})$-orbit. □

4.2.2. *$\partial$-realizable sets and the fixed subgroups of $\phi$.* We now consider the collection of conjugacy classes of subgroups $\mathrm{Fix}(\phi)$ for the geometric automorphism under consideration. Indeed, computing $\mathrm{Fix}(\phi) = \{[K_1], \dots, [K_l]\}$ from the CT $f\colon G \to G$ can be done easily [13, §8]. On the other hand, computing $\mathrm{Fix}(\phi)$ from $g\colon \Sigma \to \Sigma$ is also straightforward from its Thurston normal form: $\mathrm{Fix}(\phi)$ consists of the conjugacy classes of fundamental groups of subsurfaces on which $g$ restricts to the identity. This second characterization leads to the conclusion developed in this section: that the boundary curves of the fixed subsurfaces can be computed from a CT, and geometricity on the identity components reduces to a question of $\partial$-realizability.

We recall the details of the computation of $\mathrm{Fix}(\phi)$ from a CT $f\colon G \to G$, as a familiarity with this procedure is important in the sequel; the reader is directed to Feighn and Handel [13, §9-11] for complete details of the construction and its computability. The reader may wish to look ahead to Example 9.1.

**Definition 4.13.** Define the graph $\hat{S}(f)$ as follows. Start with the subgraph $\hat{S_1}(f)$ of $G$ consisting of all vertices in $\mathrm{Fix}(f)$ and all fixed edges. Given a linear edge

$E$ of $G$, we have $f(E) = Eu_E^d$ for some $d \neq 0$ and some root-free loop $u_E$ in $G$ that is fixed by $f$ (up to free homotopy). For each such edge, attach a "lollipop" $Y_E$ to $\hat{S}_1(f)$; $Y_E$ is the union of an edge labeled $E$ and a circle labeled $u_E$, which is attached at the initial vertex of $E$, considered as an element of $\hat{S}_1(f)$. For each geometric EG stratum $H_r$ with an indivisible Nielsen path of height $r$, choose one such indivisible Nielsen path $\rho$ (there are only two and they differ by a choice of orientation) and attach an edge path labeled by $\rho$ to $\hat{S}_1(f)$ with endpoints equal to those of $\rho$. The result is $\hat{S}(f)$.

When $v \in \mathrm{Fix}(f)$, we abuse notation and also denote by $v$ the unique vertex of $\hat{S}_1(f) \subset \hat{S}(f)$ labeled by $v$. The component of $\hat{S}(f)$ containing this vertex is denoted by $\hat{S}(f, v)$. Each component of $\hat{S}(f)$ has its fundamental group identified with a subgroup of $\mathbb{F}$ by the immersion determined by edge labels. The collection of conjugacy classes of such subgroups is precisely $\mathrm{Fix}(\phi)$. The graph $\hat{S}(f)$ is clearly computable from the data of a CT and we will use its components as computable representatives of $\mathrm{Fix}(\phi)$.

Heuristically, linear strata in a CT representing a geometric outer automorphism should correspond to Dehn twists in the associated surface. This motivates the following definition, which will record the set of Dehn twist curves in a reducing system for such an automorphism. This definition should be compared to Handel and Mosher's definition of $\mathrm{Twist}(\phi)$ [17, Definition II.2.7]. Taken as a set $\mathcal{L}_K$ is the intersection up to conjugacy of $K$ and $\mathrm{Twist}(\phi)$, however the multiset structure is necessary for determining geometricity.

**Definition 4.14.** For each $[K] \in \mathrm{Fix}(\phi)$, we define a multiset $\mathcal{L}_K$ of conjugacy classes of $K$ as follows ($\mathcal{L}$ is for "linear"). Consider each linear edge $E$ of $f\colon G \to G$ in turn. For each edge, write $f(E) = Eu_E^d$ for some root-free reduced loop $u_E$ in $G$ and some integer $d \neq 0$. Let $v$ and $v'$ be the initial and terminal endpoints of $E$ respectively, both of which are necessarily in $\mathrm{Fix}(f)$. If $v$ is in the component of $\hat{S}(f)$ corresponding to $[K]$ add $Eu_E\overline{E}$ to $\mathcal{L}_K$. If $v'$ is in the component of $\hat{S}(f)$ corresponding to $[K]$ add $u_E$ to $\mathcal{L}_K$.

Our definition of $\mathcal{L}_K$ is computable from a CT representative, for it to be useful we also must know that $\mathcal{L}_K$ is an invariant of $\phi$ independent of the choice of CT. To do this we will make use of an alternate definition, in terms of the *axes* of $\phi$. Recall that a root-free conjugacy class $\mu$ is an *axis of* $\phi$ if there are distinct principal lifts $\Phi, \Psi \in P(\phi)$ that fix a representative $u$ of $\mu$. The number of distinct principal lifts in $P(\phi)$ fixing $u$ is the *multiplicity* of $u$, denoted $m(\mu)$. This does not depend on the particular representative, $\mu$ and its multiplicity are invariants of $\phi$. For a fixed $u$ representing $\mu$ there is a unique *base lift* $\Phi_0 \in P(\phi)$, characterized as corresponding to the unique lift $\tilde{f}_0$ with fixed points in the axis of $u$ and commuting with the covering translation $\tau_u$ [12, pg. 95]. With this notion of base lift, Feighn and Handel connect the axes of an outer automorphism $\phi$ to the linear edges of a CT representative.

**Lemma 4.15** ([12, Lemma 4.40])**.** *Suppose that $\phi$ is forward rotationless and that the unoriented conjugacy class $\mu$ is an axis for $\phi$. Let $f : G \to G$ be a CT representative of $\phi$. Fix a representative circuit $u$ in $G$ for $\mu$ and let $\Phi_0$ be the base lift corresponding to the choices of $u$ and $f$. There is a bijection between the set*

*$\Phi_j \in P(\phi)$ such that $\Phi_j \neq \Phi_0$ and $\Phi_j$ fixes $u$, and the set of linear edges $E_j$ for $f$ such that $f(E_j) = E_j u^d$.*

*Remark* 4.16. The bijection depends on the choice of base lift but the collection $\{\Phi_0, \dots, \Phi_{m(\mu)-1}\} \subset P(\phi)$ depends only on $u$ and $\phi$. Given $\Phi \in P(\phi)$ that fixes $u$, this collection is equal to $P(\phi) \cap \{i_u^n \circ \Phi\}_{n\in\mathbb{Z}}$.

**Lemma 4.17.** *The multisets $\mathcal{L}_K$, as $[K]$ varies over* $\mathrm{Fix}(\phi)$*, depend only on $\phi$ and not on the CT used to compute them.*

*Proof.* We give an alternate definition of $\mathcal{L}_K$ in terms of the principal lifts and axes of $\phi$. (Compare Handel and Mosher's Fact II.2.8 [17].) For each conjugacy class in $\mathrm{Fix}(\phi)$, fix a representative subgroup. We will define a multiset $\mathcal{L}'_K$ for each representative subgroup. For each axis $\mu$ of $\phi$, pick a representative element $u$ and an automorphism $\Phi \in P(\phi)$ fixing $u$. (As we are giving a CT independent definition, $\Phi$ may not be a base lift, but as remarked this will not matter.) The fixed subgroup $K_\Phi$ is conjugate by some $v$ to a representative $K$. For each $\Psi \in P(\phi)$ not equal to $\Phi$ that fixes $u$, the fixed subgroup $K_\Psi$ is conjugate by some $w$ to a representative $K'$ (it is possible $K = K'$ if $\Psi$ and $\Phi$ are isogredient), we add $i_v(u)$ to $\mathcal{L}_K$ and $i_w(u)$ to $\mathcal{L}'_K$. Following this procedure for all axes defines the multisets $\mathcal{L}'_K$, and up to conjugacy these are well-defined and depend only on $\phi$.

We claim that $\mathcal{L}_K = \mathcal{L}'_K$.

Fix representatives of $\mathrm{Fix}(\phi)$. The axes of an automorphism $\phi$ are precisely the root-free conjugacy classes that are suffices of linear NEG edges. Let $\mu$ be an axis of $\phi$ with representative circuit $u$ and linear edges $E_1, \dots E_{m(\mu)-1}$ with suffix $u$. In Feighn and Handel's proof of Lemma 4.15 a bijection is constructed as follows. Let $v$ be the initial (and terminal) vertex of $u$ and fix a lift $\tilde{v} \in \tilde{G}$. The base lift $\Phi_0$ is realized by the lift of $f$ at $\tilde{v}$. Each linear edge $E_j$ has a unique lift $\tilde{E}_j$ terminating at $\tilde{v}$, and the bijection sends $E_j$ to the principal lift $\Phi_j$ of $f$ at the initial vertex of $\tilde{E}_j$. The fixed subgroup of $\Phi_0$ is conjugate via some $v$ to the representative $K$; each $E_j$ contributes a copy of $i_v(u)$ to $\mathcal{L}_K$ and to $\mathcal{L}'_K$ by definition. For each $j$ let $K_j$ be the fixed subgroup of $\Phi_j$, conjugate via some $w$ to its representative $K'$ the edge $E_j$ contributes $i_w(E_j u \overline{E}_j)$ to $\mathcal{L}'_K$ by definition. As verified by Feighn and Handel this correspondence is a bijection [12, Proof of Lemma 4.40]. Thus $\mathcal{L}_K = \mathcal{L}'_K$ is an invariant of $\phi$. □

It follows from the definition of a weak geometric model (in particular the fact that $G_r$ embeds into $Y$) and the EG case of Lemma 2.12 "moving up through the filtration" that each attaching map $\alpha_i$ of a lower boundary component is either a local homeomorphism, or else has image $Ew\overline{E}$ for some linear edge $E$ of $G$ and a closed loop $w$ which is a Nielsen path. We now record the elements of $\mathbb{F}$ corresponding to boundaries of the surfaces of EG strata in $G$.

**Definition 4.18.** Define a multiset $\mathcal{E}_K$ of elements of $[K] \in \mathrm{Fix}(\phi)$ as follows ($\mathcal{E}$ is for exponential). Let $H_r$ be a (necessarily geometric, c.f. Lemma 4.5) EG stratum of $f\colon G \to G$, with associated weak geometric model as above. The upper boundary component is a closed height $r$ indivisible Nielsen path, which is represented in some component of $\hat{S}(f)$. Let $S$ be the surface associated to a weak geometric model $Y$ for $H_r$. The image under $\alpha_i$ of each lower boundary component $\partial_i S$ is either a loop in some component of $\hat{S}(f)$, or an indivisible Nielsen path corresponding to a lollipop in a component $\hat{S}(f)$. The multiset $\mathcal{E}_K$ is the (multiset)-union over EG strata of $f$ of the boundary classes that are represented in the component of $\hat{S}(f)$ corresponding to $[K]$.

The reader should note that the elements of $\mathcal{L}_K$ are always root-free conjugacy classes; the definition asks us to take roots. On the other hand this is not necessarily the case for elements of $\mathcal{E}_K$. See Example 9.1.

**Definition 4.19.** The *candidate boundary multiset* of $[K] \in \mathrm{Fix}(\phi)$, denoted $\mathcal{C}_K = \mathcal{L}_K \dot{\cup} \mathcal{E}_K$ is the multiset union of the two multisets defined previously.

As observed in their definitions, both multisets in the union are computable from a CT $f$ and the graph $\hat{S}(f)$. Thus for each $[K] \in \mathrm{Fix}(\phi)$ (represented as a connected component of $\hat{S}(f)$), the multiset $\mathcal{C}_K$ is computable.

**Lemma 4.20.** *If $\phi$ is a geometric rotationless outer automorphism then for every $[K] \in \mathrm{Fix}(\phi)$, $\mathcal{C}_K$ $\partial$-realizable in $K$.*

*Proof.* Since $\phi$ is geometric, there is a bijection between $\mathrm{Fix}(\phi)$ and the fixed subsurfaces of the Thurston normal form of a realization $g : \Sigma \to \Sigma$. Fix $[K] \in \mathrm{Fix}(\phi)$ and the corresponding subsurface $\Sigma_K$. The components of $\partial\Sigma_K$ are partitioned into 3 types: those that the Thurston normal form Dehn twists around, those that separate $\Sigma_K$ from pseudo-Anosov subsurfaces, and components of $\partial\Sigma$.

Lemma 4.17 characterizes the elements of $\mathcal{L}_K$ in terms of principal lifts; the surface and outer automorphism notions of principal lift coincide, so $\mathcal{L}_K$ is also determined by the Dehn twist boundary curves of $\Sigma_K$.

It follows from Lemma 4.10 and the definition of $\mathcal{E}_K$ that the components of $\partial\Sigma_K$ that separate $\Sigma_K$ from a pseudo-Anosov subsurface are equal to $\mathcal{E}_K$.

Thus, either $\Sigma_K$ is an annulus and $\mathcal{C}_K$ is a multiset with 2 copies of the generator of $\pi_1\Sigma_K$, or $\mathcal{C}_K$ is a $\partial$-realizable set with $\Sigma_K$ witnessing the realization. □

As a consequence we further observe:

**Corollary 4.21.** *If $\phi$ is geometric, then for every $[K] \in \mathrm{Fix}(\phi)$*

(1) *$\mathcal{L}_K$ is a set – the multiplicity of every element is* 1
(2) *no element of $\mathcal{E}_K$ is a proper power.*

## 4.3. **The algorithm.**

**Proposition 4.22.** *Algorithm 1 is correct.*

*Proof.* First Algorithm 1 halts: every procedure used is an algorithm and there are finitely many components in $\hat{S}$ for any CT.

Now suppose $\phi$ is a rotationless outer automorphism. There are two possible results of running Algorithm 1 on $\phi$, we consider them in turn.

No. If the algorithm returns `No` from line 4, then, by the contrapositive of Lemma 4.8 or Corollary 4.9, $\phi$ is not geometric. If the algorithm returns `No` from line 10, then by the contrapositive of Lemma 4.20, $\phi$ is not geometric. In either case, it correctly reports non-geometric.

`Yes`. Let $f$ be the CT for $\phi$ used by the algorithm. Since the algorithm did not return from line 4, we know that every EG stratum of $f$ is geometric, so let $\{S_i\}$ be the set of surface pieces of the weak geometric models for these strata, with pseudo-Anosov homeomorphisms $g_i$; The attaching maps determine an identification of each $S_i$ with a conjugacy class of subgroup of $\mathbb{F}$ and so a marking up to choice of

**Algorithm 1** Decide if a rotationless outer automorphism is geometric.

```
1: procedure RotationlessGeometric?(φ)
2:     f ← ComputeCT(φ)
3:     if ZeroStrata?(f) ∨ NonlinearNEG?(f) ∨ NongeometricEG?(f) then
4:         return No
5:     end if
6:     Ŝ ← ComputeŜ(f)
7:     for K ∈ ConnectedComponents(Ŝ) do
8:         𝒞 ← ComputeCandidateBoundary(K, f)
9:         if ¬∂-realizable?(𝒞, K) then
10:            return No
11:        end if
12:    end for
13:    return Yes
14: end procedure
```

basepoint. Further, since the algorithm did not return from line 10, we know that for each $[K] \in \mathrm{Fix}(\phi)$ the multiset $\mathcal{C}_K$ is $\partial$-realizable in $K$; let $S_K$ be a surface witnessing this realization, marked by the choice of representative $K$.

Each element of $\mathcal{C}_K$ matches to a unique boundary component of a surface $S'_K$ or $S_i$, according to whether it came from $\mathcal{E}_K$ or $\mathcal{L}_K$. Since $\mathcal{C}_K$ is $\partial$-realizable, this identifies each boundary component of $S_K$ with a unique boundary component of some $S'_K$ or $S_i$. Moreover, if two EG pieces $S_i$ and $S_j$ have boundaries $c_i$ and $c_j$ which have conjugate proper powers, then there is some $[K] \in \mathrm{Fix}(\phi)$ such that $\{c_i, c_j\} \subset \mathcal{E}_K$. It follows from $\partial$-realizability that $c_i$ and $c_j$ are not proper powers and indeed $K = \langle c_i \rangle = \langle c_j^{\pm} \rangle$. So we can glue the surfaces $S_K$ and $S_i$ according to the boundary identifications coming from their boundary conjugacy classes in $\mathbb{F}$, in a way that extends the marking. Call the resulting marked surface $\Sigma$.

Define a homeomorphism $g : \Sigma \to \Sigma$ by $g_i$ on each $S_i$ component, the identity on each $S_K$ component, and a Dehn-twist by the twist power of $\phi$ around each curve from each $\mathcal{L}_K$. By construction $g_*$ and $\phi$ have the same set of laminations and the same twist coordinates, so by the Recognition Theorem [12] for $\mathrm{Out}(\mathbb{F})$ we conclude $g_* = \phi$, that is $\phi$ is geometric. □

**Porism 4.23.** *There is an algorithm* RotationlessGeometricWitness *that takes as input a geometric rotationless* $\phi \in \mathrm{Out}(\mathbb{F})$ *and outputs a marked surface* $\Sigma$ *and distinguished subsurface* $Q$ *such that* $\phi$ *is realized by a homeomorphism* $g$ *on* $\Sigma$, $g|_{\Sigma \setminus Q} = \mathrm{id}$ *and* $g$ *is not isotopic to the identity on any subsurface of* $Q$.

*Proof.* Observe that in the `Yes` case of the proof of Proposition 4.22 the proof describes how to construct $\Sigma$ from data computed from a CT for $\phi$, and that $Q = \Sigma \setminus (\sqcup_{[K]\in \mathrm{Fix}(\phi)} S_K^{\circ}$ is also computable from data computed from a CT for $\phi$. □

*Remark* 4.24. A consequence of Corollary 3.11 and lemma 4.10 is that the subsurface $Q$, carrying the pseudo-Anosov pieces of $\phi$ is determined by the outer automorphism $\phi$. This observation is essential for the root-finding algorithm.

## 5. Manipulating partially geometric surface pairs

Given an outer automorphism $\phi$ with rotationless power $\phi^N$, we can use Algorithm 1 to produce a marked surface pair $(\Sigma, Q)$ such that $\phi^N$ is geometric on $\Sigma$ realized by a homeomorphism $g$. Moreover $g_{\Sigma \setminus Q^\circ} = \mathrm{id}$. (If $\phi$ is finite order this produces an empty $Q$.) The subsurface $Q$ is an invariant of $\phi^N$, and a necessary condition for $\phi$ to be geometric is that $\phi$ is partially geometric on some other pair $(\Sigma', Q)$. Here we need to be precise by what we mean by $(\Sigma', Q)$.

**Definition 5.1.** Two marked surface pairs $(\Sigma, Q)$, $(\Sigma', Q')$ are *outer-equivalent* if $Q$ is homeomorphic to $Q'$ and there is a difference-of-markings map $h \colon \Sigma \to \Sigma'$ such that $h|_Q$ is a homeomorphism and the induced outer automorphism $h_*$ restricts to the identity on $\pi_1 Q$. For convenience we will abuse notation and often refer to outer-equivalent pairs $(\Sigma, Q)$ and $(\Sigma', Q)$.

Observe that if $\phi$ is a partially geometric outer automorphism on $(\Sigma, Q)$ then $\phi$ is also partially geometric on every outer-equivalent $(\Sigma', Q)$.

The primary way we will obtain outer-equivalent pairs is by deleting a connected component $K \subset \Sigma \setminus Q^\circ$ and re-attaching a different surface $K'$ along $\partial K$ such that $[\pi_1 K] = [\pi_1 K']$. We refer to this operation as *replacing a subsurface*, $K'$ is the *replacement* for $K$ and $\Sigma'$ is obtained from $\Sigma$ by *replacing* $K$.

*Remark* 5.2. If $\Sigma$ is triangulated and $K$ is a sub-triangulation replacing $K$ with a triangulated $K'$ is a computable operation.

It is clear this operation will be useful in determining if the finite-order behavior on the identity components of Thurston normal form for a rotationless power are geometric. It turns out that this operation is also useful in developing an algorithm to decide if an outer automorphism is partially geometric, so we treat the notion in this section before continuing with the algorithm.

**Lemma 5.3.** *An outer automorphism $\phi$ is partially geometric on a pair $(\Sigma, Q)$ if and only if $\partial Q$ is $\phi$-invariant and there exists an outer-equivalent surface pair $(\Sigma', Q)$ and a subsurface $Q' \supseteq Q$ such that every connected component of $\partial Q'$ has nontrivial intersection with $\partial \Sigma'$ and $\phi$ is partially geometric on $(\Sigma', Q')$. Moreover this extension is computable and $(\Sigma', Q')$ does not depend on $\phi$.*

*Proof.* First, we prove the forward direction. Since $\phi$ is partially geometric on $(\Sigma, Q)$, the boundary $\partial Q$ is $\phi$-invariant by definition.

In each component of $\Sigma \setminus Q$ contains a boundary curve of $\Sigma$ this is straightforward: for each component $c$ of $\partial Q$ that does not already meet $\partial \Sigma$ choose an arc connecting $c$ to $\partial \Sigma$, and chose this family of arcs to be disjoint. Then set $Q'$ to be the union of $Q$ and regular neighborhoods of these arcs. Since $Q'$ is isotopic to $Q$ we can modify a geometric witness for $\phi$ by a homotopy to obtain a geometric witness for $\phi$ on $(\Sigma, Q')$.

If a component $R$ of $\Sigma \setminus Q$ does not meet $\partial \Sigma$ and the fundamental group is not generated by the components of $\partial Q$ meeting $R$, replace each surface in the $\phi$-homotopy orbit of $R$ with a copy of a surface $R'$ with boundary and fundamental group of the same rank. By construction $\phi$ induces a homotopy equivalence of $\Sigma'$, and we are now in the previously considered case. If the component $R$ has fundamental group generated by the boundary, then by the Dehn-Nielsen-Baer theorem $\phi$ is homotopic to a homeomorphism on $R$, so adjust the realization of $\phi$ and take $Q' = Q \cup R$.

Observe that all of these operations can be done with finite refinements of finite triangulations, so the extension is computable.

Conversely, if such an extension exists then since $\partial Q$ is $\phi$-invariant $\phi$ has a realization as a homotopy equivalence of $\Sigma'$ that is a homeomorphism of $Q$, so $\phi$ is partially geometric on $(\Sigma', Q)$ and hence on $(\Sigma, Q)$. □

## 6. Partial geometricity and the core

In this section we give a criterion for an outer automorphism $\phi$ to be partially geometric on a surface pair $(\Sigma, Q)$ in terms of the Guirardel core. This criterion is computable using Behrstock, Bestvina, and Clay's [1] algorithm for computing certain cores, which leads to an algorithm for testing partial geometricity (Corollary 6.9). The starting point for this analysis is Guirardel's motivating observation that if $\gamma, \gamma'$ are a pair of filling curves on a closed surface $\Sigma$, then the core of the Bass-Serre trees coming from the induced splittings along $\gamma$ and $\gamma'$ is the universal cover of the square tiling dual to their minimalgeneral position intersection. Similar square tilings are central, leading to a definition.

**Definition 6.1.** A nonempty connected marked square complex $m_X\colon \mathfrak{R} \to X$ is *surface type* if there is an embedding $\eta\colon X \to \Sigma$ for a marked surface $(\Sigma, m_\Sigma)$ such that $m_{\Sigma *}^{-1}\eta_* m_{X*} = \mathrm{id} \in \mathrm{Out}(\mathbb{F})$. A nonempty subcomplex $Y \subseteq X$ of a marked square complex is *surface type* if each component of $Y$ is surface type with the induced marking.

A free $\mathbb{F}$ action on a connected square complex $\tilde{X}$ is *surface type* if it is the universal cover of a surface type marked square complex $X/\mathbb{F}$. A non-empty subcomplex $Y \subseteq X$ is *surface type* if it is surface type with respect to the $\mathrm{Stab}(Y)$ action.

*Remark* 6.2. We allow square complexes that themselves are not homeomorphic to surfaces. A rose is a surface type square complex. Two squares joined at a single vertex is also a surface type square complex.

**Definition 6.3.** Suppose $X$ is a surface type marked square complex. A *surface boundary class* of $X$ is a conjugacy class $[\gamma] \subset \mathbb{F}$ that can be represented by a connected component of the boundary of a regular neighborhood of an embedding $X$ in some marked surface $\Sigma$. The set of surface boundary classes for a particular embedding is denoted $\partial_\Sigma X$. This notion is similarly applied to complexes with free $\mathbb{F}$ action.

Note that the definition of surface boundary class depends on the embedding, a surface type complex may embed in more than one non-homeomorphic marked surface.

To connect surface structures to square complexes, we also make use of a standard combinatorial model of surfaces.

**Definition 6.4.** Suppose $G \subseteq \Sigma$ is a graph embedded in a surface $\Sigma$. This embedding induces a *ribbon structure* on $G$: an assignment of a rectangle to each edge, a polygon to each vertex, and gluing data attaching these rectangles to polygons such that the result is a cell complex homeomorphic to a regular neighborhood of $G$ in $\Sigma$. The *dual arc system* to a ribbon structure is the collection of arcs transverse to $G$, each arc intersecting a unique edge exactly once and joining the unglued sides of the corresponding rectangle.

The relationship between partial geometricity and the Guirardel core is explored in the following two propositions. The two propositions are presented separately, as the conclusion of Proposition 6.5 is significantly stronger than the hypothesis of Proposition 6.6, but together they imply a characterization of partial geometricity in Corollary 6.8.

**Proposition 6.5.** *Suppose $\phi$ is a partially geometric automorphism in* $\mathrm{Out}(\mathbb{F})$*, realized on $(\Sigma, Q)$, such that every component of $\partial Q$ meets $\partial\Sigma$. Let $G \subseteq \Sigma$ be a spine for $\Sigma$ with a subgraph $K$ that is a spine of $Q$. Let $T$ be the universal cover of $G$. For each pair of connected components $K_1, K_2$ of $K$ and each elevation $T_i$ of $K_i$ to $T$, the subcomplex $(T_1 \times T_2\phi) \cap \mathrm{Core}(T, T\phi)$ is surface type. Here $T_2\phi$ is the minimal subtree for $\phi(\mathrm{Stab}(T_2))$.*

*Proof.* Let $\mathcal{A}$ be the collection of arcs dual to the ribbon structure of $K$, representatives chosen so that the endpoints of each $\alpha \in \mathcal{A}$ are on $\partial\Sigma$. This is possible since each component of $\partial Q$ meets $\partial\Sigma$ and the dual arcs start and end on $\partial Q$.

Let $\Sigma_d$ be the 2-complex obtained by doubling $\Sigma$ along $(\Sigma \setminus Q) \cup \partial Q$. By Van Kampen's theorem[1], the fundamental group is

$$\pi_1(\Sigma_d) = \pi_1(\Sigma) *_{\pi_1((\Sigma\setminus Q)\cup\partial Q)} \pi_1(\Sigma).$$

We fix an identification of $\mathbb{F}$ with one of the two $\pi_1(\Sigma)$ factors of the amalgam, which will be used throughout the proof. The double of $\mathcal{A}$, denoted $\mathcal{A}_d$, is a a collection of embedded essential circles in $\Sigma_d$ that do not pass through any non-manifold points as no point of $\mathcal{A}$ is contained in $\Sigma \setminus Q$ by construction. Let $T$ be the universal cover of $G$, and let $A_d$ be the Bass-Serre tree of the splitting of $\pi_1(\Sigma_d)$ defined by $\mathcal{A}_d$. By construction, the minimal subtree for the fixed $\mathbb{F}$ action, $\bar{T} = A_d^{\mathbb{F}}$ of $A_d$ is the cover of the graph of groups obtained by collapsing the complement of $K$ in $G$. The tree $\bar{T}$ is a collapse of $T$. Indeed, for $\gamma \in \pi_1(\Sigma_d)$ the translation length of $\gamma$ on $A_d$ is equal to the intersection number of $\gamma$ with $\mathcal{A}_d$, and for $\gamma \in \mathbb{F}$ this is equal to the number of edges of $K$ in a cyclically reduced path in $G$ representing $\gamma$, again by construction.

Let $g : \Sigma \to \Sigma$ be a partial geometric witness for $\phi$. Then $g(K)$ is also a spine for $Q$. There is a dual arc system $\mathcal{B} = g(\mathcal{A})$ dual to the induced ribbon graph structure on $g(K)$. Since $\partial Q$ meets $\partial\Sigma$ and $\partial Q$ is $g$-invariant we can take the endpoints of $\mathcal{B}$ to lie on $\partial\Sigma$. Moreover, since $g$ preserves the decomposition of $\Sigma$, it induces a map $\hat{g} \colon \Sigma_d \to \Sigma_d$ on the double, and $\hat{g}(\mathcal{A}_d) = \mathcal{B}_d$. As before, the double of the dual arc system $\mathcal{B}_d$ is a collection of embedded essential circles in $\Sigma_d$ that do not pass through any non-manifold points. Hence they induce a splitting of $\pi_1(\Sigma_d)$ and a Bass-Serre tree $B_d$. The minimal subtree $B_d^{\mathbb{F}}$ is the Bass-Serre tree for the graph of groups obtained by collapsing the complement of $K\phi$ in $G\phi$ (recall the action on the right is by twisting the marking, we are not using a topological representative of $\phi$ here). Observe that $B_d^{\mathbb{F}} = \bar{T}\phi$ and is a collapse of $T\phi$, and that we have the following commutative diagram of $\mathbb{F}$ trees.

$$\begin{array}{ccc} T & \xrightarrow{\phi} & T\phi \\ \downarrow & & \downarrow \\ \bar{T} & \xrightarrow{\phi} & \bar{T}\phi \end{array}$$

[1] If $(\Sigma\setminus Q)\cup\partial Q$ has more than one component, this is a shorthand for the two-vertex, multi-edge graph of groups splitting of $\pi_1(\Sigma_d)$ induced by $(\Sigma \setminus Q) \cup \partial Q$

Guirardel [15, Section 2.2 Example 3] shows that the core of two Bass-Serre trees dual to curve systems on a surface is surface type and identifies the cell structure as the universal cover of the square complex dual to the intersection pattern, with connected components of the non-augmented core corresponding to the subsurfaces filled by the curve system. In fact, while Guirardel analyzes surfaces, the argument is local and applies to any collection of curves on a surface subset of a 2-complex. For clarity, we present the argument in full.

Let $p\colon \tilde{\Sigma}_d \to \Sigma_d$ be the universal cover of $\Sigma_d$. There are equivariant maps $f\colon \tilde{\Sigma}_d \to A_d$ and $g\colon \tilde{\Sigma}_d \to B_d$ since these trees are dual to the collections of curves. The components of $Q_d$ are closed surfaces with infinite fundamental group. Fix a constant curvature metric on each component of $Q_d$. This induces a metric on each component of $\tilde{Q}_d = p^{-1}(Q_d)$ making them either a hyperbolic or euclidean plane.

Suppose $a \in A_d$ is an edge of $A_d$ dual to an element $\alpha \in \pi_1(\Sigma_d)$ and $b \in B_d$ is an edge of $B_d$ dual to $\beta \in \pi_1(\Sigma_d)$. Then, by construction, $\alpha$ and $\beta$ are represented by closed curves contained in the subsurface $Q_d$. Let $\tilde{\alpha}$ and $\tilde{\beta}$ be the axes of $\alpha$ and $\beta$ in $\tilde{\Sigma}_d$ (use a single representative parallel line in the case that $\alpha$ or $\beta$ represents a curve in a torus component of $Q_d$). Analyzing the axes of other elements allows us to compute the partition for applying the 4-sets condition by analyzing connected components in the universal cover and applying the maps $f$ and $g$. There are four cases:

**Case 1:** $\tilde{\alpha}$ and $\tilde{\beta}$ are in distinct components of $\tilde{Q}_d$. Using the maps $f$ and $g$ it is clear that either $\partial^+(a)$ or $\partial^-(a)$ contains one of $\partial^\pm(b)$, so $a$ and $b$ do not satisfy the 4-sets condition of Definition 2.17.

**Case 2:** $\tilde{\alpha}$ and $\tilde{\beta}$ are in a common hyperbolic plane component of $\tilde{Q}_d$ but disjoint. Again, using the maps $f$ and $g$ it is clear that either $\partial^+(a)$ or $\partial^-(a)$ contains one of $\partial^\pm(b)$, so $a$ and $b$ do not satisfy the 4-sets condition.

**Case 3:** $\tilde{\alpha}$ and $\tilde{\beta}$ are in a common hyperbolic plane piece inside $\tilde{\Sigma}_d$ and intersect. Then, using the maps $f$ and $g$ we can see that each of the four sets

$$\partial^\pm(a) \cap \partial^\pm(b)$$

have non-empty intersection, since there are curves which witness the non-empty intersection in this component of $\partial Q_d$.

**Case 4:** $\tilde{\alpha}$ and $\tilde{\beta}$ are in a common euclidean plane component of $\tilde{Q}_d$. Then either $\tilde{\alpha}$ is parallel to $\tilde{\beta}$, in which case, up to orientation $\partial^+(a) \cap \partial^-(b) = \emptyset$, or $\tilde{\alpha}$ and $\tilde{\beta}$ intersect, in which case $a$ and $b$ satisfy the 4-sets condition with witnesses coming from the plane as in the hyperbolic case.

Thus, the squares of $\operatorname{Core}(A_d, B_d)$ are in one-to-one correspondence with $\pi_1(\Sigma_d)$ orbits of intersection points of the curve systems $\mathcal{A}_d$ and $\mathcal{B}_d$. We conclude that the squares of the core are the lift to the universal cover $\tilde{\Sigma}_d$ of the surface type square system dual to the intersection of $\mathcal{A}_d$ and $\mathcal{B}_d$.

Further, restricting the $\pi_1(\Sigma_d)$ action to the $\mathbb{F}$ action coming from the fixed copy of $\pi_1(\Sigma)$ in the van Kampen decomposition, we obtain the equivariant inclusion

$$\operatorname{Core}(\bar{T}, \bar{T}\phi) \subseteq \operatorname{Core}(A_d, B_d).$$

Since $A_d^{\mathbb{F}} = T$ and $B_d^{\mathbb{F}} = T\phi$, for each pair of components $K_1, K_2$ of $K$, and lifts $T_1, T_2$ to $T$, the intersection

$$(T_1 \times T_2\phi) \cap \operatorname{Core}(T, T\phi) = (T_1 \times T_2\phi) \cap \operatorname{Core}(A_d^{\mathbb{F}}, B_d^{\mathbb{F}})$$

is either empty or contained in a surface subset of $\tilde{\Sigma}_d$; i.e. the intersection is surface type, as required. □

**Proposition 6.6.** *Suppose there is a marked graph $G$ with covering tree $T$ and a subgraph $K$ that carries a $\phi$-invariant free factor system for $\phi \in \mathrm{Out}(\mathbb{F})$. Suppose*

- *the subcomplex $Y = \pi^{-1}(K) \subset \mathrm{Core}(T \times T\phi)/\mathbb{F} = X$ is surface type,*
- *there is an embedding such that the surface boundary classes of $Y$ are $\phi$-invariant,*
- *the graph of spaces decomposition of $X$ into $Y$ and $X \setminus Y^\circ$ is $\phi$-invariant.*

*Then $\phi$ is partially geometric on a marked surface pair $(\Sigma, Q)$.*

*Proof.* First, by hypothesis there is a homotopy equivalence $f\colon X \to X$ representing $\phi$ such that $Y$ and $X \setminus Y^\circ$ are $f$-invariant.

Let $Q$ be the regular neighborhood of an embedding of $Y$ into a surface $\Sigma$ such that $\partial Q$ is a set of $\phi$-invariant conjugacy classes. Such an object exists by hypothesis, and we may extend $f$ to $X \cup Q$. By the Dehn-Nielsen-Baer theorem (Theorem 2.3), the restriction $f|_Q$ is homotopic to a map $g$ such that $g|_Q$ is a homeomorphism. Extending this homotopy by the constant homotopy on $X \setminus Q$ we obtain a homotopy equivalence $g\colon X \to X$ such that $Q$ and $X \setminus Q$ are $g$-invariant and $Q|_g$ is a homeomorphism.

Finally, the pair $(X, Q)$ is homotopy equivalent rel $Q$ to a surface pair $(\Sigma, Q)$. Since $Q$ and $X \setminus Q$ are $g$-invariant, $g$ induces a homotopy equivalence $h$ of $\Sigma$ such that $h|_Q = g|_Q$ is a homeomorphism, as required. □

Lemma 5.3 allows us to combine Proposition 6.5 and Proposition 6.6 to characterize partially geometric outer automorphisms.

**Theorem 6.7.** *An outer automorphism $\phi \in \mathrm{Out}(\mathbb{F})$ is partially geometric if and only if there exists a marked graph $G$ with cover $T$ such that $X = \mathrm{Core}(T, T\phi)/\mathbb{F}$ has a surface type subcomplex $Y$ where $Y$ has $\phi$-invariant surface boundary classes and the splitting induced by $X \setminus Y$ and $Y$ is $\phi$-invariant.*

**Corollary 6.8.** *An automorphism in $\mathrm{Out}(\mathbb{F})$ is geometric if and only if there exists some free simplicial $\mathbb{F}$-tree $T$ such that $\mathrm{Core}(T, T\phi)$ is geometric and has $\phi$-invariant surface boundary classes.*

*Proof of Theorem 6.7.* If $\phi$ is partially geometric on $(\Sigma, Q)$ then, by Lemma 5.3 it is partially geometric on a surface pair $(\Sigma', Q')$ where every component of $\partial Q'$ intersects $\partial \Sigma'$. Let $G$ be a spine for $\Sigma'$ with a subgraph $K$ that is a spine for $Q'$ and covering tree $T$. By Proposition 6.5 the core quotient $X = \mathrm{Core}(T, T\phi)/\mathbb{F}$ has a surface type subcomplex $Y = \pi^{-1}(K)$ that has $\phi$-invariant surface boundary classes and induces a $\phi$-invariant splitting.

The converse is exactly the content of Proposition 6.6. □

Finally, the criterion of Theorem 6.7 is computable. Moreover, it is computable for a specific desired subgraph $K$, which allows us to determine if an outer automorphism is partially geometric on a given marked surface pair.

**Corollary 6.9.** *Given an outer automorphism $\phi$, a marked surface $\Sigma$ and a subsurface $Q$ there is an algorithm* PartiallyGeometric? *to decide if $\phi$ is partially geometric on $(\Sigma, Q)$.*

*Proof.* It is algorithmic to verify if $\partial Q$ is $\phi$-invariant, and report `No` if not. Next use Lemma 5.3 to compute an outer-equivalent pair $(\Sigma', Q)$ and an extension $Q' \supset Q$ such that every component of $\partial Q'$ meets $\partial\Sigma'$. By Lemma 5.3 $\phi$ is partially geometric on $(\Sigma, Q)$ if and only if it is partially geometric on $(\Sigma', Q')$. By Theorem 6.7 and Proposition 6.5, $\phi$ is partially geometric on $(\Sigma', Q')$ if and only if for any spine $G$ of $\Sigma'$ with subgraph $K$ carrying $Q'$, the subcomplex $\pi^{-1}(K)$ of the core quotient $\mathrm{Core}(T, T\phi)/\mathbb{F}$ is surface type with $\phi$ invariant surface boundary and splitting.

It is algorithmic to compute a desired spine $G$ and subgraph $K$. Moreover the core quotient $\mathrm{Core}(T, T\phi)/\mathbb{F}$ where $T$ is the cover of $G$ is finite can be computed using Behrstock, Bestvina, and Clay's algorithm [1]. Finally, it is algorithmic to verify if a finite subcomplex is surface type, there are finitely many possible surface boundary classes, and it is algorithmic to test $\phi$ invariance. □

Corollary 6.9 alone is not sufficient to determine if an outer automorphism is geometric algorithmically, as it does not offer any method for searching the space of marked surfaces in finite time. However, in combination with Algorithm 1 it can be used to reduce the problem of determining if an outer automorphism is geometric to an extension problem, which we explore next.

## 7. From partially geometric to geometric

With a method to test if an outer automorphism is partially geometric in hand, we turn our attention to deciding an extension problem: given a partially geometric outer automorphism $\phi$ realized by $g$ on the marked pair $(\Sigma, Q)$ such that

$$g^N|_{\Sigma\setminus Q^\circ} \sim \mathrm{id},$$

can it be extended to be partially geometric on a larger subsurface, partially after replacing components of $\Sigma \setminus Q$. We will solve the extension problem at hand by giving an algorithm to replace surfaces (or certify that no geometric replacements exist).

Our first extension result deals with the special case of finding a geometric replacement for a $\phi$-invariant splitting factor.

**Lemma 7.1.** *Suppose $\phi \in \mathrm{Out}(\mathbb{F})$ is partially geometric on $(\Sigma, Q)$. Let $R \subset \Sigma \setminus Q^\circ$ be a connected component of the complement such that $\phi([\pi_1 R]) = [\pi_1 R]$. There is an algorithm to produce a finite list $R_1, \ldots, R_s$ of replacements for $R$ such that*

- *$\phi$ is partially geometric on $(\Sigma_i, R_i \cup Q)$, where $\Sigma_i$ is the result of replacing $R$ with $R_i$;*
- *all possible replacements $R'$ such that $\phi$ is partially geometric on $(\Sigma', R' \cup Q)$ are listed, up to the action of the $\mathrm{Out}(\pi_1 R)$ stabilizer of $[\partial Q] \cap [\partial R]$.*

*Proof.* Let $\mathcal{C} = [\partial Q \cap \partial R]$ be the set of conjugacy classes joining $R$ to $Q$. Let $n = \mathrm{rank}(\pi_1 R)$ and $m\colon \mathfrak{R}_n \to R$ be the induced geometric marking from the inclusion $\pi_1 R \to \mathbb{F}$. Observe that $\mathcal{C}$ is $\partial$-realizable, and that *any* $\partial$-realization of $\mathcal{C}$ can be used as a replacement for $R$.

Suppose there is a replacement $R'$ of $R$ such that $\phi$ is partially geometric on $(\Sigma', R' \cup Q)$, the result of replacing $R$ with $R'$ in $\Sigma$. Then $\phi|_R$ in the $\mathrm{Out}(R)$ conjugacy class of a finite order mapping class for of the standard surface $S_{n,i}$ homeomorphic to $R$. Moreover, the conjugating element $\theta$ satisfies $\theta(\mathcal{C}) \subseteq [\partial S_{n,i}]$.

Conversely, suppose $\phi|_R$ is conjugate by $\theta$ in $\mathrm{Out}(R)$ to some $\alpha$ which is a finite order mapping class on a standard surface $S_{n,i}$, and $\theta(\mathcal{C}) \subseteq [\partial S_{n,i}]$. Then

$(S_{n,i}, \theta)$ is a marked replacement $R'$ for $R$ such that $\phi|_{R'}$ is geometric, realized by a homeomorphism representative of $\alpha$. Thus $\phi$ is partially geometric on $(\Sigma', R' \cup Q)$ as desired.

This condition allows us to use the equivariant Whitehead algorithm of Krstic, Lustig, and Vogtmann (Theorem 2.15). The restriction $\alpha = \phi|_R$ is finite order. For each standard surface $S_{n,i}$, enumerate representatives of the finitely many $\mathrm{Map}(S_{n,i})$ conjugacy classes of finite order mapping classes as $\alpha_{n,i,j}$. For each $\alpha_{n,i,j}$ use the equivariant Whitehead algorithm to decide if there is some $\theta \in \mathrm{Out}(R)$ such that $\theta\alpha\theta^{-1} = \alpha_{n,i,j}$ and $\theta(\mathcal{C}) \subset [\partial S_{n,1}]$. Use each result to report a replacement $R'$; if none are found report the empty list. ☐

As a consequence, we obtain an algorithm to handle the general finite-order case.

**Corollary 7.2.** *There is an algorithm* FiniteOrderGeometric? *to decide if a finite order outer automorphism is geometric.*

*Proof.* Use the above algorithm with $Q = \emptyset$. ☐

Note that, if $\phi$ is partially geometric on $(\Sigma, Q)$, a witness for $\phi$ homotopy permutes the complementary components of $Q$. So, it remains to handle the replacement of entire $\phi$ orbits.

**Lemma 7.3.** *Suppose $\phi \in \mathrm{Out}(\mathbb{F})$ is partially geometric on $(\Sigma, Q)$, realized by $g \colon \Sigma \to \Sigma$. Let $R_0 \subset \Sigma \setminus Q^\circ$ be a connected component of the complement such that $[\pi_1 R_0]$ is $\phi$-periodic with period $k$ and $\phi^k|_{\pi_1 R_0}$ is finite-order. Let $R_0, \ldots, R_{k-1}$ be the components visited by the forward orbit of $R_0$ under $g$.*

*There is an algorithm* GeometricOrbitReplacement? *that takes $\phi$, $\Sigma$, and $R_0$ and decides if there are replacements $R'_0, \ldots, R'_{k-1}$ such that $\phi$ is partially geometric on $(\Sigma', Q \cup R')$ where $R' = \bigcup_{i=0}^{k-1} R_i$. Moreover there is a procedure to produce $\Sigma'$ if it exists.*

*Proof.* First, apply Lemma 7.1 to $\phi^k$ and $R_0$, to obtain a list of candidate replacements $\{R_{0,i}\}$. If this list is empty, report `No`; if $\phi$ is geometric then $\phi^k$ will have at least one geometric replacement for $R_0$, and finitely many up to the action of $\mathrm{Out}(\pi_1 R_0)$.

Observe that if there is a family of replacements as in the hypotheses, $\phi([\partial R'_0])$ is a $\partial$-realizable subset of $[\pi_1 R_1]$. Since $\partial Q$ is $\phi$-invariant, a $\partial$-realization of $\phi([\partial R'_0])$ can replace $R_1$.

Moreover, $R'_0$ is homeomorphic to a candidate replacement $R_{0,i}$. Let $m', m_i$ be the respective markings so that $m = \theta m'$ for some $\theta \in \mathrm{Out}(\pi_1 R_0)$, then

$$\phi m = \phi\theta m' = \phi\theta\phi^{-1}\phi m'.$$

Thus $\phi m'$ and $\phi m_i$ differ by the action of $\mathrm{Out}(\pi_1 R_1)$. We deduce that there is a family of replacements as in the hypothesis if, and only if, for some candidate replacement $R_{0,i}$, the set $\phi^j([\partial R_{0,i}])$ is $\partial$-realizable in $\pi_1 R_j$ for all $0 < j < k$.

To finish the algorithm, for each candidate replacement $R_{0,i}$, check

$$\partial\text{-REALIZABLE?}(\phi^j([\partial R_{0,i}]), \pi_1 R_j),$$

for each $0 < j < k$. If each iterate is $\partial$-realizable, report `Yes` and provide the $\partial$-realizations of the iterates as the desired replacements. Otherwise, report `No`. Remark 5.2 assures us that $\Sigma'$ can be computed if desired. ☐

## 8. Deciding geometricity in general

**Algorithm 2** Decide if an outer automorphism is geometric.

1: **procedure** GEOMETRIC?($\phi$)
2: $\psi \leftarrow$ ROTATIONLESSPOWER($\phi$)
3: **if** $\psi = \mathrm{id}$ **then**
4: **return** FINITEORDERGEOMETRIC?($\phi$)
5: **end if**
6: **if** $\neg$ROTATIONLESSGEOMETRIC?($\psi$) **then**
7: **return** `No`
8: **end if**
9: $(\Sigma_0, Q_0) \leftarrow$ ROTATIONLESSGEOMETRICWITNESS($\psi$)
10: $(\Sigma, Q) \leftarrow$ BOUNDARYCONTACTEXTENSION($\Sigma_0$,$Q_0$)
11: **if** $\neg$PARTIALLYGEOMETRIC?($\phi$,$\Sigma$,$Q$) **then**
12: **return** `No`
13: **end if**
14: **for** $R \in$ CONNECTEDCOMPONENTS($\Sigma \setminus Q^\circ$) **do**
15: **if** $\neg$GEOMETRICORBITREPLACEMENT?($\phi$,$\Sigma$,$R$) **then**
16: **return** `No`
17: **end if**
18: **end for**
19: **return** `Yes`
20: **end procedure**

**Theorem 8.1.** *Algorithm 2 is correct.*

*Proof.* First, Algorithm 2 halts: there are finitely many connected components of $\Sigma \setminus Q^\circ$ as $\Sigma, Q$ are compact surfaces.

Next, Algorithm 2 reports the correct result. If $\phi$ is geometric, then any power is geometric; thus if the `No` result comes from line 7 it is correct. Next, since $Q$ is the subsurface where the rotationless power $\phi^N$ is non-identity, it is determined up to topological type and marking by Corollary 3.11 and Lemma 4.10, so a necessary condition for $\phi$ to be geometric is that $\phi$ is relatively geometric relative to $(\Sigma, Q)$; thus line 12 is correct. By Lemma 7.3, if $\phi$ is geometric for each homotopy orbit of connected complementary component there is a geometric replacement. So the algorithm correctly reports `No` at line 16 if any orbit fails to have a geometric replacement.

Finally, if $\phi$ is relatively geometric on $(\Sigma, Q)$ and every homotopy orbit of connected complementary component has a geometric replacement, then $\phi$ is geometric on $\Sigma'$, the result of conducting all of these geometric replacements, so the report of `Yes` is correct. □

**Porism 8.2.** *There is an algorithm* GEOMETRICWITNESS *that takes as input an outer automorphism* $\phi \in \mathrm{Out}(\mathbb{F})$ *and produces a marked surface* $\Sigma$ *where* $\phi$ *is realized by a homeomorphism.*

*Proof.* If $\phi$ is geometric, the surface $\Sigma'$ constructed in the proof of Theorem 8.1 is the desired marked surface. The process of finding and attaching geometric replacements is algorithmic (Lemma 7.3). □

## 9. Illustrative examples

**Example 9.1.** Consider the genus two surface with two punctures $\Sigma = \Sigma_{2,2}$ depicted in Figure 1 thought of as a union of a pair of pants $\Sigma' = \Sigma^2_{1,0}$ and a torus with one puncture and two boundary components $\Sigma'' = \Sigma^2_{1,1}$, glued appropriately. Let $h\colon \Sigma'' \to \Sigma''$ be a pseudo-Anosov homeomorphism and let $g\colon \Sigma \to \Sigma$ be defined by $g = h \circ D^4_\beta \circ D^2_\gamma$ where $D_c$ is the right Dehn twist about the curve $c$.

There is a nested sequence of $g$-invariant subsurfaces (starting with a neighborhood of $\alpha$) $N_\alpha \subset N_\alpha \sqcup N_\beta \subset \Sigma' \subset \Sigma$ that, on the level of fundamental groups, determines a nested sequence of $g_*$-invariant free factor systems of $\pi_1\Sigma = \mathbb{F}_5$. A CT $f\colon G \to G$ realizing this nested sequence will have six strata (see the figure for a schematic): $H_i$ for $i = 1, 2, 3$ each consist of a single fixed edge $E_i$, with the first two being disjoint loops (corresponding to $\alpha$ and $\beta$ respectively) and the third connecting these loops; $H_4$ is a linear edge, $E_4$, with $f(E_4) = E_4E_2^4$; $H_5$ is a linear edge $E_5$ with $f(E_5) = E_5(E_2\overline{E_3}E_1E_3)^2$; finally $H_6$ is a geometric EG stratum whose geometric model has two lower boundary components with attaching maps $\alpha_1(\partial_1 S) = E_4E_2\overline{E_4}$ and $\alpha_2(\partial_2 S) = E_5E_2\overline{E_3}E_1E_3\overline{E_5}$.

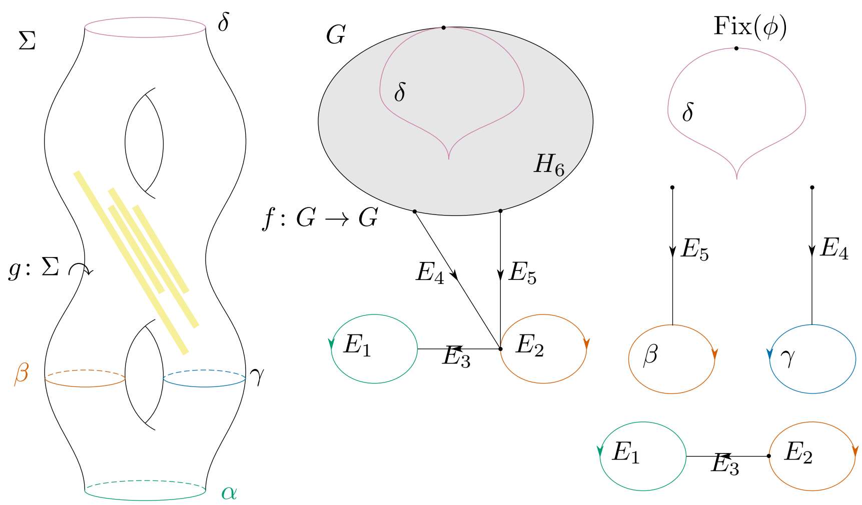


Figure 1. A surface homeomorphism $g\colon \Sigma \to \Sigma$ with associated CT $f\colon G \to G$.

The graph $\hat{S}(f)$ is shown on the right side of the figure. It has four components, each of whose fundamental groups is a subgroup of $\mathbb{F}_5$ that is preserved by a particular $\Phi \in [\phi]$; we denote these subgroups by $K_3, \ldots, K_6$ according to the stratum of the CT. We now compute the sets $\mathcal{L}_{K_i}$ and $\mathcal{E}_{K_i}$ in this example; rather than write out the edge paths, we will abuse notation and write the corresponding curve on the surface. Indeed,

$$\mathcal{L}_{K_3} = \{\beta, \gamma\} \qquad \mathcal{L}_{K_4} = \{E_4\beta\overline{E_4}\} \qquad \mathcal{L}_{K_5} = \{E_5\gamma\overline{E_5}\} \qquad \mathcal{L}_{K_6} = \varnothing.$$

And

$$\mathcal{E}_{K_3} = \varnothing \qquad \mathcal{E}_{K_4} = \{E_4\beta\overline{E_4}\} \qquad \mathcal{E}_{K_5} = \{E_5\gamma\overline{E_5}\} \qquad \mathcal{E}_{K_6} = \{\delta\}.$$

The reader may find it instructive to consider how these sets change when $\Sigma$ is replaced with: the 2-complex obtained by "making a duplicate copy of $\Sigma$", or the 2-complex obtained by cutting $\Sigma$ along $\beta$ and then re-gluing the resulting upper boundary $\beta^+$ to the lower boundary $\beta^-$ via with a 2-1 map of the circle.

**Example 9.2.** Consider the following reducible automorphism:

$$\begin{aligned} \phi : & a \to ab^5 \\ & b \to b \end{aligned}$$

Let $T$ be the universal cover of the rose whose loops are labeled $a$ and $b$. The quotient of the core $\mathrm{Core}(T, T\phi)/\mathbb{F}$ is shown in Figure 2, this was computed with [8]; the red edge paths of length 3 are identifed. The projection from this core to the horizontal direction is a homotopy equivalence (the edge labelled $a$ is the diagonal of a rectangle) which provides a marking.

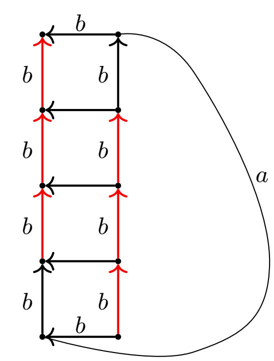


FIGURE 2. $\mathrm{Core}(T, T\phi)/\mathbb{F}$ for a typical Dehn twist.

The core is shown in Figure 3 and can be embedded in a surface, the edge labels indicate the horizontal marking (roughly) to guide the embedding.

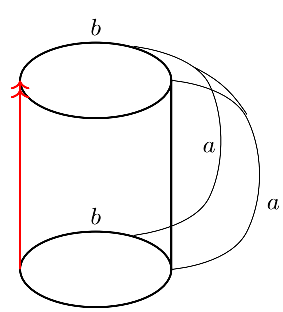


FIGURE 3. The surface $S$ in which the Guirardel core embeds.

There is only one boundary component in this surface, which is associated via the marking to the word $ab^{-1}a^{-1}b$ (half of the top and bottom $b$ . The surface is homeomorphic to a torus with one boundary component. One can check that:

$$\phi(ab^{-1}a^{-1}b) = ab^{-1}a^{-1}b.$$

## References


[1] Jason Behrstock, Mladen Bestvina, and Matt Clay, *Growth of intersection numbers for free group automorphisms*, J. Topol. **3** (2010), no. 2, 280–310, DOI 10.1112/jtopol/jtq008. MR2651361

[2] Edgar A. Bering IV, *Compatible Trees and Outer Automorphisms of a Free Group*, 2017.

[3] Mladen Bestvina and Mark Feighn, *Outer Limits* (1992), preprint.

[4] M. Bestvina and M. Handel, *Train-tracks for surface homeomorphisms*, Topology **34** (1995), no. 1, 109–140, DOI 10.1016/0040-9383(94)E0009-9. MR1308491

[5] Andrew J. Casson and Steven A. Bleiler, *Automorphisms of surfaces after Nielsen and Thurston*, London Mathematical Society Student Texts, vol. 9, Cambridge University Press, Cambridge, 1988. MR0964685

[6] Matt Clay, Yulan Qing, and Kasra Rafi, *Uniform fellow traveling between surgery paths in the sphere graph*, Algebr. Geom. Topol. **17** (2017), no. 6, 3751–3778, DOI 10.2140/agt.2017.17.3751. MR3709659

[7] Marshall M. Cohen and Martin Lustig, *Very small group actions on* **R***-trees and Dehn twist automorphisms*, Topology **34** (1995), no. 3, 575–617, DOI 10.1016/0040-9383(94)00038-M. MR1341810

[8] Thierry Coulbois, `train-track`.

[9] Marc Culler and Karen Vogtmann, *Moduli of graphs and automorphisms of free groups*, Invent. Math. **84** (1986), no. 1, 91–119, DOI 10.1007/BF01388734. MR0830040

[10] Albert Fathi, François Laudenbach, and Valentin Poénaru, *Thurston's work on surfaces*, Mathematical Notes, vol. 48, Princeton University Press, Princeton, NJ, 2012. Translated from the 1979 French original by Djun M. Kim and Dan Margalit. MR3053012

[11] Mark Feighn and Michael Handel, *Abelian subgroups of* $\mathrm{Out}(F_n)$, Geom. Topol. **13** (2009), no. 3, 1657–1727, DOI 10.2140/gt.2009.13.1657. MR2496054

[12] ———, *The recognition theorem for* $\mathrm{Out}(F_n)$, Groups Geom. Dyn. **5** (2011), no. 1, 39–106, DOI 10.4171/GGD/116. MR2763779

[13] ———, *Algorithmic constructions of relative train track maps and CTs*, Groups Geom. Dyn. **12** (2018), no. 3, 1159–1238, DOI 10.4171/GGD/466. MR3845002

[14] Koji Fujiwara, *On the outer automorphism group of a hyperbolic group*, Israel J. Math. **131** (2002), 277–284, DOI 10.1007/BF02785862. MR1942313

[15] Vincent Guirardel, *Cœur et nombre d'intersection pour les actions de groupes sur les arbres*, Ann. Sci. École Norm. Sup. (4) **38** (2005), no. 6, 847–888, DOI 10.1016/j.ansens.2005.11.001 (French, with English and French summaries). MR2216833

[16] Radhika Gupta and Derrick Wigglesworth, *Loxodromics for the cyclic splitting complex and their centralizers*, Pacific J. Math. **301** (2019), no. 1, 107–142, DOI 10.2140/pjm.2019.301.107. MR4007375

[17] Michael Handel and Lee Mosher, *Subgroup decomposition in* $\mathrm{Out}(F_n)$, Mem. Amer. Math. Soc. **264** (2020), no. 1280, vii+276, DOI 10.1090/memo/1280. MR4089372

[18] John Hempel, *3-manifolds*, AMS Chelsea Publishing, Providence, RI, 2004. Reprint of the 1976 original. MR2098385

[19] Sava Krstić, Martin Lustig, and Karen Vogtmann, *An equivariant Whitehead algorithm and conjugacy for roots of Dehn twist automorphisms*, Proc. Edinb. Math. Soc. (2) **44** (2001), no. 1, 117–141, DOI 10.1017/S0013091599000061. MR1879214

[20] Roger C. Lyndon and Paul E. Schupp, *Combinatorial group theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 89, Springer-Verlag, Berlin-New York, 1977. MR0577064

[21] C. Maclachlan and W. J. Harvey, *On mapping-class groups and Teichmüller spaces*. 4, Proc. London Math. Soc. (3) **30** (1975), no. part, 496–512, DOI 10.1112/plms/s3-30.4.496. MR374414

[22] John D. McCarthy, *Normalizers and centralizers of pseudo-Anosov mapping classes* (1994), preprint.

[23] Jakob Nielsen, *Investigations in the Topology of Closed Orientable Surfaces, I, II, and III*, Jakob Nielsen: collected mathematical papers., 1986, pp. xii + 459. Edited and with a preface by Vagn Lundsgaard Hansen. MR0865335

[24] Abe Shenitzer, *Decomposition of a group with a single defining relation into a free product*, Proc. Amer. Math. Soc. **6** (1955), 273–279, DOI 10.2307/2032354. MR69174

[25] G. A. Swarup, *Decompositions of free groups*, J. Pure Appl. Algebra **40** (1986), no. 1, 99–102, DOI 10.1016/0022-4049(86)90032-0. MR825183

[26] Ying-Qing Wu, *Canonical reducing curves of surface homeomorphism*, Acta Math. Sinica (N.S.) **3** (1987), no. 4, 305–313, DOI 10.1007/BF02559911. MR930761

[27] Kaidi Ye, *When is a polynomially growing automorphism of $F_n$ geometric?* (2016), preprint, available at arXiv:1605.07390.

Department of Mathematics & Statistics, San José State University
*Email address*: edgar.bering@sjsu.edu

Department of Mathematics, University of Tennessee at Knoxville
*Email address*: yqing@utk.edu

http://www.drwiggle.com
*Email address*: derrick.wigglesworth@gmail.com