THE NON-PERIPHERAL CURVE GRAPH AND DIVERGENCE IN BIG MAPPING CLASS GROUPS

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ABSTRACT. We introduce a numerical invariant $\zeta(\Sigma)$ measuring the end-complexity of Σ and use it to organize coarse-geometric features of $\mathrm{Map}(\Sigma)$. Our main tool is the *non-peripheral curve graph* $C_{\mathrm{np}}(\Sigma)$, whose vertices are those essential simple closed curves that cannot be pushed out of every compact subsurface, with edges given by disjointness. Assuming $\mathrm{Map}(\Sigma)$ is CB-generated and $\zeta(\Sigma) \geq 5$, we prove that $C_{\mathrm{np}}(\Sigma)$ is connected, has infinite diameter, is Gromov hyperbolic, and that the $\mathrm{Map}(\Sigma)$-action has unbounded orbits. As applications, we show that if $\zeta(\Sigma) \geq 4$ then $\mathrm{Map}(\Sigma)$ has infinite coarse rank, and if $\zeta(\Sigma) \geq 5$ then $\mathrm{Map}(\Sigma)$ has at most quadratic divergence, hence is one-ended.

CONTENTS

1. INTRODUCTION

Let Σ be a connected, orientable, infinite-type surface such that the end space of Σ is stable (Definition 2.1). Let $\mathrm{Map}(\Sigma)$ denote its mapping class group: the group of orientation-preserving homeomorphisms up to isotopy. Equipped with the compact-open topology, $\mathrm{Map}(\Sigma)$ is a Polish group [Ros21]. In contrast to the finite-type case, $\mathrm{Map}(\Sigma)$ is never countably generated. Nevertheless, for many infinite-type surfaces it admits a robust large-scale geometry in the sense of Rosendal: if $\mathrm{Map}(\Sigma)$ is *CB-generated* (i.e. generated by a coarsely bounded neighborhood of the identity together with finitely many additional elements), then any two word metrics coming from CB-generating sets are quasi-isometric, so the quasi-isometry type of $\mathrm{Map}(\Sigma)$ is well-defined [Ros21, MR23].

CB-generation and coarse geometry. In [MR23] Mann and Rafi give a classification of those infinite-type surfaces Σ for which $\mathrm{Map}(\Sigma)$ is CB-generated. In this setting, one can equip $\mathrm{Map}(\Sigma)$ with a word metric coming from any coarsely bounded generating set, and the resulting quasi-isometry type is independent of the choice of CB generators [Ros21, MR23].

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This classification opens the door to a systematic study of quasi-isometry invariants for big mapping class groups, in direct analogy with the coarse geometry of finitely generated groups.

Several recent works develop this viewpoint. Grant–Rafi–Verberne [GRV] investigate quasi-isometry invariants such as asymptotic dimension and introduce *essential shifts* (shift maps that are countable and cannot be pushed to infinity); existence of essential shifts implies the existence of high-dimensional quasi-flats in $\mathrm{Map}(\Sigma)$. Related results on asymptotic dimension for various classes of big mapping class groups appear in the recent preprint [KS25], where Kopreski–Shaji construct arc-and-curve models for locally bounded Polish subgroups of $\mathrm{Map}(\Sigma)$, and use these models to show (in particular) that the asymptotic dimension of a CB-generated $\mathrm{Map}(\Sigma)$ is infinite unless $\mathrm{Map}(\Sigma)$ is CB. We emphasize that asymptotic dimension alone does not detect coarse rank: in general, infinite asymptotic dimension does not rule out the possibility that the coarse rank is finite. In a different direction, Horbez–Qing–Rafi [HQR22] classify when (CB-generated) big mapping class groups admit nonelementary actions on hyperbolic spaces and derive applications. It turns out that $\mathrm{Map}(\Sigma)$ has a non-trivial action on a Gromov hyperbolic space if and only if Σ contains non-displaceable subsurfaces. These results relate the topology of Σ to the large-scale geometry of $\mathrm{Map}(\Sigma)$.

A complexity for end spaces. In this paper we introduce a numerical invariant $\zeta(\Sigma)$ (defined in Section 2.5, see Definition 2.3), which measures the "end-complexity" of Σ in a way that is tailored to the CB setting. Concretely, $\zeta(\Sigma)$ is defined using the minimal anchor surface K_0 from [MR23] such that the stabilizer of K_0 is a CB subgroup of $\mathrm{Map}(\Sigma)$. One should think of $\zeta(\Sigma)$ as an analogue of the finite-type complexity parameter "number of boundary components" (or punctures), which strongly influences the coarse geometry of $\mathrm{Map}(S)$ for finite-type surfaces. We show that $\zeta(\Sigma)$ predicts several coarse-geometric features of $\mathrm{Map}(\Sigma)$.

Main results. The first application concerns coarse rank. For finitely generated mapping class groups, the (coarse) rank is the largest n such that $\mathbb{Z}^n$ quasi-isometrically embeds into the group, and it is finite for finite-type mapping class groups [BM08]. In the present setting, CB-generation gives a well-defined quasi-isometry type of word metrics, and hence coarse rank is again a quasi-isometry invariant.

Theorem A (Mapping class groups with infinite coarse rank)**.** *Suppose Σ is stable,* $\mathrm{Map}(\Sigma)$ *is CB-generated, and $\zeta(\Sigma) \geq 4$. Then* $\mathrm{Map}(\Sigma)$ *has infinite coarse rank.*

In forthcoming work-in-progress by Bar–Natan - Verberne - Schaeffer–Cohen announce a proof that $\mathrm{Map}(\Sigma)$ is Gromov hyperbolic and not CB if and only if $\zeta(\Sigma) \leq 3$ and $\mathrm{Map}(\Sigma)$ has no essential shifts. Together with the results in [GRV], this yields the following dichotomy in the CB-generated setting.

Corollary B. *Suppose Σ is stable and* $\mathrm{Map}(\Sigma)$ *is CB-generated. Then* $\mathrm{Map}(\Sigma)$ *is either Gromov hyperbolic or it has infinite coarse rank.*

Our second application concerns divergence, a quasi-isometry invariant that measures how long detours must be in order to connect two points while avoiding a large ball. Gersten introduced divergence as a quasi-isometry invariant in [Ger94A]; in Euclidean space divergence is linear, while in δ-hyperbolic spaces divergence is at least exponential. Over the past two decades it has become clear that quadratic divergence is remarkably common: Gersten constructed CAT(0) examples with quadratic divergence in [Ger94A], and in [Ger94B] he showed that fundamental groups of closed geometric 3-manifolds have divergence that is either linear, quadratic, or exponential, with quadratic divergence corresponding to graph manifold groups; this characterization was strengthened by Kapovich–Kleiner–Leeb [KKL98]. In higher rank, Drutu–Mozes–Sapir [DMS09] develop a robust definition of divergence for geodesic metric spaces and conjecture that many higher rank lattices have linear divergence; and Behrstock–Charney [BC12] show that right-angled Artin groups have either linear or quadratic divergence, with the linear case characterized by the defining graph being a join. For mapping class groups and Teichmüller space of finite-type surfaces, quadratic divergence was proved Behrstock [Beh06] and Duchin–Rafi [DR09] who give a flexible detour construction; we will follow the approach of [DR09].

Theorem C (Quadratic divergence bound). *Suppose Σ is stable,* $\mathrm{Map}(\Sigma)$ *is CB-generated, and $\zeta(\Sigma) \geq 5$. Then* $\mathrm{Map}(\Sigma)$ *has at most quadratic divergence.*

As a geometric consequence, we obtain one-endedness (in the sense of ends of the Cayley graph associated to any CB word metric).

Corollary D. *Suppose Σ is stable,* $\mathrm{Map}(\Sigma)$ *is CB-generated, and $\zeta(\Sigma) \geq 5$. Then* $\mathrm{Map}(\Sigma)$ *is one-ended.*

See also [OQW25], where one-endedness is established for avenue surfaces.

A ζ-based picture. A recurring theme is that $\zeta(\Sigma)$ organizes coarse geometry of $\mathrm{Map}(\Sigma)$ much like finite-type complexity organizes the geometry of $\mathrm{Map}(S)$. The following table summarizes the picture for stable surfaces. Recall that, if there is an essential shift, then $\mathrm{Map}(\Sigma)$ has infinite coarse rank [GRV].

Complexity $\zeta(\Sigma)$	Group properties of $\mathrm{Map}(\Sigma)$
$\zeta(\Sigma) = 1$	$\mathrm{Map}(\Sigma)$ is always coarsely bounded [MR23].
$\zeta(\Sigma) = 2$	$\mathrm{Map}(\Sigma)$ is coarsely bounded if and only if there are no essential shifts.
$\zeta(\Sigma) = 3$	$\mathrm{Map}(\Sigma)$ is not coarsely bounded. Moreover, if there are no essential shifts then $\mathrm{Map}(\Sigma)$ is Gromov hyperbolic (Bar-Natan–Verberne–Shaeffer-Coehn, in progress).
$\zeta(\Sigma) = 4$	$\mathrm{Map}(\Sigma)$ has infinite coarse rank (Theorem A).
$\zeta(\Sigma) \geq 5$	$\mathrm{Map}(\Sigma)$ is one-ended and has at most quadratic divergence (Theorem C and Corollary D).

The non-peripheral curve graph. The main tool in this paper is a new curve graph adapted to CB coarse geometry. In the finite-type setting, Masur–Minsky's curve graph is central: the action of $\mathrm{Map}(S)$ on the curve graph captures much of the large-scale geometry of $\mathrm{Map}(S)$ and underlies powerful hierarchical structure. For infinite-type surfaces, finding useful analogues has been a major theme; recent years have produced several hyperbolic graphs with big mapping class group actions, including [DFV18, BNV23, HQR22] among others.

A naive adaptation of the classical curve graph does not directly serve coarse geometry in the big setting. Indeed, the "usual" curve graph on Σ (vertices all essential curves, edges disjointness) is algebraically interesting (see, for instance, [DFV18, MR23] and the discussion therein), but in the infinite-type cases it has finite diameter (in fact diameter 2), so the action of $\mathrm{Map}(\Sigma)$ on this graph does not detect large-scale features of $\mathrm{Map}(\Sigma)$.

Our starting point is that, in the CB framework, the stabilizer of a sufficiently large compact subsurface is a coarsely bounded subset of $\mathrm{Map}(\Sigma)$, i.e. it is *small* from the viewpoint of coarse geometry. Therefore, to build a curve-type graph that sees unbounded geometry, one should focus on curves that cannot be pushed out of every compact region. In the finite-type case, the word *peripheral* means the curve bounds a once-punctured disk; equivalently, it can be pushed into a cusp away from all compact sets. For infinite type we adopt the latter point of view: a curve (or compact subsurface) is *peripheral* if for every compact subsurface $K \subset \Sigma$ there exists $g \in \mathrm{Map}(\Sigma)$ such that $g(\alpha) \cap K = \emptyset$. Curves that fail this are *non-peripheral*. This definition is tightly related to the notion of *essential shift* from [GRV]: a shift map is geometrically essential only when its support cannot be pushed into an end neighborhood, i.e. when it is supported on a non-peripheral region. This philosophy also explains why non-peripheral subsurfaces and curves are the natural objects to examine in CB coarse geometry.

We define the *non-peripheral curve graph* $C_{\mathrm{np}}(\Sigma)$ to be the graph whose vertices are non-peripheral simple closed curves and whose edges connect disjoint pairs. Our first main structural result is that, at sufficiently high end-complexity, $C_{\mathrm{np}}(\Sigma)$ becomes a useful Masur–Minsky type object.

Theorem E (Geometry of $C_{\mathrm{np}}(\Sigma)$)**.** *Let Σ be stable,* $\mathrm{Map}(\Sigma)$ *be CB-generated, and suppose $\zeta(\Sigma) \geq 5$. Then $C_{\mathrm{np}}(\Sigma)$ is connected and has infinite diameter,* $\mathrm{Map}(\Sigma)$ *acts on $C_{\mathrm{np}}(\Sigma)$ with unbounded orbits, and $C_{\mathrm{np}}(\Sigma)$ is Gromov hyperbolic.*

A related study of Qing–Thomas [QT25] (using and adapting the unicorn-path technology of Hensel–Przytycki–Webb [HPW15]) establishes uniform hyperbolicity of $C_{\mathrm{np}}(\Sigma)$ in a broad range of cases; we expect further structural parallels with the curve graph in finite type.

Questions. This paper introduces $C_{\mathrm{np}}(\Sigma)$ as a basic tool and uses it for coarse rank and divergence. Much remains to be understood, and we conclude with several questions.

Question 1.1 (Lower bounds for divergence)**.** *When $\zeta(\Sigma) \geq 5$, is the divergence of* $\mathrm{Map}(\Sigma)$ *bounded* below *by a quadratic function? Equivalently, does there exist a pair of geodesic rays in* $\mathrm{Map}(\Sigma)$ *whose divergence grows at least quadratically?*

In the finite-type setting, quadratic lower bounds follow from strong contraction properties of pseudo-Anosov axes [DR09]. It would be interesting to know whether appropriate analogues exist for big mapping class groups in the range $\zeta(\Sigma) \geq 5$.

Question 1.2 (When is $C_{\mathrm{np}}(\Sigma)$ hyperbolic?)**.** *Find necessary and sufficient topological conditions on Σ for $C_{\mathrm{np}}(\Sigma)$ to be Gromov hyperbolic.*

When Σ has finite genus, it is natural to introduce a combined complexity

$$\xi(\Sigma) := 3g(\Sigma) - 3 + \zeta(\Sigma),$$

which plays the role of the usual finite-type complexity $3g - 3 + n$. It is plausible that $\xi(\Sigma)$ is the correct parameter for an "if and only if" hyperbolicity statement for $C_{\mathrm{np}}(\Sigma)$ in the finite-genus setting, analogous to the finite-type case.

Question 1.3 (Quadratic divergence beyond $\zeta \geq 5$)**.** *Are there finite-genus surfaces with* $\zeta(\Sigma) = 4$ *for which* $\mathrm{Map}(\Sigma)$ *has quadratic divergence? More generally, can one characterize (topologically) when* $\mathrm{Map}(\Sigma)$ *has quadratic divergence in terms of* $\xi(\Sigma)$*?*

Question 1.4 (Automorphisms of $C_{\mathrm{np}}(\Sigma)$)**.** *Assume* Σ *is stable,* $\mathrm{Map}(\Sigma)$ *is CB-generated, and* $\zeta(\Sigma) \geq 5$*. Is every simplicial automorphism of* $C_{\mathrm{np}}(\Sigma)$ *induced by an element of* $\mathrm{Map}(\Sigma)$*? Equivalently, is* $\mathrm{Aut}(C_{\mathrm{np}}(\Sigma)) \cong \mathrm{Map}(\Sigma)$ *in this range?*

Organization of the paper. In Section 2.5 we recall the CB framework, the notion of stable surfaces, and the anchor-surface technology from [MR23], and we define the end-complexity $\zeta(\Sigma)$. In Section 3 we introduce peripheral and non-peripheral compact subsurfaces, develop subsurface-projection length functions, and prove Theorem A on infinite coarse rank when $\zeta(\Sigma) \geq 4$. In Section 4 we define the graph $C_{\mathrm{np}}(\Sigma)$, prove connectivity and infinite diameter, and establish Gromov hyperbolicity, proving Theorem E. Finally, in Section 6 we use $C_{\mathrm{np}}(\Sigma)$ together with chains of commuting twists and a "persistent twist" detour argument inspired by [DR09] to obtain the quadratic upper bound on divergence (Theorem C), and then deduce one-endedness (Corollary D).

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2. Preliminaries

2.1. **Surfaces and mapping class group.** A *surface* Σ is a connected 2-dimensional topological manifold, i.e., a second-countable Hausdorff 2-dimensional space with no boundary. In this paper, we assume all surfaces to be orientable. The *mapping class group* of Σ is defined as the group $\mathrm{Map}(\Sigma)$ of all isotopy classes of orientation-preserving homeomorphisms of Σ. The group $\mathrm{Map}(\Sigma)$ is equipped with the quotient topology of the compact-open topology on the group $\mathrm{Homeo}^+(\Sigma)$ of all orientation-preserving homeomorphisms of Σ.

In this paper, a *subsurface* S of a surface Σ is a connected closed subset of Σ that is a manifold with boundary whose boundary consists of a finite number of pairwise non-intersecting simple closed curves, such that none of these boundary curves bounds a disk or a once-punctured disk in Σ. A surface Σ is *of finite type* if its fundamental group is finitely generated. We always assume Σ has infinite type. Similarly, a subsurface S of Σ is of finite type if its fundamental group is finitely generated.

2.2. **The space of ends.** The space of ends of a surface Σ is defined to be the inverse limit of the system of components of complements of compact subsets of Σ. Intuitively, each end corresponds to a way of leaving every compact subset of Σ (see [Ric63] for details). Let $\mathrm{End}(\Sigma)$ be the end space of Σ and $\mathrm{End}^g(\Sigma) \subseteq \mathrm{End}(\Sigma)$ be the subspace of $\mathrm{End}(\Sigma)$ consisting of non-planar ends. Also, let $\mathrm{genus}(\Sigma)$ be the genus of Σ (possibly infinite). By a theorem of Richards [Ric63], connected, orientable surfaces Σ are classified up to homeomorphism by the triple $(\mathrm{genus}(\Sigma), \mathrm{End}(\Sigma), \mathrm{End}^g(\Sigma))$.

Given a subsurface $S \subseteq \Sigma$, the space of ends of S is defined similarly and is denoted by $\mathrm{End}(S)$. The embedding of S in Σ gives a natural embedding of $\mathrm{End}(S)$ into $\mathrm{End}(\Sigma)$.

Every subsurface $S \subseteq \Sigma$ of finite type determines a finite partition Π_S of the ends of Σ where each part of the partition is the space of ends of a connected component of $\Sigma - S$. Given two subsets $X, Y \subseteq \mathrm{End}(\Sigma)$, we say that a subsurface K *separates* X and Y if X and Y belong to distinct subsets of the partition Π_K.

Definition 2.1. For $x \in \mathrm{End}(\Sigma)$, we call a neighborhood U of x *stable* if for any smaller neighborhood $U' \subset U$ of x, there is a homeomorphism from U to U' fixing x. (See [BDR25, Proposition 3.2] for equivalent definitions of a stable neighborhood). We say Σ is *stable* if every end $x \in \mathrm{End}(\Sigma)$ is stable.

In this paper, we always assume the surface Σ is stable. For an end $x \in \mathrm{End}(\Sigma)$, we denote the orbit of x by

$$E(x) = \{\phi(x) \mid \phi \in \mathrm{Map}(\Sigma)\}.$$

We say x and x' are of the same type if $x' \in E(x)$. This defines a partial order on the set of ends as follows. We say $x \preceq y$ if $E(x)$ accumulates to y. It was shown in [MR23] that $x \preceq y$ and $y \preceq x$ implies $x \in E(y)$.

Let $\mathcal{M}(\mathrm{End}(\Sigma))$ be the set of maximal ends, that is, the set of points $x \in \mathrm{End}(\Sigma)$ such that if $x \preceq y$ then $y \in E(x)$. For a stable surface, $\mathcal{M}(\mathrm{End}(\Sigma))$ is non-empty and consists of finitely many different types. We refer to elements of $\mathcal{M}(\mathrm{End}(\Sigma))$ as *maximal ends*. For every maximal end x, either $E(x)$ is finite or it is a Cantor set. We refer to these as *maximal ends of finite type* and *maximal ends of Cantor type*, respectively.

2.3. **Curves and curve graphs.** A simple closed curve α on a surface Σ is a free homotopy class of an essential simple closed curve. Here, *simple* means α has a representative that does not self-intersect and *essential* means that α does not bound a disk or a once-punctured disk. For a subsurface S of Σ, a curve α is *in* S if it has a representative that is contained in S and if α is essential in S, that is, α is not parallel to a boundary component of S.

The curve graph of Σ, denoted by $\mathcal{C}(\Sigma)$, is defined [Har81] to be the graph whose vertex set is the set of curves in Σ and whose edges are pairs of distinct curves in Σ that have disjoint representatives. Similarly, for a finite type surface S of Σ, let the curve graph of S, $\mathcal{C}(S)$, be the graph whose vertex set is the set of curves in S and whose edges are pairs of distinct curves in S that have disjoint representatives. Then $\mathcal{C}(S)$ is an induced subgraph of the curve graph $\mathcal{C}(\Sigma)$.

We equip the curve graph with a metric where every edge has length 1. We denote the associated distance by d_Σ and d_S, that is, for curves α and β in S, $d_S(\alpha, \beta)$ is the smallest number n such that there is a sequence

$$\alpha = \alpha_0, \alpha_1 \dots, \alpha_n = \beta$$

where α_i and α_{i+1} are disjoint for $i = 0, \dots, n-1$.

Recall that, for finite type surfaces S, the curve graph $\mathcal{C}(S)$ has infinite diameter [Har81] and is Gromov hyperbolic [MM00b], and the mapping class group acts on $\mathcal{C}(S)$ by isometries with unbounded orbits [Har81]. For infinite type surfaces, the curve graph (defined as in the case of finite type surfaces) has bounded diameter [BDR25].

2.4. **Markings on a surface.** We say a set of curves $\{\alpha_i\}$ in Σ *fill* a subsurface S if α_i are contained in S and for every curve β in S, there exists a curve α_i such that

$$\beta \cap \alpha_i \neq \emptyset.$$

A *marking* μ_S of a finite-type surface S is a collection of simple closed curves

$$\mu_S = \{\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n\}$$

such that:

- The curves $\{\alpha_i\}$ form a pants decomposition of S.

- $\alpha_i \cap \beta_j = \emptyset$ whenever $i \neq j$.
- For any i, either α_i and β_i fill a punctured torus and $|\alpha_i \cap \beta_i| = 1$, or they fill a four-times punctured sphere and $|\alpha_i \cap \beta_i| = 2$.

It follows from the definition that a marking μ_S fills the subsurface S. See [MM00b] for more details.

2.5. **Coarse geometry of big mapping class groups.** Let G be a Polish topological group. A subset $A \subset G$ is coarsely bounded, abbreviated CB, if every compatible left-invariant metric on G gives A finite diameter. CB sets behave in many ways like finite sets and, by work of Rosendal [Ros21], the theory of uncountable groups that are generated by CB sets resembles the theory of finitely generated groups.

Theorem 2.2 ([Ros21])**.** *Let G be a Polish group that has both a CB neighborhood of the identity and is generated by a CB subset. Then the identity map is a quasi-isometry between G endowed with any two word metrics associated to symmetric, CB generating sets. We then say G is a CB-generated group.*

In [MR23], Mann and Rafi gave descriptions of stable surfaces Σ where $\mathrm{Map}(\Sigma)$ has a CB neighborhood of the identity and surfaces where $\mathrm{Map}(\Sigma)$ is generated by a CB subset. We recall/summarize these statements here (see Proposition 5.5 and Theorem 5.7 in [MR23]).

Since the topology of $\mathrm{Map}(\Sigma)$ comes from the compact-open topology on $\mathrm{Homeo}^+(\Sigma)$, a neighborhood of the identity in $\mathrm{Map}(\Sigma)$ can be described as the stabilizer of a compact subsurface K. Following [MR23], we define

$$\mathcal{V}_K = \big\{\phi \in \mathrm{Map}(\Sigma) \mid \phi(K) = K \quad \text{and} \quad \phi|_K = \mathrm{id}\big\}. \tag{1}$$

Then every end of Σ is an end of some component of $\Sigma - K$, that is, K partitions $\mathrm{End}(\Sigma)$ into finitely many disjoint subsets. We say a component S of $\Sigma - K$ contains an end $x \in \mathrm{End}(\Sigma)$ if $x \in \mathrm{End}(S)$.

For $\mathcal{V}_K$ to be CB, K has to be large enough. This can be made precise by examining the way K decomposes $\mathrm{End}(\Sigma)$. Essentially, we need each component of $\Sigma - K$ to contain ends of at most one maximal type. Also, if $x \in \mathcal{M}(\mathrm{End}(\Sigma))$ and $E(y)$ accumulates to x, we need $E(y)$ also to have an accumulation point in some component of $\Sigma - K$ other than the one containing x. In particular, for $x \in \mathcal{M}(\mathrm{End}(\Sigma))$, if $E(x)$ is a Cantor set, since $E(x)$ accumulates to itself, we need there to be at least two components of $\Sigma - K$ that contain points of $E(x)$. (This is to ensure part (iii) of Theorem 5.7 in [MR23] holds).

In more detail, there has to exist a surface of finite type K such that the connected components of $\Sigma - K$ partition $\mathrm{End}(\Sigma)$ as

$$\mathrm{End}(\Sigma) = \bigsqcup_{A \in \mathcal{A}} A \;\sqcup\; \bigsqcup_{P \in \mathcal{P}} P,$$

such that the following holds:

(i) Each connected component of $\Sigma - K$ has one or infinitely many ends and zero or infinite genus.

(ii) For $A \in \mathcal{A}$, the set of maximal points $\mathcal{M}(A)$ is either a single point or a Cantor set; points in $\mathcal{M}(A)$ are all of the same type and they are maximal in $\mathrm{End}(\Sigma)$. Furthermore,

$$\mathcal{M}(\mathrm{End}(\Sigma)) = \bigsqcup_{A \in \mathcal{A}} \mathcal{M}(A).$$

(iii) For each $P \in \mathcal{P}$, there exists some $A \in \mathcal{A}$ such that P is homeomorphic to a clopen subset of A.
(iv) For $y \in \mathrm{End}(\Sigma)$, if $E(y)$ has an accumulation point in A, it also has an accumulation point outside of A.

We refer to a subsurface K with the above properties as an *anchor surface*. In fact, we fix a minimal anchor surface K_0, meaning an anchor surface where the genus and the number of connected components of $\Sigma - K_0$ are minimal.

Let us examine how this minimality can be achieved. If Σ has finite genus, then the genus of K_0 has to equal the genus of Σ because, in this case, every component of $\Sigma - K_0$ has to be genus zero. Otherwise (meaning if the genus of Σ is zero or infinite), K_0 has genus zero since all the genus can be pushed near some non-planar end of Σ.

Furthermore, as mentioned before, the set $\mathcal{M}(\mathrm{End}(\Sigma))$ has finitely many types; some of them are isolated in $\mathcal{M}(\mathrm{End}(\Sigma))$ and some are Cantor types. In the minimal case, each set of Cantor type maximal ends should appear in exactly two sets in $\mathcal{A}$ and each isolated point is contained in its own component. Hence, the minimum value for $|\mathcal{A}|$ is

$$|\mathcal{A}| = 2 \cdot \big(\#\text{ of Cantor types in } \mathcal{M}(\mathrm{End}(\Sigma))\big) + \big(\#\text{ of isolated points in } \mathcal{M}(\mathrm{End}(\Sigma))\big). \tag{2}$$

If $E(y)$ accumulates to $x \in \mathcal{M}(\mathrm{End}(\Sigma))$ where $E(x)$ is a Cantor set, then $E(y)$ accumulates to every point in $E(x)$. Hence $E(y)$ has accumulation points in at least two sets in $\mathcal{A}$. But it is possible that y is not maximal and $E(y)$ accumulates to a single isolated maximal point x which is contained in $A \in \mathcal{A}$. The set $E(y)$ has to have an accumulation point outside of A, but it may not have an accumulation point in any other $A' \neq A$. This could happen, for example, if $\overline{E(y)}$ (the closure of $E(y)$) is a Cantor set and

$$\overline{E(y)} = E(y) \cup \{x\}.$$

Then $E(y)$ accumulates to itself. Or $E(z)$ could accumulate to points in $E(y)$, where $E(y)$ is as above. In this case, $E(y)$ has to appear in some $P \in \mathcal{P}$. In this case, we say y *uniquely accumulates to the maximal end* x or we say x is the *unique maximal accumulation point of* y.

Assume $x \in A$ is the unique maximal accumulation point of $E(y)$ and the other (non-maximal) accumulation point of $E(y)$ is in P, and that $x' \in A'$ is the unique maximal accumulation point of $E(y')$ and the other accumulation point of $E(y')$ is in P'. Then

$$E(y) \cap A' = \emptyset, \qquad \text{and} \qquad E(y') \cap A = \emptyset.$$

Also,

$$E(y) \cap P' = \emptyset, \qquad \text{and} \qquad E(y') \cap P = \emptyset,$$

because if P intersects both $E(y)$ and $E(y')$ then it cannot be contained in any $A'' \in \mathcal{A}$. This means, in the minimal case,

$$|\mathcal{P}| = \#\big(\text{isolated maximal ends that are unique maximal accumulation points}\big). \tag{3}$$

To summarize, for every maximal end of Cantor type x, we divide $E(x)$ between sets A_x^1 and A_x^2 such that $A_x^1 \sqcup A_x^2$ contains a stable neighborhood of every point in $E(x)$. If $x \in \mathcal{M}(\mathrm{End}(\Sigma))$ is isolated, we choose a stable neighborhood A_x of x. We let $\mathcal{A}$ be the collection of these sets. For every isolated maximal end $x \in A_x$ that is a unique maximal accumulation point, we form a set P_x containing representatives of all accumulation points of sets $E(y)$ where $E(y)$ uniquely accumulates to the maximal end x. Then $\mathcal{P}$ is the collection of such sets P_x. Every other remaining point is not maximal and can be added to some $A \in \mathcal{A}$ or $P \in \mathcal{P}$.

Definition 2.3. The end-complexity $\zeta(\Sigma)$ of an infinite type surface Σ is the minimum number of boundary components of an anchor surface. That is,

$$\begin{aligned}\zeta(\Sigma) = 2 \cdot \#\big(&E(x) \text{ where } E(x) \subset \mathcal{M}(\mathrm{End}(\Sigma)) \text{ is a Cantor set}\big)+\\ &+ \#\big(\text{of isolated points in } \mathcal{M}(\mathrm{End}(\Sigma))\big)+\\ &+ \#\big(\text{isolated maximal ends that are unique maximal accumulation points}\big).\end{aligned}$$

Example 2.4. We now examine this definition in a specific example. Let Σ be the surface depicted in Figure 1. The set of maximal ends $\mathcal{M}(\Sigma)$ consists of two isolated points x_A and x_C and a Cantor set of non-planar points. The point x_A is an accumulation of point of two different Cantor types, $E(y')$ that are accumulated by punctures and $E(y)$ that are not. Since the points $E(y)$ and $E(y')$ uniquely accumulate to x_A (they do not accumulate to any other maximal end) the anchor surface K_0 has to separate some of $E(y)$ and $E(y')$ from A and place them in a set P. Also, K_0 has to separate the Cantor set of non-planar points into two sets $B \sqcup B'$. Since x_C is isolated in $\mathrm{End}(\Sigma)$ and the set $C = \{x_C\}$ contains only one point.

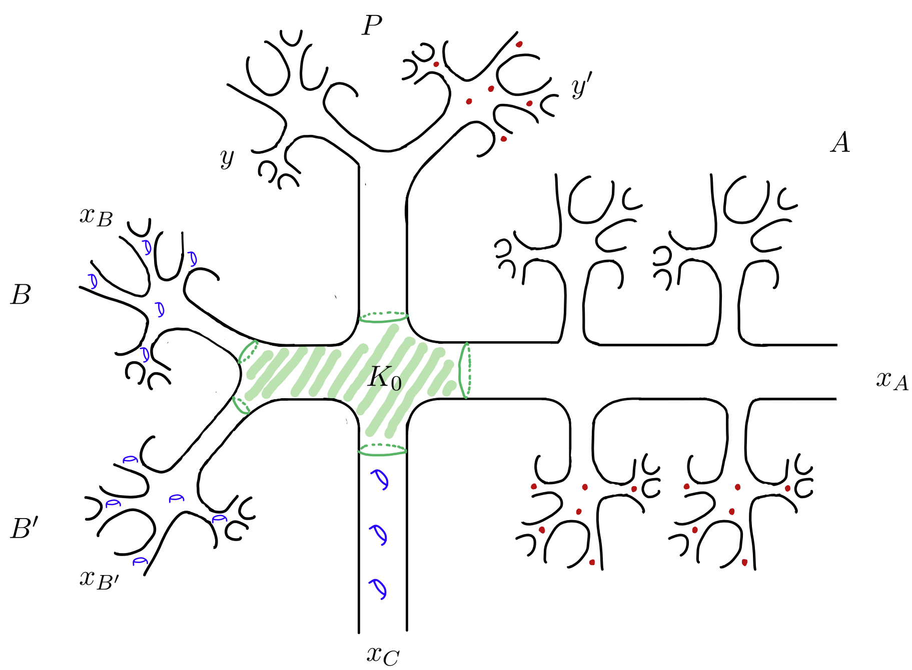

FIGURE 1. A surface Σ with $\zeta(\Sigma) = 5$.

That is, $\mathcal{M}(\Sigma)$ contans two isolated maximal ends and a cantor set of maximal end and one of the isolated maximal ends is unique accumulation point. Hence $\zeta(\Sigma) = 5$.

So far, we have produced a CB neighborhood of the identity $\mathcal{V}_{K_0}$ in $\mathrm{Map}(\Sigma)$. To find a CB set that generates $\mathrm{Map}(\Sigma)$, we need further assumptions.

Theorem 2.5 ([MR23], Theorem 1.6)**.** *(Classification of CB generated mapping class groups). For a stable surface Σ with locally (but not globally) CB mapping class group,* $\mathrm{Map}(\Sigma)$ *is CB generated if and only if* $\mathrm{End}(\Sigma)$ *is finite rank and not of limit type.*

The definition of terms in the above theorem is not relevant to us and we refer the reader to [MR23]. Instead, we give a description of a CB generating set assuming $\mathrm{End}(\Sigma)$ is finite rank and not of limit type.

Lemma 2.6 ([MR23], Lemma 6.10). *Assume that* $\mathrm{Map}(\Sigma)$ *is locally CB and that* $\mathrm{End}(\Sigma)$ *does not have limit type. Then:*

- *For every $A \in \mathcal{A}$, there is a point $x_A \in \mathcal{M}(A)$ and a neighborhood $N(x_A) \subset A$ containing x_A such that $A - N(x_A)$ contains a representative of every type in $A - \{x_A\}$.*
- *For every pair $A, B \in \mathcal{A}$, there is a clopen set $W_{A,B} \subset A - N(x_A)$ with the property that $E(z) \cap W_{A,B} \neq \emptyset$ if and only if*
$$E(z) \cap (A - \{x_A\}) \neq \emptyset \qquad \text{and} \qquad E(z) \cap (B - \{x_B\}) \neq \emptyset.$$
- *For every $A \in \mathcal{A}$, there is a clopen set $W_A \subset (A - N(x_A))$ with the property that if $E(z) \cap A - \{x_A\} \neq \emptyset$ and, for all $B \neq A$, $E(z) \cap B = \emptyset$, then $E(z) \cap W_A \neq \emptyset$.*
- *The sets $W_{A,B}$ and W_A give a decomposition of $A - N(x_A)$:*
$$A - N(x_A) = \left(\bigsqcup_{B \in (\mathcal{A} - \{A\})} W_{A,B} \right) \sqcup \left(\bigsqcup_{A \in \mathcal{A}} W_A \right).$$

These sets $W_{A,B}$ will allow us to define some shift maps between A and B. It follows from [MR23, Section 6.4] that the self-similarity of stable neighborhoods allows us to extend these sets into infinite sequences. Since A is stable and $x_A \in A$, the space $A - \{x_A\}$ consists of infinitely many disjoint copies of the fundamental domain $A - N(x_A)$. Similarly, if $P \in \mathcal{P}$ is associated to A, then P is homeomorphic to a clopen subset of $A - N(x_A)$ (by property (iii) of the anchor surface).

We can therefore decompose the entire end space (excluding the maximal ends x_A) into disjoint orbits (see also [BDR25, Proposition 3.2]).

Lemma 2.7 (Decomposition of Ends). *There exists a decomposition of the space of ends*
$$\mathrm{End}(\Sigma) - \{x_A \mid A \in \mathcal{A}\} = \left(\bigsqcup_{A \in \mathcal{A}} \bigsqcup_{B \in \mathcal{A} - \{A\}} \mathcal{O}_{A,B} \right) \sqcup \left(\bigsqcup_{A \in \mathcal{A}} \mathcal{O}_A \right),$$
where the sets $\mathcal{O}_{A,B}$ and $\mathcal{O}_A$ are unions of disjoint clopen sets indexed by integers:

- *For every pair of distinct $A, B \in \mathcal{A}$, the set $\mathcal{O}_{A,B}$ is a disjoint union*
$$\mathcal{O}_{A,B} = \bigsqcup_{k \in \mathbb{Z}} W^k_{A,B},$$
where each $W^k_{A,B}$ is homeomorphic to the set $W_{A,B}$ defined in Lemma 6.10. Furthermore, we can index these sets such that for $k \leq 0$, $W^k_{A,B} \subset A$, and for $k > 0$, $W^k_{A,B} \subset B$.
- *For every $A \in \mathcal{A}$, the set $\mathcal{O}_A$ is a disjoint union*
$$\mathcal{O}_A = \bigsqcup_{k \in \mathbb{Z}} W^k_A,$$
where each W^k_A is homeomorphic to W_A. Furthermore, for $k \leq 0$, $W^k_A \subset A$, and for $k > 0$, the sets W^k_A cover the ends in the sets $P \in \mathcal{P}$ associated with A.

With this decomposition fixed, we can now define the elements of our generating set.

2.6. **The generating set.** We define a finite set of mapping classes $\mathcal{G}$ consisting of shift maps, maximal end permutations, and local generators.

Shift maps. The decomposition above provides natural tracks along which we can shift the ends of the surface.

- **Inter-region shifts ($f_{A,B}$):** For every distinct pair $A, B \in \mathcal{A}$, let $f_{A,B}$ be a homeomorphism supported on a subsurface containing $\mathcal{O}_{A,B}$ that satisfies:
$$f_{A,B}(W_{A,B}^k) = W_{A,B}^{k+1} \quad \text{for all } k \in \mathbb{Z},$$
and is the identity everywhere else. This map effectively shifts a copy of the shared types $W_{A,B}$ from the region A into the region B.
- **Unique accumulation shifts (f_A):** For every $A \in \mathcal{A}$ such that W_A is non-empty (i.e., there are P sets associated with A), let f_A be a homeomorphism supported on a subsurface containing $\mathcal{O}_A$ that satisfies:
$$f_A(W_A^k) = W_A^{k+1} \quad \text{for all } k \in \mathbb{Z},$$
and is the identity everywhere else. This map shifts ends from A into the corresponding sets $P \in \mathcal{P}$.

Maximal end permutations. Let $\mathrm{Sym}(\mathcal{A})$ be the group of permutations of the set $\mathcal{A}$. We consider the subgroup of permutations σ such that for all $A \in \mathcal{A}$, the maximal end x_A is of the same topological type as $x_{\sigma(A)}$. For each such σ, we fix a homeomorphism h_σ such that:
$$h_\sigma(A) = \sigma(A) \quad \text{and} \quad h_\sigma(x_A) = x_{\sigma(A)}.$$
We include a finite set of such maps generating this subgroup.

Local generators. Finally, we need to generate the mapping class group of the anchor surface. Let $K \supseteq K_0$ be a compact subsurface of finite type large enough to contain the boundaries of the shift maps' supports restricted to the "0-level" (specifically, the boundaries separating W^0 and W^1 for all shift maps). Let $\mathcal{L}$ be a standard finite generating set for $\mathrm{Map}(K)$, consisting of:

- Dehn twists along a finite set of simple closed curves in K.
- If K has boundary components that are permutable, half-twists braiding these boundary components.

Theorem 2.8. *Let Σ be a stable surface of infinite type and let K_0 be the minimal anchor surface we fixed above. Then the set*
$$\mathcal{G} = \{f_{A,B} \mid A, B \in \mathcal{A}, A \neq B\} \cup \{f_A \mid A \in \mathcal{A}\} \cup \{h_\sigma\} \cup \mathcal{L} \cup \mathcal{V}_{K_0}$$
is a coarsely bounded (CB) generating set for $\mathrm{Map}(\Sigma)$.

3. Non-peripheral subsurfaces and the coarse rank of the mapping class group

In this section we introduce the notion of *non-peripheral* compact subsurfaces of an infinite-type surface Σ, and use them to produce quasi-flats in $\mathrm{Map}(\Sigma)$ when $\mathrm{Map}(\Sigma)$ is CB-generated (see Section 2.5). The main result is Theorem 3.18, which states that, when $\zeta(\Sigma) \geq 4$, $\mathrm{Map}(\Sigma)$ has infinite coarse rank.

3.1. **Peripheral and non-peripheral subsurfaces.** For finite-type surfaces, a curve is called peripheral if it bounds a once-punctured disk (or equivalently, if it can be pushed into a cusp). For infinite-type surfaces there is no canonical finite set of cusps. Our definition is a direct analogue: a compact subsurface is peripheral if it can be pushed away from *every* compact region.

Definition 3.1 (Peripheral and non-peripheral subsurfaces)**.** A compact subsurface $R \subset \Sigma$ is *peripheral* if for every compact subsurface $K \subset \Sigma$ there exists $g \in \mathrm{Map}(\Sigma)$ such that

$$g(R) \cap K = \emptyset.$$

Otherwise, R is *non-peripheral*.

Recall that in Section 2.5 we fixed a minimal anchor surface K_0 (a compact subsurface of finite type) with the properties described there. The next lemma shows that to test peripheralness it suffices to test disjointness from K_0.

Lemma 3.2. *A compact subsurface $R \subset \Sigma$ is peripheral if and only if there exists $g \in \mathrm{Map}(\Sigma)$ such that*

$$g(R) \cap K_0 = \emptyset.$$

Proof. If R is peripheral, apply Definition 3.1 with $K = K_0$ to obtain some $g \in \mathrm{Map}(\Sigma)$ with $g(R) \cap K_0 = \emptyset$.

Conversely, suppose there exists $g_0 \in \mathrm{Map}(\Sigma)$ such that $g_0(R) \cap K_0 = \emptyset$. Let $K \subset \Sigma$ be any compact subsurface. We will find $h \in \mathrm{Map}(\Sigma)$ such that $h(R) \cap K = \emptyset$.

Since K_0 is an anchor surface (Section 2.5), the components of $\Sigma - K_0$ correspond to the clopen sets in the partition

$$\mathrm{End}(\Sigma) = \bigsqcup_{A \in \mathcal{A}} A \;\sqcup\; \bigsqcup_{P \in \mathcal{P}} P$$

from Section 2.5. Let Σ_A be the component of $\Sigma - K_0$ containing $g_0(R)$, where $A \in \mathcal{A}$ is the corresponding end space. By Lemma 2.7 (Section 2.6) and the definition of the shift maps, there exists a mapping class $s \in \mathrm{Map}(\Sigma)$ which is a word in the shift maps $\{f_A\}$ and $\{f_{A,B}\}$ and has the following property: for a fixed stable neighborhood $N(x_A) \subset A$ (as in Lemma 6.10 of [MR23] recalled in Section 2.5), we have

$$\mathrm{End}\big(s^m(\Sigma_A)\big) \;\subset\; N(x_A) \quad \text{for all sufficiently large } m,$$

and hence $s^m(\Sigma_A)$ eventually leaves every compact subset of Σ. In particular, there exists m such that $s^m(\Sigma_A) \cap K = \emptyset$.

Set $h = s^m \circ g_0$. Then $h(R) \subset s^m(\Sigma_A)$, so $h(R) \cap K = \emptyset$. Since K was arbitrary, Definition 3.1 implies that R is peripheral. □

Thus we obtain a convenient equivalent characterization.

Definition 3.3 (Non-peripheral subsurfaces)**.** A compact subsurface $R \subset \Sigma$ is non-peripheral if and only if for every $g \in \mathrm{Map}(\Sigma)$ we have

$$g(R) \cap K_0 \neq \emptyset.$$

Remark 3.4. Recall from [MR23] that a compact subsurface R is *non-displaceable* if $R \cap g(R) \neq \emptyset$ for every $g \in \mathrm{Map}(\Sigma)$ (see also Section 2.5). Non-displaceable subsurfaces are a basic source of unbounded coarse geometry for big mapping class groups (e.g. [MR23, Proposition 2.8]). Every non-displaceable subsurface is non-peripheral, but non-peripheral subsurfaces form a larger class and will be more flexible for producing quasi-flats.

3.2. **Coarse rank and quasi-flats.** As discussed in Section 2.5, a CB-generated Polish group admits a well-defined quasi-isometry type of word metrics coming from symmetric CB generating sets [Ros21]. Following [GRV], we use this to define a rank notion for big mapping class groups.

Definition 3.5 (Coarse rank [GRV])**.** Assume $\mathrm{Map}(\Sigma)$ is CB-generated (Section 2.5). Equip $\mathrm{Map}(\Sigma)$ with a word metric $d_{\mathcal{G}}$ associated to any symmetric CB generating set $\mathcal{G}$ (e.g. the set from Theorem 2.8). The *coarse rank* of $\mathrm{Map}(\Sigma)$ is the largest $n \geq 0$ for which there exists a quasi-isometric embedding $\mathbb{Z}^n \hookrightarrow (\mathrm{Map}(\Sigma), d_{\mathcal{G}})$. If such n is unbounded, we say $\mathrm{Map}(\Sigma)$ has *infinite coarse rank*.

3.3. **Length functions and subsurface projection distances.** A convenient way to obtain lower bounds on word metrics is via length functions.

Definition 3.6. Let G be a topological group. A *length function* on G is a function $\ell\colon G \to [0, \infty)$ such that

- $\ell(\mathrm{id}) = 0$;
- $\ell(g) = \ell(g^{-1})$ for all $g \in G$;
- $\ell(gh) \leq \ell(g) + \ell(h)$ for all $g, h \in G$.

Remark 3.7. By [Ros21, Proposition 2.7(5)], every length function on a Polish group is bounded on every coarsely bounded (CB) subset (see Section 2.5). In particular, if $\mathcal{G}$ is a CB generating set for G, then $\sup_{s\in\mathcal{G}} \ell(s) < \infty$ and ℓ gives a uniform lower bound for the word metric.

We now recall subsurface projections and the associated projection distances in curve graphs. These constructions go back to Masur–Minsky and are standard; we follow [MM00b]. Curve graphs and their metrics were defined in Section 2.5.

Subsurface projections and projection distances. In this subsection we recall the coarse definition of the projection π_S to a non-annular subsurface S and the induced projection distance.

Let S be a finite-type subsurface of Σ (possibly an annulus). For a curve α on Σ (Section 2.5), the *subsurface projection* $\pi_S(\alpha)$ is defined whenever α intersects S essentially.

Definition 3.8 (Subsurface projection for non-annular subsurfaces)**.** Assume S is a finite-type subsurface which is *not* an annulus. If a curve α has a representative intersecting S essentially, put α and ∂S in minimal position and consider the collection of essential arcs of $\alpha \cap S$. For each such arc, perform surgery with ∂S (equivalently: take the boundary components of a regular neighborhood of the arc together with ∂S) to obtain a finite set of essential curves in S. Define $\pi_S(\alpha) \subset \mathcal{C}(S)$ to be the union of these curves. If α is disjoint from S (or $\alpha \cap S$ has no essential arc component), we set $\pi_S(\alpha) = \emptyset$.

If P is a multicurve (e.g. $P = \partial R$ for some subsurface R), define

$$\pi_S(P) \ := \ \bigcup_{\alpha\in P} \pi_S(\alpha),$$

and if μ is a marking on a finite-type surface (Section 2.5), define

$$\pi_S(\mu) \ := \ \bigcup_{\gamma\in\mu} \pi_S(\gamma),$$

where the union is over all base curves and transversals of μ.

Remark 3.9. Different choices of representatives and surgeries change $\pi_S(\alpha)$ only by a uniformly bounded amount in $\mathcal{C}(S)$; in particular $\mathrm{diam}_{\mathcal{C}(S)}(\pi_S(\alpha))$ is uniformly bounded (depending only on the topological type of S). Thus subsurface projections are *coarsely well-defined.* See [MM00b] for details.

Definition 3.10 (Projection distance)**.** Let S be a finite-type subsurface which is not an annulus, and let X, Y be curves, multicurves, or markings. If $\pi_S(X) \neq \emptyset$ and $\pi_S(Y) \neq \emptyset$, define

$$d_S(X,Y) \ := \ \mathrm{diam}_{\mathcal{C}(S)}\big(\pi_S(X) \cup \pi_S(Y)\big) = \sup\big\{d_S(x,y) \mid x \in \pi_S(X),\ y \in \pi_S(Y)\big\}. \tag{4}$$

If $\pi_S(X) = \emptyset$ or $\pi_S(Y) = \emptyset$, then the distance is not defined.

Annular projections and twisting. In this subsection we recall annular projections and the associated twisting distance.

When S is an annulus with core curve γ, subsurface projection is taken to the *annular arc graph* $\mathcal{C}(\gamma)$, whose vertices are isotopy classes (rel. endpoints) of essential arcs in the annular cover $\widetilde{S}_\gamma$ corresponding to γ, with endpoints on $\partial\widetilde{S}_\gamma$, and whose edges join disjoint arcs. For a curve α intersecting γ, the projection $\pi_\gamma(\alpha)$ is the set of lifts of α to $\widetilde{S}_\gamma$ that are arcs connecting the two components of $\partial\widetilde{S}_\gamma$. For a multicurve P and a marking μ, define $\pi_\gamma(P)$ and $\pi_\gamma(\mu)$ by taking unions over constituent curves as in Definition 3.8. Again, see [MM00b] for details.

Remark 3.11. As in the non-annular case, $\pi_\gamma(\alpha)$ is coarsely well-defined and has uniformly bounded diameter in $\mathcal{C}(\gamma)$. In particular, the distance between two projections is well-defined up to a bounded additive error.

Definition 3.12 (Relative twisting / annular distance)**.** Let γ be a curve and let $\mathcal{C}(\gamma)$ be the annular arc graph. For curves, multicurves, or markings X, Y with $\pi_\gamma(X), \pi_\gamma(Y) \neq \emptyset$, define the *relative twisting* (annular projection distance) about γ by

$$\mathrm{tw}_\gamma(X,Y) \ := \ d_\gamma\big(\pi_\gamma(X), \pi_\gamma(Y)\big) \ := \ \mathrm{diam}_\gamma\left(\pi_\gamma(X) \cup \pi_\gamma(Y)\right).$$

If either projection is empty, we set $\mathrm{tw}_\gamma(X,Y) = 0$.

Remark 3.13. The main fact we will use about relative twisting is that a Dehn twist T_γ acts by a translation on $\mathcal{C}(\gamma)$, and hence $\mathrm{tw}_\gamma\big(X, T_\gamma^n(X)\big)$ grows coarsely linearly in $|n|$ for any X with $\pi_\gamma(X) \neq \emptyset$.

3.4. **A subsurface-projection length function.** We now package the above projection distances into a length function on $\mathrm{Map}(\Sigma)$.

Let $\mathcal{R} = \{R_1, \dots, R_n\}$ be a collection of pairwise disjoint non-peripheral compact subsurfaces of Σ. By Lemma 3.2 and Definition 3.3, every translate $g(R_i)$ intersects K_0 essentially.

Definition 3.14. Let $\mathcal{R} = \{R_1, \dots, R_n\}$ be disjoint non-peripheral compact subsurfaces of Σ. Fix a marking μ on K_0 (Section 2.5). Define $L^\mu_{\mathcal{R}}\colon \mathrm{Map}(\Sigma) \to [0,\infty)$ by

$$L^\mu_{\mathcal{R}}(f) \ := \ \sup_{g \in \mathrm{Map}(\Sigma)} \ \sum_{i=1}^{n} d_{g(R_i)}\big(\mu, f(\mu)\big), \tag{5}$$

when f is not the identity. For the identity element we let $L^\mu_{\mathcal{R}}(\mathrm{id}) = 0$. We suppress μ from the notation and write $L_{\mathcal{R}}$ when μ is fixed.

Lemma 3.15. *$L_{\mathcal{R}}$ is a length function in the sense of Definition 3.6.*

Proof. Nonnegativity and $L_{\mathcal{R}}(\mathrm{id}) = 0$ are immediate from the definition. For symmetry, we use $\mathrm{Map}(\Sigma)$–equivariance of projections (and of annular projections) to observe that for any finite-type subsurface Y and any $f \in \mathrm{Map}(\Sigma)$,

$$d_Y\big(\mu, f^{-1}(\mu)\big) \;=\; d_{f(Y)}\big(\mu, f(\mu)\big).$$

Therefore

$$\begin{aligned} L_{\mathcal{R}}(f^{-1}) &= \sup_g \sum_i d_{g(R_i)}\big(\mu, f^{-1}(\mu)\big) = \sup_g \sum_i d_{fg(R_i)}\big(\mu, f(\mu)\big) \\ &= \sup_h \sum_i d_{h(R_i)}\big(\mu, f(\mu)\big) = L_{\mathcal{R}}(f), \end{aligned}$$

where we reparametrized $h = fg$.

For subadditivity, fix $f, h \in \mathrm{Map}(\Sigma)$ and apply the triangle inequality in each $\mathcal{C}(g(R_i))$ (or annular arc graph):

$$d_{g(R_i)}\big(\mu, fh(\mu)\big) \;\leq\; d_{g(R_i)}\big(\mu, f(\mu)\big) \;+\; d_{g(R_i)}\big(f(\mu), fh(\mu)\big).$$

Summing over i and taking suprema gives

$$L_{\mathcal{R}}(fh) \;\leq\; \sup_g \sum_i d_{g(R_i)}\big(\mu, f(\mu)\big) \;+\; \sup_g \sum_i d_{g(R_i)}\big(f(\mu), fh(\mu)\big).$$

Using equivariance again, $d_{g(R_i)}(f(\mu), fh(\mu)) = d_{f^{-1}g(R_i)}(\mu, h(\mu))$, and since g ranges over all of $\mathrm{Map}(\Sigma)$ so does $f^{-1}g$. Since the subsurfaces R_i are non-peripheral, all the distances exist. Thus the second supremum equals $L_{\mathcal{R}}(h)$, proving

$$L_{\mathcal{R}}(fh) \leq L_{\mathcal{R}}(f) + L_{\mathcal{R}}(h).$$ □

3.5. **Quasi-isometrically embedded abelian subgroups.** We now use $L_{\mathcal{R}}$ to build quasi-isometric embeddings of $\mathbb{Z}^k$ from families of disjoint non-peripheral subsurfaces.

Proposition 3.16. *Assume that* $\mathrm{Map}(\Sigma)$ *is CB-generated. If $\mathcal{R} = \{R_1, \ldots, R_k\}$ is a family of k pairwise disjoint non-peripheral compact subsurfaces of Σ, then* $\mathrm{Map}(\Sigma)$ *contains a quasi-isometrically embedded copy of $\mathbb{Z}^k$.*

Proof. Fix a marking μ on K_0 and consider the length function $L_{\mathcal{R}} = L^{\mu}_{\mathcal{R}}$ from Definition 3.14. Let d_A be the word metric coming from any symmetric CB generating set A; for instance one may take the CB generating set $\mathcal{G}$ from Theorem 2.8 (Section 2.6). By Remark 3.7, there exists $M < \infty$ such that $L_{\mathcal{R}}(a) \leq M$ for all $a \in A$. Hence for all $g \in \mathrm{Map}(\Sigma)$ we have

$$d_A(e, g) \;\geq\; \frac{1}{M}\, L_{\mathcal{R}}(g). \tag{6}$$

For each i, choose a mapping class g_i supported on R_i such that g_i acts loxodromically on $\mathcal{C}(R_i)$. Concretely: if $\zeta(R_i) \geq 2$, take g_i pseudo-Anosov on R_i; if R_i is an annulus with core curve α_i, take $g_i = T_{\alpha_i}$. Since R_i is non-peripheral,

$$d_{R_i}\big(\mu, g_i(\mu)\big) \;>0.$$

Replacing g_i by a power if necessary, we may assume

$$d_{R_i}\big(\mu, g_i^n(\mu)\big) \;\geq\; |n| \qquad \text{for all } n \in \mathbb{Z}, \tag{7}$$

using positive translation length for pseudo-Anosov elements and the translation action of Dehn twists on annular arc graphs (see, e.g., [MM00b]).

Since the R_i are pairwise disjoint, the mapping classes g_i commute and generate a subgroup isomorphic to $\mathbb{Z}^k$. Define

$$\Phi\colon \mathbb{Z}^k \to \mathrm{Map}(\Sigma), \qquad (a_1, \dots, a_k) \mapsto g_1^{a_1} \cdots g_k^{a_k}.$$

The map Φ is Lipschitz (with respect to the ℓ^1 metric on $\mathbb{Z}^k$) since $d_A(e, g_i^n) \le |n|\, d_A(e, g_i)$.

For the lower bound, apply (6) and then evaluate $L_{\mathcal{R}}$ using $g = \mathrm{id}$ in the supremum:

$$\begin{aligned} d_A\big(\Phi(\mathbf{a}), \Phi(\mathbf{b})\big) = d_A\big(e, \Phi(\mathbf{a}-\mathbf{b})\big) \ &\ge\ \frac{1}{M}\, L_{\mathcal{R}}\big(\Phi(\mathbf{a}-\mathbf{b})\big) \\ &\ge \frac{1}{M} \sum_{i=1}^{k} d_{R_i}\left(\mu, g_i^{a_i-b_i}(\mu)\right) \ \ge\ \frac{1}{M} \sum_{i=1}^{k} |a_i - b_i|, \end{aligned}$$

where the last inequality is (7). Hence Φ is a quasi-isometric embedding of $\mathbb{Z}^k$ into $\mathrm{Map}(\Sigma)$. □

3.6. **Infinite coarse rank.** We now show that, under a mild hypothesis on the end-complexity, Σ contains arbitrarily large families of pairwise disjoint non-peripheral curves.

Lemma 3.17. *Assume $\zeta(\Sigma) \ge 4$. Then there exists an essential separating curve $\gamma \subset K_0$ such that γ is non-peripheral. More precisely, one may choose γ so that each component of $\Sigma - \gamma$ contains ends from at least two distinct members of $\mathcal{A} \cup \mathcal{P}$ (in the sense of Section 2.5).*

Proof. Since $\zeta(\Sigma) \ge 4$, the minimal anchor surface K_0 has at least four boundary components (Section 2.5). As K_0 is finite type, there exists an essential separating curve $\gamma \subset K_0$ so that each component of $K_0 - \gamma$ contains at least two boundary components of K_0. Equivalently, each component of $\Sigma - \gamma$ contains ends coming from at least two distinct complementary components of $\Sigma - K_0$, hence from at least two distinct sets in $\mathcal{A} \cup \mathcal{P}$. Denote the components of $\Sigma - \gamma$ by Σ' and Σ''.

We claim that such a γ is non-peripheral. Suppose for contradiction that γ is peripheral. Then by Lemma 3.2 there exists $g \in \mathrm{Map}(\Sigma)$ such that $g(\gamma) \cap K_0 = \emptyset$, hence $g(\gamma)$ is contained in a single component Σ_C of $\Sigma - K_0$, where $C \in \mathcal{A} \cup \mathcal{P}$ and $\mathrm{End}(\Sigma_C) = C$ (notation as in Section 2.5). That means the ends in one component of $\Sigma - \gamma$ (say Σ') are entirely mapped into C. We show this contradicts the minimality of K_0.

We start by recalling the structure of $\mathcal{A} \cup \mathcal{P}$ for a minimal K_0. The maximal ends $\mathcal{M}(\Sigma)$ are distributed among sets in $\mathcal{A}$, with each maximal end of Cantor type appearing in exactly two such sets and each isolated maximal end appearing in a separate set. Also, for every $P \in \mathcal{P}$, there is a point $y \in P$ and $A \in \mathcal{A}$ such that $\mathcal{M}(A) = x_A$ is an isolated point and x_A is the only maximal end that is an accumulation point of $E(y)$ (i.e., $E(y)$ uniquely accumulates to x_A). In particular $E(y) \subset A \cup P$. Also, recall that for any $z \in \mathrm{End}(\Sigma)$ and $g \in \mathrm{Map}(\Sigma)$, we have

$$g(E(z)) = E(z).$$

We argue in several cases.

Assume there are $A, B \in \mathcal{A}$ such that $A \sqcup B \subset \mathrm{End}(\Sigma')$ and hence $g(A \sqcup B) \subset C$. From the above discussion, we see that $g(A \sqcup B) \subset C$ is possible only if $\mathcal{M}(A)$ and $\mathcal{M}(B)$ are both of the same Cantor type $E(x)$ and $E(x) = \mathcal{M}(A) \cup \mathcal{M}(B)$. This means C must be either A or B

(say A). But then $g(\mathrm{End}(\Sigma''))$ contains B, which is a contradiction since $E(x)\cap\mathrm{End}(\Sigma'') = \emptyset$ (recall that A and B are the only components containing $E(x)$).

Next assume there is $A \in \mathcal{A}$ and $P \in \mathcal{P}$ such that $A \sqcup P \subset \mathrm{End}(\Sigma')$. Then C has to be homeomorphic to A. After composing g with some finite order element $h_\sigma \in \mathcal{G}$, we can assume $C = A$. Let $y \in P$ be the end such that $E(y)$ uniquely accumulates to some maximal end x_B. Since P can be mapped inside A, we must have $B = A$. Which means

$$E(y) \cap \mathrm{End}(\Sigma'') = \emptyset.$$

This is a contradiction since $g(A \cup P) \subset A$ implies

$$g(\mathrm{End}(\Sigma'')) \supset P \supset E(y) = g(E(y)).$$

Finally assume there are $P_1, P_2 \in \mathcal{P}$ such that $P_1 \sqcup P_2 \subset \mathrm{End}(\Sigma')$. Then, for $i = 1, 2$, there is a point $y_i \in P_i$ such that $E(y_i)$ uniquely accumulates to x_{A_i} for $A_i \in \mathcal{A}$. By minimality of K_0, y_1 and y_2 are different types, otherwise, we could replace them with $P = P_1 \cup P_2$ and reduce the number of elements in $\mathcal{P}$. Therefore, there does not exist a set in $\mathcal{A} \cup \mathcal{P}$ that intersects both $E(y_1)$ and $E(y_2)$. Hence, $g(P_1 \sqcup P_2) \subset C$ is not possible.

Since we arrived at a contradiction in all 3 cases, no such g exists, and γ is non-peripheral. □

Theorem 3.18. *Assume Σ is a stable infinite-type surface such that* $\mathrm{Map}(\Sigma)$ *is CB-generated and $\zeta(\Sigma) \geq 4$. Then for every integer $k \geq 1$ there exist k pairwise disjoint non-peripheral curves $\gamma_1, \ldots, \gamma_k$ in Σ. Consequently,* $\mathrm{Map}(\Sigma)$ *has infinite coarse rank.*

Proof. Let K_0 be the fixed minimal anchor surface from Section 2.5, and let $\mathcal{G}$ be the CB generating set from Theorem 2.8 (Section 2.6). Since Σ has infinite type, the end space contains infinitely many ends, which implies (by the self-similarity structure of stable surfaces) that at least one of the sets $W_{A,B}$ or W_A in the decomposition from Lemma 2.7 is non-empty. Hence, at least one of the shift maps in $\mathcal{G}$ is nontrivial. Fix such a shift map f (either $f = f_{A,B}$ for some $A \neq B$ or $f = f_A$ for some A and P as in Section 2.6).

By Lemma 3.17, choose a non-peripheral essential separating curve $\gamma \subset K_0$ that separates A from B in the first case and separates A from P in the second case. The curve γ intersects the support of f so that γ and $f(\gamma)$ are distinct. More precisely, there is a connected subsurface $X \subset \Sigma$ supporting f and a bi-infinite family of pairwise disjoint finite-type subsurfaces $\{X^j\}_{j\in\mathbb{Z}} \subset X$ with $f(X^j) = X^{j+1}$; choosing γ to separate $X^{\leq 0}$ from $X^{\geq 1}$ inside X, we have that $f(\gamma)$ is disjoint from γ, and hence the curves

$$\gamma_j \ := \ f^j(\gamma) \qquad (j \in \mathbb{Z})$$

are pairwise disjoint.

Finally, non-peripheralness is invariant under the action of $\mathrm{Map}(\Sigma)$; therefore, the γ_j are pairwise disjoint non-peripheral curves. Applying Proposition 3.16 to the family $\mathcal{R} = \{R_1, \ldots, R_k\}$ (viewing each R_i as an annular subsurface with core curve γ_i) yields a quasi-isometric embedding $\mathbb{Z}^k \hookrightarrow \mathrm{Map}(\Sigma)$. Since k is arbitrary, the coarse rank is infinite. □

Example 3.19. The surface Σ depicted in Figure 2 has two types of maximal ends, both are cantor types. Hence $\zeta(\Sigma) = 4$. The curve α is non-peripheral, but it intersects any other non-peripheral curve (curves disjoint from α can always be pushed away from every compact set. Hence, C_{np} is not connected. The curve β is another example of a non-peripheral curve. There is a shift map between A and A' that shifts some part of the cantor set of ends in A to

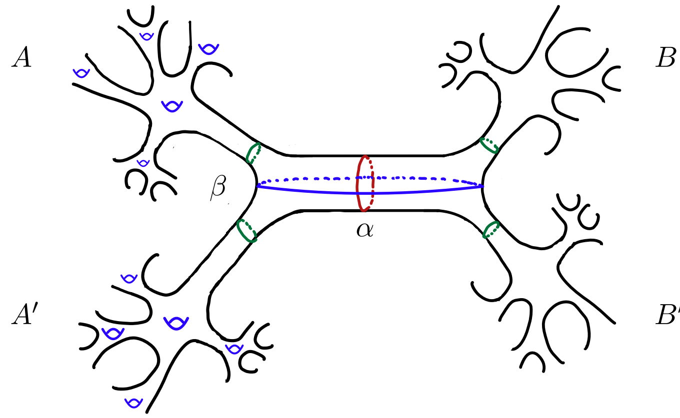

FIGURE 2. When $\zeta(\Sigma) = 4$ the graph C_{np} may not be connected. But we can still find arbitrarily large number of disjoint non-peripheral curves.

A'. The support of this shift map intersects β and send it to a curve that is disjoint from β. Therefore, Σ still infinity many disjoint non-peripheral curves and thus it has infinite coarse rank.

4. The non-peripheral curve graph

In this section, we restrict our attention to a non-peripheral curves and we define a curve graph built only from non-peripheral curves. We will prove a basic connectivity result when $\zeta(\Sigma) \geq 5$.

4.1. **Definition and first examples.** Recall that a *curve* on Σ means an essential simple closed curve, i.e. it does not bound a disk or once–punctured disk (Section 2.5). A curve is *non-peripheral* if its annular neighborhood is non-peripheral in the sense of Definition 3.3.

Definition 4.1 (Non-peripheral curve graph)**.** The *non-peripheral curve graph* $C_{\mathrm{np}}(\Sigma)$ is the graph whose vertex set is the set of isotopy classes of non-peripheral curves in Σ, and with an edge between distinct vertices if they admit disjoint representatives. We equip $C_{\mathrm{np}}(\Sigma)$ with the path metric in which each edge has length 1.

For small values of $\zeta(\Sigma)$ the graph $C_{\mathrm{np}}(\Sigma)$ can be empty or highly disconnected. For instance, if Σ is the latter surface (a two ended infinite genus surface) then $\zeta(\Sigma) = 2$ and there are no non-peripheral curves and $C_{\mathrm{np}}(\Sigma) = \emptyset$.

(Add some examples with $\zeta = 4$ here).

4.2. **Connectivity when $\zeta(\Sigma) \geq 5$.** We now prove that $C_{\mathrm{np}}(\Sigma)$ is connected provided $\zeta(\Sigma)$ is large enough.

Theorem 4.2. *Assume Σ is stable,* $\mathrm{Map}(\Sigma)$ *is CB-generated, and* $\zeta(\Sigma) \geq 5$. *Then the graph* $C_{\mathrm{np}}(\Sigma)$ *is connected.*

We first proceed to charaterized non-peripheral curves by the following technical lemma.

Definition 4.3. We say a clopen subset $X \subset \mathrm{End}(\Sigma)$ is *small* if there exist $A \in \mathcal{A}$ and $g \in \mathrm{Map}(\Sigma)$ such that $g(X) \subset A$. Note that, since every $P \in \mathcal{P}$ fits inside some $A \in \mathcal{A}$. Hence, if $g(X) \subset P$ then X is also small.

Lemma 4.4. *Let Σ be a connected, orientable surface of infinite type.*

(1) If α is a separating *peripheral curve then for some component Σ' of $\Sigma - \alpha$ the clopen set $\mathrm{End}(\Sigma') \subset \mathrm{End}(\Sigma)$ is small (in the sense of Definition 4.3).*
(2) If α is non-separating*, then α is peripheral if and only if Σ has infinite genus.*

Proof. If α is separating and peripheral then there is $g \in \mathrm{Map}(\Sigma)$ with $g(\alpha) \cap K_0 = \emptyset$. Then $g(\alpha)$ lies in a single component Σ_C of $\Sigma - K_0$, where $C = \mathrm{End}(\Sigma_C) \in \mathcal{A} \cup \mathcal{P}$. Since $g(\alpha)$ is separating, one complementary component of $\Sigma - g(\alpha)$ is contained in Σ_C. Hence, $g(\mathrm{End}(\Sigma_i)) \subset C$ for some $i \in \{1, 2\}$, and thus $\mathrm{End}(\Sigma_i)$ is small.

To see the second assertion, assume Σ has infinite genus. Then there is sequence α_i of non-separating curves exiting a non-planar end. For any other non-separating α, $\Sigma - \alpha$ is homeomorphic to $\Sigma - \alpha_i$ since the two surfaces have the same genus, the end space and the same number of boundary component. Hence, α can be mapped to α_i and hence it can be moved away from K_0.

If Σ has finite genus, then K_0 carries all the genus. Any curve disjoint from K lies in a planar neighborhood of the ends, hence is separating. Therefore, a non-separating curve can never be mapped to a curve disjoint from K_0. This finishes the proof. □

Lemma 4.5. *Assume $\zeta(\Sigma) \geq 5$ and let*

$$\mathrm{End}(\Sigma) \;=\; X_1 \sqcup X_2 \sqcup X_3 \sqcup X_4$$

be a decomposition of the space of ends into four clopen subsets such that $(X_1 \sqcup X_2)$ is not small and $(X_3 \sqcup X_4)$ is not small. Then there is $1 \leq i \leq 4$ such that both

$$X_i \qquad \text{and} \qquad \mathrm{End}(\Sigma) - X_i$$

are not small.

Proof. Define the end-complexity $\zeta(X)$ of a clopen set $X \subset \mathrm{End}(\Sigma)$ as follows.

- Add 1 for every isolated maximal end $x_A \in X$.
- Add 1 if X contains a point in $E(x)$ where x is a Cantor-type maximal end, but add 2 if $E(x)$ is entirely contained in X.
- Add 1 for every $A \in \mathcal{A}$ such that there exists $y \in \mathrm{End}(\Sigma)$ with $E(y)$ uniquely accumulating to x_A and $E(y) \cap X \neq \emptyset$.

Note that if $\zeta(X) \geq 2$, then X is not small. We claim that, for disjoint clopen sets $X, X' \subset \mathrm{End}(\Sigma)$, we have

$$\zeta(X) \;\leq\; \zeta(X \sqcup X') \;\leq\; \zeta(X) + \zeta(X').$$

The first inequality is immediate. To see the second one, note that any contribution to $\zeta(X \sqcup X')$ is also a contribution to $\zeta(X)$ or to $\zeta(X')$.

To finish the proof, we notice that

$$\sum_{i=1}^{4} \zeta(X_i) \geq \zeta(\Sigma) \geq 5.$$

Hence, for some i, $\zeta(X_i) \geq 2$. Also, $(\mathrm{End}(\Sigma) - X_i)$ contains either $(X_1 \sqcup X_2)$ or $(X_3 \sqcup X_4)$ and hence it is not small. This finishes the proof. □

Proof of Theorem 4.2. Choose a curve $\gamma \in C_{\mathrm{np}}(\Sigma)$. We show that for every $\alpha \in C_{\mathrm{np}}(\Sigma)$, there is a path in $C_{\mathrm{np}}(\Sigma)$ obtained from surgery between α and γ connecting α to γ in $C_{\mathrm{np}}(\Sigma)$.

The argument is similar to [Sch, Section 4] and it proceeds by induction on the intersection number between α and γ. As a base case, assume $i(\alpha,\gamma)=1$. Then the curve β obtained from surgery between α and γ bounds a once-punctured torus T with one boundary component that contains γ. If β can be moved out of every compact set, so can T and hence γ, which is not possible (since γ is non-peripheral). Hence $\beta \in C_{\mathrm{np}}(\Sigma)$ and therefore $d_{C_{\mathrm{np}}}(\alpha,\gamma)=2$.

If the intersection number between α and γ is larger than 1, we follow α starting from an intersection point with γ until it hits γ again. If α comes back to γ on the same side of γ as when we started, we continue following α until we have one more intersection point. That is, we find a sub-arc ω of α that starts on one side of γ, ends on the other side of γ, and whose interior is either disjoint from γ or intersects γ once.

If the interior of ω is disjoint from γ, then ω together with a sub-arc of γ forms a curve β that intersects γ once. This means γ and β are both non-separating and hence β is non-peripheral (by Lemma 4.4, if one non-separating curve is non-peripheral, then they are all non-peripheral). The curve β intersects α fewer times than γ does and $d_{C_{\mathrm{np}}}(\gamma,\beta)=2$. Therefore, by induction, there is a path connecting β to α and thus to γ.

Assume the interior of ω intersects γ once. Then $\gamma\cup\omega$ fill a four–punctured sphere R. We argue in two cases. If γ is non-separating, then some boundary component β of R is also non-separating and hence non-peripheral, and β intersects α less than γ does. Similarly, if α is non-separating we argue the same way, switching the roles of α and γ.

Finally assume that the interior of ω intersects γ once and that both α and γ are separating. Then $\omega\cup\gamma$ decomposes the ends of Σ into four clopen sets $X_1,\ldots,X_4$. We choose the indices such that γ gives the decomposition

$$\mathrm{End}(\Sigma)=(X_1\sqcup X_2)\bigsqcup(X_3\sqcup X_4).$$

Since γ is separating and non-peripheral, by Lemma 4.4 the sets $(X_1\sqcup X_2)$ and $(X_3\sqcup X_4)$ are not small, and the hypotheses of Lemma 4.5 hold. Now Lemma 4.5 implies that one of the curves β obtained by surgery between γ and ω also decomposes $\mathrm{End}(\Sigma)$ into two clopen pieces that are not small, and hence β is non-peripheral by Lemma 4.4. Then $d_{C_{\mathrm{np}}}(\beta,\gamma)=1$ and β intersects α fewer times than γ does. Again, we are done by induction. □

5. Hyperbolicity of the non-peripheral curve graph

In this section we prove that, when $\zeta(\Sigma)\geq 5$, the non-peripheral curve graph $C_{\mathrm{np}}(\Sigma)$ is Gromov hyperbolic. The argument compares $C_{\mathrm{np}}(\Sigma)$ to the grand arc graph(constructed in [BNV23]) and uses a general "coning–off/electrification" criterion (Proposition 2.6 of [KR14]).

5.1. **Grand arcs and their graph.** Recall that $\mathcal{M}(\Sigma)$ denotes the space of maximal ends of Σ, and that the *grand splitting* partitions $\mathcal{M}(\Sigma)$ into finitely many *maximal types* – self-similar equivalence classes of maximal ends. In this case, self-smilar equivalence classes of maximal ends means that singletons in $\mathcal{M}(\Sigma)$ are each considered their own separate maximal type. In particular, $\mathcal{M}(\Sigma)$ is the disjoint union of these maximal types (see [BNV23]).

A *grand arc* is a properly embedded essential arc in Σ whose two ideal endpoints lie in *distinct maximal types*, taken up to isotopy rel. endpoints. The *grand arc graph* $\mathcal{GA}(\Sigma)$ is the graph whose vertices are grand arcs and whose edges join grand arcs that can be realized disjointly.

Lemma 5.1. *Assume $\zeta(\Sigma)\geq 5$. Then there are at least three distinct maximal types, hence $\mathcal{GA}(\Sigma)$ is nonempty. Moreover, $\mathcal{GA}(\Sigma)$ is Gromov hyperbolic.*

Proof. By the definition of $\zeta(\Sigma)$ (Section 2.5) and stability, $\zeta(\Sigma) \geq 5$ forces $\mathcal{M}(\Sigma)$ to contain at least three distinct maximal types. In particular, grand arcs exist.

Since Σ is stable, there are only finitely many maximal types. Hence the hypotheses of [BNV23, Theorem 1.1] apply, and $\mathcal{GA}(\Sigma)$ is Gromov hyperbolic. □

5.2. **From grand arcs to non-peripheral curves.** Fix $\zeta(\Sigma) \geq 5$ for the rest of this section. Given a grand arc ω with endpoints in two distinct maximal types, choose disjoint *stable neighborhoods* U_A, U_B of its endpoints in the sense of Section 2.5 (so $\partial U_A \cup \partial U_B$ is a multicurve and U_A, U_B are sufficiently small in the end space). Let $\alpha = \alpha(\omega; U_A, U_B)$ be any essential simple closed curve obtained as follows: take a regular neighborhood of $\omega \cup \partial U_A \cup \partial U_B$ and let α be a boundary component which separates $U_A \cup U_B$ from the complement. We call such an α a *small neighborhood curve* of ω.

Lemma 5.2. *For U_A, U_B chosen sufficiently small (depending on ω), every small neighborhood curve $\alpha(\omega; U_A, U_B)$ is non-peripheral.*

Proof. Let $\alpha = \alpha(\omega; U_A, U_B)$ be as above and write $\Sigma \setminus \alpha = Y_1 \sqcup Y_2$ with $U_A \cup U_B \subset Y_1$. By construction, Y_1 contains ends from two distinct maximal types, so for U_A, U_B sufficiently small the end set $\mathrm{End}(Y_1)$ is not *small* in the sense of Definition 4.3. Since there are at least three maximal types (Lemma 5.1), the complementary region Y_2 contains a maximal end of a third type, and hence $\mathrm{End}(Y_2)$ is also not small. Therefore neither complementary component of $\Sigma \setminus \alpha$ has small end space, so α cannot be pushed off of every compact set. Equivalently, α is non-peripheral. □

Although the curve $\alpha(\omega; U_A, U_B)$ depends on the auxiliary choices, its ambiguity is uniformly bounded in $C_{\mathrm{np}}(\Sigma)$.

Lemma 5.3. *For every grand arc ω, the set of all small neighborhood curves of ω has diameter at most 2 in $C_{\mathrm{np}}(\Sigma)$.*

Proof. Let $\alpha = \alpha(\omega; U_A, U_B)$ and $\alpha' = \alpha(\omega; U'_A, U'_B)$ be two such curves. Choose smaller stable neighborhoods $U^*_A \subset U_A \cap U'_A$ and $U^*_B \subset U_B \cap U'_B$ and let $\beta = \alpha(\omega; U^*_A, U^*_B)$. By construction, β can be realized disjointly from both α and α' (it is supported in a thinner regular neighborhood of ω). Hence in $C_{\mathrm{np}}(\Sigma)$,

$$d_{C_{\mathrm{np}}}(\alpha, \alpha') \leq d_{C_{\mathrm{np}}}(\alpha, \beta) + d_{C_{\mathrm{np}}}(\beta, \alpha') \leq 2.$$ □

5.3. **A hybrid graph and quasi-isometry to $C_{\mathrm{np}}(\Sigma)$.** Let $\mathcal{Y}$ be the graph with vertex set

$$V(\mathcal{Y}) = V(C_{\mathrm{np}}(\Sigma)) \sqcup V(\mathcal{GA}(\Sigma)),$$

and with edges defined by disjointness: two vertices are joined by an edge if the corresponding curve/arc representatives can be realized disjointly (in particular, $C_{\mathrm{np}}(\Sigma)$ appears as an induced subgraph of $\mathcal{Y}$).

Lemma 5.4. *The inclusion $C_{\mathrm{np}}(\Sigma) \hookrightarrow \mathcal{Y}$ is a quasi-isometry. More precisely:*

- *every vertex of $\mathcal{Y}$ lies at distance ≤ 1 from $C_{\mathrm{np}}(\Sigma)$; and*
- *there exists $A \geq 1$ such that for all $\alpha, \beta \in V(C_{\mathrm{np}}(\Sigma))$,*

$$d_{C_{\mathrm{np}}}(\alpha, \beta) \leq A\, d_{\mathcal{Y}}(\alpha, \beta) + A.$$

Proof. Let $\omega \in V(\mathcal{GA}(\Sigma))$. By Lemma 5.2, choose a small neighborhood curve $\alpha \in V(C_{\mathrm{np}}(\Sigma))$ disjoint from ω, so $d_{\mathcal{Y}}(\omega, \alpha) = 1$. Thus $C_{\mathrm{np}}(\Sigma)$ is 1–dense in $\mathcal{Y}$.

For the distance comparison, let $\gamma = (v_0, \dots, v_m)$ be a $\mathcal{Y}$–path from $\alpha = v_0$ to $\beta = v_m$ with $v_i \in V(\mathcal{Y})$. Whenever v_i is a grand arc, replace it by a choice of small neighborhood curve $\widehat{v}_i \in V(C_{\mathrm{np}}(\Sigma))$. Using Lemma 5.3, we can choose $\widehat{v}_i$ so that $\widehat{v}_i$ is disjoint from both neighbors v_{i-1} and v_{i+1} (after replacing them if necessary), hence the replacement increases path length by at most a uniform additive constant per arc vertex. In particular, we obtain a path in $C_{\mathrm{np}}(\Sigma)$ from α to β of length $\leq Am + A$ for some uniform A. Therefore $d_{C_{\mathrm{np}}}(\alpha, \beta) \leq A\, d_{\mathcal{Y}}(\alpha, \beta) + A$. □

Consequently, to prove hyperbolicity of $C_{\mathrm{np}}(\Sigma)$ it is enough to prove hyperbolicity of $\mathcal{Y}$.

5.4. **Electrifying the grand arc graph.** For each $\alpha \in V(C_{\mathrm{np}}(\Sigma))$, let $\mathcal{GA}_\alpha \subset \mathcal{GA}(\Sigma)$ denote the induced subgraph spanned by all grand arcs disjoint from α.

Lemma 5.5. *There exists $C \geq 0$ such that for every $\alpha \in V(C_{\mathrm{np}}(\Sigma))$, the subgraph $\mathcal{GA}_\alpha$ is C–quasiconvex in $\mathcal{GA}(\Sigma)$.*

Proof. Let $\omega_1, \omega_2 \in V(\mathcal{GA}_\alpha)$. In [BNV23], unicorn paths between grand arcs are constructed and shown to be uniform (unparameterized) quasi-geodesics in $\mathcal{GA}(\Sigma)$. Moreover, a unicorn surgery between two arcs disjoint from α can be performed so that every intermediate arc remains disjoint from α (since the surgery is supported in a regular neighborhood of $\omega_1 \cup \omega_2$, and α is disjoint from both endpoints). Hence the entire unicorn path from ω_1 to ω_2 lies in $\mathcal{GA}_\alpha$.

Since $\mathcal{GA}(\Sigma)$ is hyperbolic (Lemma 5.1), uniform quasi-geodesics fellow travel geodesics with uniform constants. Therefore any geodesic $[\omega_1, \omega_2]$ in $\mathcal{GA}(\Sigma)$ lies in a uniform neighborhood of the unicorn path, hence in a uniform neighborhood of $\mathcal{GA}_\alpha$. This proves uniform quasiconvexity. □

Let $\widehat{\mathcal{GA}}(\Sigma)$ be the graph obtained from $\mathcal{GA}(\Sigma)$ by *electrifying* the family $\{\mathcal{GA}_\alpha\}_{\alpha \in V(C_{\mathrm{np}}(\Sigma))}$, i.e. by adding an edge between any two vertices of $\mathcal{GA}_\alpha$ for each α.

Lemma 5.6. *The graph $\widehat{\mathcal{GA}}(\Sigma)$ is Gromov hyperbolic.*

Proof. By Lemma 5.1, $\mathcal{GA}(\Sigma)$ is hyperbolic. By Lemma 5.5, the family $\{\mathcal{GA}_\alpha\}$ is uniformly quasiconvex. Therefore, [KR14, Proposition 2.6] implies that the electrified graph $\widehat{\mathcal{GA}}(\Sigma)$ is hyperbolic. □

Now consider the *coned–off* version $\mathcal{Y}_0$ defined as follows: its vertex set is $V(\mathcal{GA}(\Sigma)) \sqcup V(C_{\mathrm{np}}(\Sigma))$, it contains all edges of $\mathcal{GA}(\Sigma)$, and for each $\alpha \in V(C_{\mathrm{np}}(\Sigma))$ and each grand arc $\omega \in V(\mathcal{GA}_\alpha)$ we add an edge between α and ω (thus $\mathcal{Y}_0$ is obtained from $\mathcal{GA}(\Sigma)$ by coning off each $\mathcal{GA}_\alpha$ with a cone vertex labelled by α).

Lemma 5.7. *The inclusion $\mathcal{GA}(\Sigma) \hookrightarrow \mathcal{Y}_0$ is a quasi-isometry, and $\mathcal{Y}_0$ is Gromov hyperbolic.*

Proof. In $\widehat{\mathcal{GA}}(\Sigma)$, if $\omega, \omega' \in \mathcal{GA}_\alpha$ then $d_{\widehat{\mathcal{GA}}}(\omega, \omega') = 1$ (by an added electrifying edge), while in $\mathcal{Y}_0$ we have $d_{\mathcal{Y}_0}(\omega, \omega') \leq 2$ via the path $\omega - \alpha - \omega'$. Conversely, any path in $\mathcal{Y}_0$ between arc vertices can be pushed into $\widehat{\mathcal{GA}}(\Sigma)$ by replacing each length–2 subpath $\omega - \alpha - \omega'$ by the electrifying edge between ω and ω'. Hence the identity on arc vertices is a quasi-isometry between $\widehat{\mathcal{GA}}(\Sigma)$ and the arc-vertex subgraph of $\mathcal{Y}_0$.

Finally, every cone vertex $\alpha \in V(C_{\mathrm{np}}(\Sigma))$ is adjacent to some grand arc: since α is compact, we can choose maximal ends in distinct maximal types far from α and connect them by a grand arc disjoint from α, so $\mathcal{GA}_\alpha \neq \emptyset$. Thus $V(\mathcal{GA}(\Sigma))$ is 1–dense in $\mathcal{Y}_0$. It follows that the inclusion $\mathcal{GA}(\Sigma) \hookrightarrow \mathcal{Y}_0$ is a quasi-isometry.

Since $\widehat{\mathcal{GA}}(\Sigma)$ is hyperbolic (Lemma 5.6) and hyperbolicity is a quasi-isometry invariant, $\mathcal{Y}_0$ is hyperbolic. □

5.5. **Hyperbolicity of $\mathcal{Y}$ and of $C_{\mathrm{np}}(\Sigma)$.** Recall that $\mathcal{Y}$ has the same vertex set as $\mathcal{Y}_0$ but, in addition, it also contains the curve–curve disjointness edges of $C_{\mathrm{np}}(\Sigma)$.

Lemma 5.8. *The identity map on vertices induces a quasi-isometry $\mathcal{Y}_0 \to \mathcal{Y}$. In particular, $\mathcal{Y}$ is Gromov hyperbolic.*

Proof. The map $\mathcal{Y}_0 \to \mathcal{Y}$ is 1–Lipschitz (we only add edges). It suffices to show that every edge between disjoint curves in $\mathcal{Y}$ is realized by a uniformly bounded path in $\mathcal{Y}_0$.

Let $\alpha, \beta \in V(C_{\mathrm{np}}(\Sigma))$ be disjoint. Since α and β are compact and disjoint, we can choose maximal ends in distinct maximal types so that one lies in a complementary component of $\Sigma \setminus (\alpha \cup \beta)$ on one side and the other lies in a complementary component on the other side. Then there is a grand arc ω connecting these ends which is disjoint from $\alpha \cup \beta$. Hence $\omega \in \mathcal{GA}_\alpha \cap \mathcal{GA}_\beta$, so in $\mathcal{Y}_0$ we have a path $\alpha - \omega - \beta$ of length 2. Therefore, for curve–curve edges in $\mathcal{Y}$ we have $d_{\mathcal{Y}_0}(\alpha, \beta) \leq 2$. This proves the quasi-isometry claim.

Since $\mathcal{Y}_0$ is hyperbolic (Lemma 5.7), so is $\mathcal{Y}$. □

We can now state the main theorem of this section.

Theorem 5.9. *Assume $\zeta(\Sigma) \geq 5$. Then the non-peripheral curve graph $C_{\mathrm{np}}(\Sigma)$ is Gromov hyperbolic.*

Proof. By Lemma 5.8, the hybrid graph $\mathcal{Y}$ is hyperbolic. By Lemma 5.4, $C_{\mathrm{np}}(\Sigma)$ is quasi-isometric to $\mathcal{Y}$. Since hyperbolicity is a quasi-isometry invariant, $C_{\mathrm{np}}(\Sigma)$ is hyperbolic. □

Remark 5.10. The grand arc graph and $C_{\mathrm{np}}(\Sigma)$ need not be quasi-isometric. For example, when $|\mathcal{P}| = 2$ and $|\mathcal{A}| \geq 3$ (in the notation of Section 2.5), we have a non-peripheral curve $\alpha \subset K_0$ which separates sets in $\mathcal{P}$ from sets in $\mathcal{A}$. Let W be the component of $K_0 \setminus \alpha$ that is on the side of α containing sets in $\mathcal{A}$. That is, the boundary of W is the union of α and the boundaries of surfaces Σ_A, $A \in \mathcal{A}$. Note that W is a witness for the grand arc graph, since any arc from x_A to x_B has to intersect $\partial\Sigma_A$ and hence W. Therefore, the pseudo-Anosov on W acts loxodromically on $\mathcal{GA}(\Sigma)$. Hence, the set $\mathcal{GA}_\alpha$ of grand arcs that are disjoint from α has an infinite diameter in $\mathcal{GA}(\Sigma)$. This phenomenon is compatible with the argument above: the electrification step uses quasi-convexity of the subgraphs $\mathcal{GA}_\alpha$, not bounded diameter.

6. Divergence of big mapping class groups

In this section we give a quadratic bound for the divergence of big mapping class groups. Divergence is a quasi-isometry invariant of geodesic metric spaces which measures the length of the shortest detours avoiding large balls. Although divergence is most often studied for finitely generated groups, the definition applies in the present setting because $\mathrm{Map}(\Sigma)$ is CB-generated and hence admits a well-defined quasi-isometry type of word metrics (Section 2.5).

6.1. **Divergence.** We recall the definition from [DMS09]. (In [DMS09] this is stated for proper geodesic spaces; the definition below makes sense for any geodesic metric space, and this is the level at which we use it.)

Definition 6.1 (Divergence)**.** Let (X, d_X) be a geodesic metric space. Fix constants $0 < \rho < 1$ and $T > 0$. For a pair of points $a, b \in X$ and a point $c \in X - \{a, b\}$, define the *divergence of* (a, b) *relative to* c to be the length of the shortest path from a to b in X avoiding the open ball $B(c, r)$ of radius

$$r = \rho \cdot \min \big(d_X(c, a), d_X(c, b)\big) - T.$$

If no such path exists, define this divergence to be ∞. The *divergence of the pair* (a, b) is the supremum of these quantities over all $c \in X - \{a, b\}$. Finally, define

$$\mathrm{Div}_X(R; \rho, T) = \sup \Big\{ \mathrm{Div}_X(a, b; \rho, T) \ \Big| \ d_X(a, b) \leq R \Big\},$$

where $\mathrm{Div}_X(a, b; \rho, T)$ denotes the divergence of the pair (a, b) (i.e. the supremum over c). As usual, we consider divergence functions up to the equivalence relation

$$f \preceq g \iff \exists C > 1 \ \forall R \geq 0 : \ f(R) \leq C\, g(CR) + CR + C,$$

and

$$f \equiv g \iff f \preceq g \quad \text{and} \quad g \preceq f.$$

6.2. **Quadratic upper bound.** Our goal is to prove that if $\zeta(\Sigma) \geq 5$ then $\mathrm{Map}(\Sigma)$ has at most quadratic divergence.

Theorem 6.2. *Assume* Σ *is stable,* $\mathrm{Map}(\Sigma)$ *is CB-generated, and* $\zeta(\Sigma) \geq 5$*. Equip* $\mathrm{Map}(\Sigma)$ *with the word metric* $d_{\mathcal{S}}$ *associated to any CB generating set. Then for every* $0 < \rho < 1$ *and* $T > 0$

$$\mathrm{Div}_{(\mathrm{Map}(\Sigma), d_{\mathcal{S}})}(R; \rho, T) \preceq R^2.$$

In other words, the divergence of $\mathrm{Map}(\Sigma)$ *is at most quadratic.*

Remark 6.3. By [DMS09, Section 3], divergence is a quasi-isometry invariant, and for CB-generated Polish groups the quasi-isometry type of a word metric does not depend on the particular symmetric CB generating set (Section 2.5, [Ros21]). Thus it is enough to work with $d_{\mathcal{S}}$ for the specific generating set $\mathcal{S}$ from Theorem 2.8. Also, the equivalence class of divergence does not depend on ρ and T ([DMS09, Section 3]). Hence we keep can use any pair (ρ, T).

The left translation is an isometry of $d_{\mathcal{S}}$, so any triple (a, b, c) can be moved so that $c = \mathrm{id}$. Since we are taking supremum over all triples (a, b, c), we can as well take the supremum over all triples where $c = \mathrm{id}$. Also, following [DMS09, Section 3], we can assume $d_{\mathcal{S}}(\mathrm{id}, a) \asymp d_{\mathcal{S}}(\mathrm{id}, b) \asymp d_{\mathcal{S}}(a, b) \asymp R$. Therefore, Theorem 6.2 follows from Theorem 6.4 below:

Theorem 6.4. *Assume* $\zeta(\Sigma) \geq 5$*. There exist constants* $M_0 > 0$ *and* $T_0 > 0$ *(depending only on* Σ*) such that the following holds for all* $R > 0$*. If* $g_1, g_2 \in \mathrm{Map}(\Sigma)$ *satisfy*

$$R \leq \|g_i\|_{\mathcal{S}} \leq 2R \qquad (i = 1, 2),$$

then there exists a path in $(\mathrm{Map}(\Sigma), d_{\mathcal{S}})$ *from* g_1 *to* g_2 *which is disjoint from the ball*

$$B(\mathrm{id}, R/2 - T_0)$$

about the identity and has a length at most $M_0\, R^2$*.*

6.3. **Step 1: a twist length function.** We use the annular specialization of the length functions from Section 3.4 (Definition 3.14).

Fix once and for all a marking μ on K_0 (Section 2.5). For a non-peripheral curve α, let A_α denote the annulus with core α, and define

$$L_\alpha(g) := \sup_{h\in \mathrm{Map}(\Sigma)} \mathrm{tw}_{h(\alpha)}\big(\mu, g(\mu)\big), \tag{8}$$

where $\mathrm{tw}_{h(\alpha)}$ is the annular distance from Definition 3.12 (Section 3.3). This is the special case of $L_{\mathcal{R}}$ with $\mathcal{R} = \{A_\alpha\}$.

Lemma 6.5. *For every non-peripheral curve α, the function L_α is a length function on* $\mathrm{Map}(\Sigma)$*. Moreover, there exists $M_\alpha < \infty$ such that*

$$\|g\|_{\mathcal{S}} \geq \frac{1}{M_\alpha} L_\alpha(g) \qquad \text{for all } g \in \mathrm{Map}(\Sigma). \tag{9}$$

Proof. The fact that L_α is a length function is exactly Lemma 3.15 applied to $\mathcal{R} = \{A_\alpha\}$ (Section 3.4). Since $\mathcal{S}$ is CB, by [Ros21, Proposition 2.7(5)] and Remark 3.7, L_α is bounded on $\mathcal{S}$: set

$$M_\alpha := \sup_{s\in\mathcal{S}} L_\alpha(s) < \infty.$$

Then (9) follows from subadditivity: if $g = s_1 \cdots s_n$ with $s_i \in \mathcal{S}$, then

$$L_\alpha(g) \leq \sum_{i=1}^{n} L_\alpha(s_i) \leq n M_\alpha,$$

hence $\|g\|_{\mathcal{S}} = n \geq L_\alpha(g)/M_\alpha$. □

Lemma 6.6. *Let α be a non-peripheral curve and let D_α be the Dehn twist about α. There exists $c_\alpha > 0$ such that for all $n \in \mathbb{Z}$,*

$$L_\alpha(D_\alpha^n) \geq c_\alpha\, |n|.$$

Proof. Since α is non-peripheral, every translate $h(\alpha)$ intersects K_0 essentially (Definition 3.3), hence $\pi_{h(\alpha)}(\mu) \neq \emptyset$ and the twisting distance $\mathrm{tw}_{h(\alpha)}(\mu, D_\alpha^n(\mu))$ is defined. Taking $h = \mathrm{id}$ in (8) gives

$$L_\alpha(D_\alpha^n) \geq \mathrm{tw}_\alpha\big(\mu, D_\alpha^n(\mu)\big),$$

and by Remark 3.13 the right-hand side grows coarsely linearly in $|n|$. □

6.4. **Step 2: a projection to $C_{\mathrm{np}}(\Sigma)$.** Assume throughout this subsection that $\zeta(\Sigma) \geq 5$ so that $C_{\mathrm{np}}(\Sigma)$ is connected (Theorem 4.2 in Section 4.2). Fix a base vertex $\alpha_0 \in C_{\mathrm{np}}(\Sigma)$ and define

$$\pi : \mathrm{Map}(\Sigma) \to C_{\mathrm{np}}(\Sigma), \qquad \pi(g) := g(\alpha_0). \tag{10}$$

Remark 6.7. If α_0' is another base curve in $C_{\mathrm{np}}(\Sigma)$, then for every g,

$$d_{C_{\mathrm{np}}}\big(g(\alpha_0), g(\alpha_0')\big) = d_{C_{\mathrm{np}}}(\alpha_0, \alpha_0')$$

because $\mathrm{Map}(\Sigma)$ acts by graph automorphisms on $C_{\mathrm{np}}(\Sigma)$. Thus changing the base curve changes π by a uniformly bounded amount.

6.5. **Step 3: π is Lipschitz.**

Lemma 6.8. *There exists $L \geq 1$ such that for every generator $s \in \mathcal{S}$,*

$$d_{C_{\rm np}}\big(\alpha_0, s(\alpha_0)\big) \leq L.$$

Consequently, for all $g, h \in \mathrm{Map}(\Sigma)$,

$$d_{C_{\rm np}}\big(\pi(g), \pi(h)\big) \leq L\, d_{\mathcal{S}}(g, h).$$

Proof. The second statement follows from the first by writing $h = gs_1 \cdots s_n$ with $n = d_{\mathcal{S}}(g, h)$ and using the triangle inequality in $C_{\rm np}(\Sigma)$:

$$d_{C_{\rm np}}\big(g(\alpha_0), h(\alpha_0)\big) \leq \sum_{i=1}^{n} d_{C_{\rm np}}\big(gs_1 \cdots s_{i-1}(\alpha_0), gs_1 \cdots s_i(\alpha_0)\big) = \sum_{i=1}^{n} d_{C_{\rm np}}\big(\alpha_0, s_i(\alpha_0)\big) \leq nL.$$

Thus it remains to prove the existence of L.

Fix $\alpha_0 \subset K_0$ once and for all. For each of the finitely many types of generators in Theorem 2.8 (Section 2.6) one checks that the geometric intersection number $i(\alpha_0, s(\alpha_0))$ is uniformly bounded independent of $s \in \mathcal{S}$:

- elements of $\mathcal{V}_{K_0}$ fix K_0 pointwise, hence fix α_0;
- local generators in $\mathcal{L}$ are supported in a fixed finite-type subsurface, so they change α_0 by a uniformly bounded amount; and
- the shift maps and the finitely many maximal-end permutations have supports and boundary data contained in a fixed finite-type subsurface (as in the definition of $\mathcal{L}$), so their effect on α_0 is controlled as well.

Given a uniform bound on $i(\alpha_0, s(\alpha_0))$, the surgery argument from Section 4.2 produces a uniformly bounded path in $C_{\rm np}(\Sigma)$ from α_0 to $s(\alpha_0)$. This yields the desired L. □

6.6. **Step 4: commuting Dehn twists for generators.**

Lemma 6.9. *There exists a constant $K \geq 1$ with the following property. For every $s \in \mathcal{S}$ there exists a curve $\alpha_s \in C_{\rm np}(\Sigma)$ such that:*

- *$d_{C_{\rm np}}(\alpha_0, \alpha_s) \leq K$; and*
- *the Dehn twist D_{α_s} commutes with s.*

Proof. We choose α_s by a case-by-case inspection of the generators in Theorem 2.8 (Section 2.6).

If $s \in \mathcal{V}_{K_0}$, take $\alpha_s = \alpha_0$, so s fixes α_0 and hence commutes with D_{α_0}. If $s \in \mathcal{L}$ is supported in the fixed finite-type region used to define $\mathcal{L}$, take α_s to be a non-peripheral curve disjoint from that region; such a curve exists in the five-holed sphere guaranteed by $\zeta(\Sigma) \geq 5$ (Section 4.2). For the finitely many maximal-end permutations and shift maps in $\mathcal{S}$, choose α_s among finitely many curves in the fixed finite-type region where these maps meet K_0, so that α_s is preserved by s. In all cases, $s(\alpha_s) = \alpha_s$ and hence s commutes with D_{α_s}.

Since there are only finitely many types of generators, we may take K to be the maximum of the (finite) distances $d_{C_{\rm np}}(\alpha_0, \alpha_s)$ arising from the choices above. □

6.7. Step 5: a "linked" word decomposition.

Proposition 6.10. *There exists $C_0 \geq 1$ with the following property. For every $g_1, g_2 \in \mathrm{Map}(\Sigma)$ there exist an integer $n \geq 1$, a sequence of curves*

$$\alpha_1, \alpha_2, \dots, \alpha_n \in C_{\mathrm{np}}(\Sigma),$$

and a sequence of generators $s_1, \dots, s_n \in \mathcal{S} \cup \{\mathrm{id}\}$ such that:

- *D_{α_i} commutes with s_i for every i;*
- *$s_1 s_2 \cdots s_n = g_1^{-1} g_2$; and*
- *$n \leq C_0 \, \|g_1^{-1} g_2\|_{\mathcal{S}}$.*

Moreover, the curves α_i can be chosen from a finite subset $\mathcal{F} \subset C_{\mathrm{np}}(\Sigma)$ depending only on Σ and $\mathcal{S}$.

Proof. Write $g_1^{-1} g_2$ as a word of minimal length in $\mathcal{S}$:

$$g_1^{-1} g_2 = t_1 t_2 \cdots t_m, \qquad t_j \in \mathcal{S}, \quad m = \|g_1^{-1} g_2\|_{\mathcal{S}}.$$

For each t_j, apply Lemma 6.9 to obtain a curve $\beta_j \in C_{\mathrm{np}}(\Sigma)$ such that D_{β_j} commutes with t_j.

By construction in Lemma 6.9, the set of possible curves β_j is finite: each generator $t \in \mathcal{S}$ falls into one of finitely many types, and in each type we choose α_t from a fixed finite list. Let $\mathcal{B} \subset C_{\mathrm{np}}(\Sigma)$ denote the (finite) set of all such curves.

For each ordered pair $(\beta, \beta') \in \mathcal{B} \times \mathcal{B}$, fix once and for all an edge path $P(\beta, \beta')$ in $C_{\mathrm{np}}(\Sigma)$ from β to β'. Let $\mathcal{F}$ be the union of the vertex sets of these paths. Since $\mathcal{B}$ is finite and each $P(\beta, \beta')$ is finite, the set $\mathcal{F}$ is finite.

We now build the required word decomposition by concatenating, for each j, the fixed path $P(\beta_j, \beta_{j+1})$ (inserting identity generators along that path), and then inserting the generator t_j at the vertex β_j. Identity commutes with all twists, and D_{β_j} commutes with t_j by construction. The resulting word $s_1 \cdots s_n$ equals $t_1 \cdots t_m = g_1^{-1} g_2$, and the total number n of letters is bounded by $n \leq C_0 m$ where C_0 is the maximum, over $(\beta, \beta') \in \mathcal{B}^2$, of the length of $P(\beta, \beta')$ plus 1. Finally, every curve label α_i is a vertex of some $P(\beta, \beta')$, hence lies in the finite set $\mathcal{F}$. □

Remark 6.11. In Proposition 6.10 all twist curves belong to the fixed finite set $\mathcal{F}$. Consequently, any constants attached to these curves can be chosen uniformly.

For each $\alpha \in \mathcal{F}$ let M_α be as in Lemma 6.5, and let c_α be the constant from Lemma 6.6. In addition, fix an additive constant $B_\alpha \geq 0$ so that for every $m \in \mathbb{Z}$,

$$\mathrm{tw}_\alpha \left(\mu, D_\alpha^m(\mu)\right) \geq |m| - B_\alpha,$$

and so that $\pi_\alpha(\mu)$ has diameter at most B_α in the annular curve graph of α. (Any such choice works, and we only use that these constants exist and are uniform on $\mathcal{F}$.) Define

$$M := \max_{\alpha \in \mathcal{F}} M_\alpha, \qquad c := \min_{\alpha \in \mathcal{F}} c_\alpha, \qquad B := \max_{\alpha \in \mathcal{F}} B_\alpha, \qquad A := \max\{2B, 1\}.$$

Finally, for each $\alpha \in \mathcal{F}$ let (q_α, Q_α) be the quasi-geodesic constants from Lemma 6.12 for the twist line $m \mapsto g D_\alpha^m$. Set

$$q := \max_{\alpha \in \mathcal{F}} q_\alpha, \qquad Q := \max_{\alpha \in \mathcal{F}} Q_\alpha, \qquad D_0 := \max_{\alpha \in \mathcal{F}} \|D_\alpha\|_{\mathcal{S}}.$$

Fix a constant $k \geq \max(3, M)$ so that

$$\frac{1+2q^2}{k} \leq \frac{1}{2}, \qquad \text{and} \qquad k \geq \frac{4D_0 M}{c}.$$

All subsequent estimates will use the constants M, c, B, A, k (and implicitly q, Q).

6.8. **Step 6: detours using commuting Dehn twists.** We now complete the proof of Theorem 6.4 using the standard detour construction (compare [DR09, Section 4]).

Lemma 6.12. *Let $\alpha \in C_{\mathrm{np}}(\Sigma)$ and let D_α be the Dehn twist about α. The map*

$$\mathbb{Z} \to (\mathrm{Map}(\Sigma), d_{\mathcal{S}}), \qquad m \mapsto g D_\alpha^m$$

is a quasi-isometric embedding, with constants depending only on α and $\mathcal{S}$.

Proof. Since right-multiplication by g is an isometry, it suffices to treat the map $m \mapsto D_\alpha^m$. The upper bound is immediate from subadditivity:

$$d_{\mathcal{S}}(D_\alpha^m, D_\alpha^n) = \|D_\alpha^{m-n}\|_{\mathcal{S}} \leq |m-n| \, \|D_\alpha\|_{\mathcal{S}}.$$

For the lower bound, apply Lemma 6.5 and Lemma 6.6:

$$d_{\mathcal{S}}(D_\alpha^m, D_\alpha^n) = \|D_\alpha^{m-n}\|_{\mathcal{S}} \geq \frac{1}{M_\alpha} L_\alpha(D_\alpha^{m-n}) \geq \frac{c_\alpha}{M_\alpha} |m-n|.$$

This gives the desired quasi-isometric embedding. □

Lemma 6.13 (Extending branch and uniform twist escape)**.** *Let $\mathcal{F} \subset C_{\mathrm{np}}(\Sigma)$ and the constants M, k, A, c, q, Q, B be as in Remark 6.11. There exists a constant $T_1 \geq 0$ (depending only on Σ and $\mathcal{S}$) such that the following hold.*

(1) *(*Extending branch*) For every $\alpha \in \mathcal{F}$ and every $g \in$* Map(Σ)*, there exists a sign $\varepsilon \in \{\pm 1\}$ such that*

$$\|g D_\alpha^{\varepsilon m}\|_{\mathcal{S}} \geq \frac{1}{k} \|g\|_{\mathcal{S}} - T_1 \qquad \text{for all } m \geq 0. \tag{11}$$

(2) *(*Uniform linear lower bound: one twist*) For every $\alpha \in \mathcal{F}$, every $g \in$* Map(Σ)*, and every $m \in \mathbb{Z}$,*

$$\|g D_\alpha^m\|_{\mathcal{S}} \geq \frac{|m| - M \|g\|_{\mathcal{S}} - A}{k}. \tag{12}$$

In particular, for the sign ε from (1) *and every $m \geq 0$,*

$$\|g D_\alpha^{\varepsilon m}\|_{\mathcal{S}} \geq \frac{m - M \|g\|_{\mathcal{S}} - A}{k}.$$

(3) *(*Uniform linear lower bound: two disjoint twists*) If $\alpha, \beta \in \mathcal{F}$ are disjoint, then for every $g \in$* Map(Σ) *and every $m, n \in \mathbb{Z}$,*

$$\|g D_\alpha^m D_\beta^n\|_{\mathcal{S}} \geq \frac{\max\{|m|, |n|\} - M \|g\|_{\mathcal{S}} - A}{k}. \tag{13}$$

Proof. We begin with (1). Set

$$D_0 := \max_{\alpha \in \mathcal{F}} \|D_\alpha\|_{\mathcal{S}} < \infty, \qquad \lambda := \frac{c}{M} > 0.$$

Fix $\alpha \in \mathcal{F}$ and $g \in \mathrm{Map}(\Sigma)$, and write $R := \|g\|_{\mathcal{S}}$. Assume for contradiction that both rays $\{gD_\alpha^m\}_{m\geq 0}$ and $\{gD_\alpha^{-m}\}_{m\geq 0}$ meet the ball $B(\mathrm{id}, R/k)$. Then there exist integers $m_+, m_- \geq 0$ such that

$$\|gD_\alpha^{m_+}\|_{\mathcal{S}} \leq \frac{R}{k} \qquad \text{and} \qquad \|gD_\alpha^{-m_-}\|_{\mathcal{S}} \leq \frac{R}{k}.$$

By the triangle inequality,

$$\|D_\alpha^{m_+}\|_{\mathcal{S}} = d_{\mathcal{S}}(g, gD_\alpha^{m_+}) \geq R - \frac{R}{k}, \qquad \|D_\alpha^{m_-}\|_{\mathcal{S}} = d_{\mathcal{S}}(g, gD_\alpha^{-m_-}) \geq R - \frac{R}{k}.$$

Since $\|D_\alpha^m\|_{\mathcal{S}} \leq |m|\, \|D_\alpha\|_{\mathcal{S}} \leq |m| D_0$, we obtain

$$m_+ \geq \frac{(1-1/k)R}{D_0} \qquad \text{and} \qquad m_- \geq \frac{(1-1/k)R}{D_0}.$$

On the other hand, using left-invariance and Lemmas 6.6 and 6.5,

$$d_{\mathcal{S}}(gD_\alpha^{-m_-}, gD_\alpha^{m_+}) = \|D_\alpha^{m_+ + m_-}\|_{\mathcal{S}} \geq \frac{1}{M}\, L_\alpha(D_\alpha^{m_+ + m_-}) \geq \lambda\,(m_+ + m_-).$$

But since both points lie in $B(\mathrm{id}, R/k)$ we also have

$$d_{\mathcal{S}}(gD_\alpha^{-m_-}, gD_\alpha^{m_+}) \leq \frac{2R}{k}.$$

Combining the last three displays yields

$$\frac{2R}{k} \geq \lambda\,(m_+ + m_-) \geq \frac{2\lambda(1-1/k)R}{D_0}.$$

For k large enough (as fixed in Remark 6.11), this is impossible once R is large. Enlarging T_1 if necessary makes (11) valid for all R, and the desired sign ε is the direction of the ray that avoids $B(\mathrm{id}, R/k)$.

We now prove (2). Fix $\alpha \in \mathcal{F}$ and $g \in \mathrm{Map}(\Sigma)$, and set $\beta = g(\alpha)$ so that $gD_\alpha^m = D_\beta^m g$. Taking $h = g$ in the definition of L_α gives

$$L_\alpha(gD_\alpha^m) \geq \mathrm{tw}_\beta\left(\mu,\, D_\beta^m g(\mu)\right).$$

Since D_β acts on the annular curve graph of β by translation, and the projection sets $\pi_\beta(\mu)$ and $\pi_\beta\big(g(\mu)\big)$ have diameter bounded by the uniform constant B (Remark 6.11), we have

$$\mathrm{tw}_\beta\left(g(\mu), D_\beta^m g(\mu)\right) \geq |m| - B.$$

By the triangle inequality in the annular graph,

$$\mathrm{tw}_\beta\left(\mu, D_\beta^m g(\mu)\right) \geq |m| - B - \mathrm{tw}_\beta\left(\mu, g(\mu)\right).$$

Again taking $h = g$ in (8) shows $\mathrm{tw}_\beta(\mu, g(\mu)) \leq L_\alpha(g)$, and Remark 6.11 gives $L_\alpha(g) \leq M\,\|g\|_{\mathcal{S}}$. Therefore

$$L_\alpha(gD_\alpha^m) \geq |m| - M\,\|g\|_{\mathcal{S}} - B.$$

Finally, Lemma 6.5 and $k \geq M$ imply

$$\|gD_\alpha^m\|_{\mathcal{S}} \geq \frac{1}{M}\, L_\alpha(gD_\alpha^m) \geq \frac{|m| - M\,\|g\|_{\mathcal{S}} - B}{M} \geq \frac{|m| - M\,\|g\|_{\mathcal{S}} - A}{k},$$

which is (12).

For (3), assume $\alpha, \beta \in \mathcal{F}$ are disjoint and write $\delta = g(\beta)$. Then $gD_\alpha^m D_\beta^n = D_{g(\alpha)}^m D_\delta^n g$ and the curves $g(\alpha)$ and δ are disjoint, so the twists commute. Taking $h = g$ in the definition of L_β gives

$$L_\beta(gD_\alpha^m D_\beta^n) \geq \mathrm{tw}_\delta\left(\mu,\, D_{g(\alpha)}^m D_\delta^n g(\mu)\right).$$

Since $g(\alpha)$ is disjoint from δ, the twist $D_{g(\alpha)}^m$ is supported away from the annulus A_δ. Hence it moves the projection to the annular curve graph of δ by at most a uniform constant (absorbed into B), and we obtain

$$\mathrm{tw}_\delta\left(\mu,\, D_{g(\alpha)}^m D_\delta^n g(\mu)\right) \geq \mathrm{tw}_\delta\left(\mu,\, D_\delta^n g(\mu)\right) - B.$$

Applying the estimate from (2) (with β in place of α and using the same uniform constants) yields

$$\mathrm{tw}_\delta\left(\mu,\, D_\delta^n g(\mu)\right) \geq |n| - M\,\|g\|_{\mathcal{S}} - B,$$

and therefore

$$L_\beta(gD_\alpha^m D_\beta^n) \geq |n| - M\,\|g\|_{\mathcal{S}} - 2B.$$

As above, $k \geq M$ and $A \geq 2B$ give

$$\|gD_\alpha^m D_\beta^n\|_{\mathcal{S}} \geq \frac{|n| - M\,\|g\|_{\mathcal{S}} - A}{k}.$$

Swapping the roles of α and β gives the same lower bound with $|m|$ in place of $|n|$, and combining the two yields (13). □

Proof of Theorem 6.4. Fix $R > 0$ and let $g_1, g_2 \in \mathrm{Map}(\Sigma)$ satisfy $R \leq \|g_i\|_{\mathcal{S}} \leq 2R$. Set $h = g_1^{-1}g_2$ so that $\|h\|_{\mathcal{S}} \leq 4R$.

Apply Proposition 6.10 to obtain curves $\alpha_1, \ldots, \alpha_n \in C_{\mathrm{np}}(\Sigma)$ and elements $s_1, \ldots, s_n \in \mathcal{S} \cup \{\mathrm{id}\}$ with $s_1 \cdots s_n = h$, D_{α_i} commuting with s_i, and

$$n \leq C_0\,\|h\|_{\mathcal{S}} \leq 4C_0R.$$

By construction in Proposition 6.10 (inserting identity steps along fixed edge paths in $C_{\mathrm{np}}(\Sigma)$), we may assume that consecutive curves are either equal or adjacent in $C_{\mathrm{np}}(\Sigma)$; in particular, whenever $\alpha_i \neq \alpha_{i+1}$ the curves α_i and α_{i+1} are disjoint and hence the twists D_{α_i} and $D_{\alpha_{i+1}}$ commute.

Let $K_{\mathrm{tw}} \geq 1$ be a constant (to be fixed below) and set

$$N := \lceil K_{\mathrm{tw}} R \rceil.$$

Choose $\varepsilon_1, \varepsilon_n \in \{\pm 1\}$ as in Lemma 6.13(1) applied to the pairs (α_1, g_1) and (α_n, g_2), and define

$$v := g_1 D_{\alpha_1}^{\varepsilon_1 N}, \qquad w := g_2 D_{\alpha_n}^{\varepsilon_n N}.$$

By Lemma 6.13(1), the path from g_1 to v along the twist ray is disjoint from $B(\mathrm{id}, R/k - T_1)$, and similarly the path from g_2 to w is disjoint from $B(\mathrm{id}, R/k - T_1)$. Thus it remains to connect v to w by a path outside the same ball.

Set $u_0 = \mathrm{id}$ and $u_i = s_1 \cdots s_i$ for $1 \leq i \leq n$, so $u_n = h$ and $g_1 u_n = g_2$. For $0 \leq i \leq n$ set

$$p_i := g_1 u_i.$$

Then $p_0 = g_1$ and $p_n = g_2$, and for each i we have $p_i = p_{i-1}s_i$. Since $\|u_i\|_{\mathcal{S}} \leq \|h\|_{\mathcal{S}} \leq 4R$, we have

$$\|p_i\|_{\mathcal{S}} = \|g_1 u_i\|_{\mathcal{S}} \leq \|g_1\|_{\mathcal{S}} + \|u_i\|_{\mathcal{S}} \leq 6R.$$

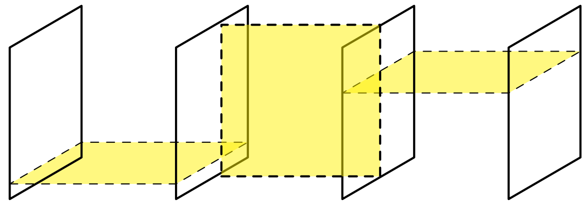

FIGURE 3. Chain of flats form alternating product region

We now build a detour path from $v = p_0 D_{\alpha_1}^{\varepsilon_1 N}$ to $w = p_n D_{\alpha_n}^{\varepsilon_n N}$ by induction. For $1 \leq i \leq n-1$, we connect

$$p_{i-1} D_{\alpha_i}^{\varepsilon_i N} \quad \text{to} \quad p_i D_{\alpha_{i+1}}^{\varepsilon_{i+1} N}$$

by the following sequence of moves:

(1) move from $p_{i-1} D_{\alpha_i}^{\varepsilon_i N}$ to $p_i D_{\alpha_i}^{\varepsilon_i N}$ by right-multiplying by s_i (using that s_i commutes with D_{α_i});
(2) add the new large twist, moving along the ray

$$p_i D_{\alpha_i}^{\varepsilon_i N} \to p_i D_{\alpha_i}^{\varepsilon_i N} D_{\alpha_{i+1}}^{\varepsilon_{i+1} N};$$

(3) remove the previous large twist *after* the new twist is present, moving along the ray

$$p_i D_{\alpha_i}^{\varepsilon_i N} D_{\alpha_{i+1}}^{\varepsilon_{i+1} N} \to p_i D_{\alpha_{i+1}}^{\varepsilon_{i+1} N}.$$

If $\alpha_i = \alpha_{i+1}$, we interpret steps (2)–(3) as the obvious cancellation.

This produces a path from v to w whose length is bounded by

$$n + 2nN \leq 4C_0 R + 8C_0 RN \preceq R^2,$$

since $N \asymp R$.

It remains to verify that all vertices on the constructed path stay outside a ball of radius $\rho R - T_0$. Fix i, and consider the vertices occurring in the three steps above. Step (1) only right-multiplies by a generator, and step (2) and step (3) only move along twist rays. At every point in steps (2) and (3), there is at least one twist factor of exponent at least N present. Therefore, by Lemma 6.13(2)–(3), applied with $g = p_i$ and with either one twist or a pair of disjoint twists, we obtain a uniform lower bound of the form

$$\|x\|_{\mathcal{S}} \geq \frac{N - M\,\|p_i\|_{\mathcal{S}} - A}{k} \geq \frac{K_{\text{tw}} R - 6MR - A}{k}$$

for every vertex x that lies in steps (2) or (3). Choosing K_{tw} large enough compared to M, k, A, we can ensure that the right-hand side is at least $\rho R - T_0$. Since the steps (1) contribute only a uniformly bounded backtracking, increasing T_0 if needed yields that the entire path is disjoint from $B(\text{id}, \rho R - T_0)$.

Finally, concatenating the twist-ray path from g_1 to v, the $O(R^2)$ detour path from v to w, and the reverse of the twist-ray path from g_2 to w gives a path from g_1 to g_2 disjoint from $B(\text{id}, \rho R - T_0)$ and of length at most $M_0 R^2$ for a uniform M_0. This completes the proof of Theorem 6.4. □

6.9. **End space of big mapping class groups.** It is a classical quest to investigate the end space of groups. If end space is not empty, the trichotomy of 1,2 or infinity ends is established by the theorem of Freudenthal–Hopf. Such a tricotomy has been established further for countably and compactly generated groups by [Cor19] and for all non-locally finite graphs by [OP22]. In the case of CB-generated big mapping class groups, end space can be defined as the inverse limit of connected components of the associated Cayley graph under exhaustion by bounded balls. This question has been largely open until recently. [OQW25] showed that for avenue surface the associated group is one-ended. Now we offer an answer for all surfaces satisfying the assumptions of Theorem C:

Corollary 6.14. *Suppose* Σ *is a stable surface and* $\mathrm{Map}(\Sigma)$ *is CB-generated. Suppose in additionally that* $\zeta(\Sigma) \geq 5$*, then with respect to any CB-generating set* $\mathrm{Map}(\Sigma)$ *is one-ended.*

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