THE NUMBER OF ENDS OF BIG MAPPING CLASS GROUPS

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Abstract. We analyze the number of ends of the mapping class group of a stable avenue surface. We prove that the mapping class group is one-ended whenever the stable avenue surface has at least one end of discrete type. Our method is to show that the associated translatable curve graph, which is quasi-isometric to the mapping class group, is one-ended.

1. Introduction

A central theme in geometric group theory is the *large-scale geometry* of groups. By this we mean the metric structure that is preserved under quasi-isometries—maps which preserve distances up to a uniformly bounded multiplicative and additive error (Section 2.1). In a 1983 ICM address [Gro83], Gromov set forth a far-reaching program advocating the study of finitely generated groups as geometric objects and their classification up to quasi-isometry. Among the earliest and most classical quasi-isometric invariants is the *number of ends*. Informally speaking, the number of ends of a simplicial graph is the number of topologically distinct ways to escape to infinity, and the number of ends of a finitely generated group is defined to be the number of ends of its Cayley graph with respect to some (any) finite generating set (Section 2.2). Freudenthal [Fre30] and Hopf [Hop43] independently proved that the number of ends of a finitely generated infinite group is always either one, two, or infinite, where the group has exactly two ends if and only if it contains an infinite cyclic subgroup of finite index. A celebrated theorem of Stallings [Sta68, Sta71] provides a complete characterization of finitely generated groups with infinitely many ends. It states that a finitely generated group has more than one end if and only if it splits over finite groups, or equivalently, it admits an action on a simplicial tree with finite edge-stabilizers and without edge-inversions or a global fixed vertex. More generally, infinite countable groups, even when not finitely generated, enjoy the same trichotomy on their number of ends [Cor19].

It is a classical fact that the mapping class groups of finite-type surfaces, except for some cases when the surface has low complexity, are all one-ended (many even have geodesic divergence bounded above by a quadratic function [DR09]). So the goal of this paper is to investigate the number of ends of *big mapping class groups*, i.e., mapping class groups of infinite-type surfaces (Section 2.3). Interest in big mapping class groups grew significantly after Bavard [Bav16] proved that the ray graph is Gromov hyperbolic, analogous to the famous theorem of Masur–Minsky [MM99] that the curve graph is Gromov hyperbolic. Since then, much effort has been made to understand big mapping class groups and to compare them with the better-studied mapping class groups of finite-type surfaces (see for example, [Lon25, PW25, MV22]).

Geometric group theory typically concerns finitely (or compactly) generated groups because it is well-known that the word metrics associated to any two finite (or compact) generating sets are quasi-isometric to each other. Big mapping class groups are not finitely or countably infinitely generated, nor are they compactly generated. So a priori, they do not have a well-defined geometry. However, Rosendal [Ros22] developed ideas which allow for the study of the large-scale geometry of topological groups which are not locally compact, let alone finitely generated. Specifically, Rosendal proved that for groups which admit a *coarsely bounded* (CB) generating set (Section 2.5), it is also the case that the word metrics associated to any two CB generating sets are quasi-isometric to each other. We remark that a subset of a locally compact space is CB if and only if it is compact, so this notion does indeed generalize older ones. Recent work of Mann–Rafi [MR23] applied this idea to classify (many of) the big mapping class groups which are CB generated, and therefore have a well-defined quasi-isometry type. Using their tools and classification, several results have shown that these groups admit a rich geometric structure ([GRV, HQR22]).

Within this framework, one can now reasonably define the number of ends of a CB generated group to be the number of ends of its Cayley graph with respect to some (any) CB generating set. Then it is

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natural to ask: *How many ends do CB generated big mapping class groups have?* A theorem in [OP22] states that the number of ends of a connected, unbounded, (coarsely) transitive graph is one, two, or infinite, generalizing the theorem of Freudenthal–Hopf. The theorem statement includes an assumption that the graph is locally finite, but in fact, this assumption is unnecessary as it is never used in the proof. The Cayley graph of a CB generated mapping class group is connected and transitive. So even though it is locally infinite (in general, the degree of each vertex is uncountable), we may conclude the following.

Observation 1.1. *The number of ends of a CB generated mapping class group is zero, one, two, or infinite. The number of ends is zero if and only if the group is CB.*

A surface Σ of infinite type falls into one of two classes depending on whether or not it contains a finite-type *non-displaceable subsurface*—a subsurface which intersects its image under every homeomorphism of Σ. According to recent work, the large-scale geometry of the big mapping class group $\mathrm{Map}(\Sigma)$ often depends on whether or not Σ contains a non-displaceable subsurface of finite type. For example, if Σ does contain a non-displaceable subsurface of finite type, then $\mathrm{Map}(\Sigma)$ is not coarsely bounded [MR23] and it admits a continuous and nonelementary action by isometries on a Gromov hyperbolic space [HQR22] (the authors also prove that under an extra condition, $\mathrm{Map}(\Sigma)$ admits such an action only if Σ contains a non-displaceable subsurface). On the other hand, if Σ does not contain a non-displaceable subsurface and $\mathrm{Map}(\Sigma)$ is not CB, then $\mathrm{Map}(\Sigma)$ has infinite asymptotic dimension [GRV].

In an effort to further classify the number of ends of big mapping class groups, we focus on those surfaces Σ which do not contain a non-displaceable subsurface. Since we are attempting to analyze the large-scale geometry of $\mathrm{Map}(\Sigma)$, we are only interested in the surfaces for which $\mathrm{Map}(\Sigma)$ is CB generated, but not CB itself. A broad class of surfaces which satisfy these conditions is the class of *avenue surfaces* (Section 2.6). Avenue surfaces were defined in [HQR22], and the only additional condition imposed on them is a weak topological condition called *tameness* on the space of ends of Σ. Tameness is rather technical and lengthy to define, so we choose to work within the slightly more restrictive, but much more easily defined, class of *stable* surfaces. Stable surfaces form a substantial subclass of tame surfaces, and it has become somewhat standard to work with them. For example, several other works in the literature [BNV23, GRV, BDR] similarly restrict to stable surfaces in order to conveniently work within the framework developed in [MR23]. That being said, we expect our results to also hold in the setting of surfaces with tame end spaces.

Every avenue surface has two distinct ends e_+ and e_- which are maximal with respect to the partial pre-order $\preceq$ on $\mathrm{End}(\Sigma)$ defined in [MR23]. It follows from [SC24] that for every end x which is an immediate predecessor of e_+ (and e_-), the equivalence class of x is either a discrete set or a totally disconnected perfect set, where the equivalence relation is defined by $\preceq$. In the first case, we say the end x is of *discrete type*. We can now state our main theorem.

Theorem. *If Σ is a stable avenue surface with at least one discrete-type end, then* $\mathrm{Map}(\Sigma)$ *is one-ended.*

The most basic example of a surface covered by this theorem is the bi-infinite flute surface. This surface indeed has exactly two maximal ends, and all other ends, represented by punctures, form a single equivalence class homeomorphic to $\mathbb{Z}$. On the other hand, one surface which is not covered by our theorem is the Jacob's ladder surface, which has exactly two ends, each accumulated by genus. It may seem at first that the Jacob's ladder surface should be no more difficult to handle than the bi-infinite flute surface. However, one significant difference is that in the bi-infinite flute surface, curves cannot meaningfully "interact" with the punctures, since the punctures are not points on the surface, but in the Jacob's ladder surface, curves may "interact" with the handles in a highly non-trivial way. We do not know whether our methods may be generalized to deal with this additional layer of complexity, so we ask the following.

Question 1.2. *Is the mapping class group of the Jacob's ladder surface one-ended? More generally, if Σ is a stable avenue surface without any discrete-type ends, is* $\mathrm{Map}(\Sigma)$ *one-ended?*

A common strategy to study mapping class groups is to construct appropriate metric spaces on which the groups act with good geometric properties. For finite-type surfaces, these spaces are often graphs whose vertices are represented by curves and arcs on the surface, and the fundamental example is the curve graph defined in [Har81]. Avenue surfaces fall in the broader class of *translatable surfaces*, for which Schaffer-Cohen [SC24] introduced the *translatable curve graph*, whose vertices correspond to simple closed curves which separate the two maximal ends, and whose edges connect pairs of vertices whenever the corresponding curves bound a subsurface homeomorphic to one of a pre-chosen finite collection of subsurfaces (Section 2.7). The translatable curve graph is not only an analogue of the curve graph in the setting of translatable surfaces; it is even quasi-isometric to the associated mapping class group [SC24]. So every property which is invariant under quasi-isometry holds for the mapping class group precisely when it holds for the translatable curve graph. Therefore, we directly analyze the translatable curve graph in order to determine the number of ends of the mapping class group. To this end, we construct an operation called *lassoing* that takes a vertex in the graph to an adjacent vertex and, more generally, takes a (well-behaved) path to an adjacent path. Furthermore, we find some lower bounds, which we call *flux* and *Hamming distance*, on the graph distance which are crucial to guaranteeing that points are sufficiently far from a given basepoint. Section 3 presents a proof of the theorem in the case of the bi-infinite flute, capturing many of the key ideas, and the theorem is proved in full in Section 4.

We finish with some more questions that arose during our investigation.

Open Questions. This paper concerns only those surfaces which do not contain a non-displaceable subsurface of finite type. This, of course, raises the following question.

Question 1.3. *Can one classify the number of ends of* $\mathrm{Map}(\Sigma)$ *when* Σ *is an infinite-type surface which contains a non-displaceable subsurface of finite type?*

In particular, this question is open for the specific surface studied in [Bav16] that initiated much of the recent interest in big mapping class groups.

Question 1.4. *Let* $\Sigma = \mathbb{R}^2 \setminus \mathcal{C}$ *be the plane minus a Cantor set. How many ends does* $\mathrm{Map}(\Sigma)$ *have?*

Schaffer-Cohen proved that $\mathrm{Map}(\Sigma)$ is quasi-isometric to the loop graph [SC24, Theorem 6.5], which, in turn, is quasi-isometric to the ray graph [Bav16, Proposition 3.11]. Bavard–Walker gave a characterization of the Gromov boundary of the ray graph in [BW18]. However, their results do not immediately determine the number of ends of the ray graph.

A question that approaches the subject from another angle is:

Question 1.5. *Does there exist a big mapping class group with more than one end?*

Recall that a finitely generated group is two-ended if and only if it contains an infinite cyclic subgroup of finite index. This motivates the next question.

Question 1.6. *Does there exist a big mapping class group with exactly two ends? If one exists, what can be said about its algebraic structure?*

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2. Preliminaries and set-up

2.1. **Quasi-isometry.** Let $f\colon (X, d_X) \to (Y, d_Y)$ be a map on metric spaces. If $L \geq 1$ and $A \geq 0$, we say that f is an (L, A)-*quasi-isometric embedding* if for every $a, b \in X$,

$$\frac{1}{L} d_X(a, b) - A \leq d_Y(f(a), f(b)) \leq L d_X(a, b) + A.$$

An (L, A)-quasi-isometric embedding f is an (L, A)-*quasi-isometry* if there is an (L', A')-quasi-isometric embedding $g : Y \to X$ such that $d_X(g \circ f, \mathrm{Id}_X) < \infty$ and $d_Y(f \circ g, \mathrm{Id}_Y) < \infty$, and we call g a *quasi-inverse* of f. Equivalently, an (L, A)-quasi-isometric embedding f is an (L, A)-quasi-isometry if it is *coarsely surjective*, that is, if there is a $C \geq 0$ such that the image of f is C-dense in Y. A map $f\colon X \to Y$ is a *quasi-isometry* between X and Y if it is an (L, A)-quasi-isometry for some $L \geq 1, A \geq 0$. Two metric spaces X and Y are *quasi-isometric* if there exists a quasi-isometry between them.

2.2. **Number of ends of a graph.** A *graph* X is a pair of sets (V, E) where E is a set of subsets of V containing exactly two elements. That is

$$E \subset \{e \mid e \subset V, |e| = 2\}.$$

We call $V = V(X)$ the set of *vertices* and $E = E(X)$ the set of *edges*. Given an edge $\{x, y\}$, we call x and y the *endpoints* of $\{x, y\}$ and we say that x and y are *adjacent* or *neighbors*. A *graph isomorphism* between graphs X and Y is a bijection $f : V(X) \to V(Y)$ such that x and y are adjacent if and only if $f(x)$ and $f(y)$ are adjacent. A *graph automorphism* of a graph X is a graph isomorphism from X to itself. A graph is *vertex-transitive* (or *transitive*) if its automorphism group acts transitively on its vertices. We emphasize that we do not assume our graphs are locally finite. In fact, every vertex in our graphs of interest will have uncountable degree.

A *path* between two vertices x and y is a sequence $(v_0, v_1, \dots, v_n)$ of vertices such that $v_0 = x$, $v_n = y$, and consecutive vertices are connected by an edge, i.e. $\{v_i, v_{i+1}\} \in E$ for $i = 0, \cdots, n-1$. A graph is *connected* if any two vertices can be connected by a path. Any connected graph X is also a metric space via the path metric which defines the distance between two vertices to be the length of a shortest path between them. If $S \subset V(X)$, then the *subgraph of X induced by S* is the graph whose vertex set is S and whose edge set is the subset of edges in $E(X)$ that have both endpoints in S. We re-use the symbol S to denote this induced subgraph, and we denote by both $X \setminus S$ and S^{c} the subgraph of X induced by $V(X) \setminus S$.

For a connected graph X and subset $S \subset V(X)$, denote by $U(X, S)$ (or $\pi_0^u(S^{\mathsf{c}})$) the set of unbounded connected components of the subgraph $X \setminus S$. Define the *number of ends* of X, denoted by $e(X)$, by

$$e(X) = \sup\{|U(X, B)| : B \text{ is a bounded subset of } X\}.$$

The number of ends turns out to be invariant under quasi-isometry. This statement is a well-known fact under conditions such as local finiteness, or local compactness for more general spaces, and proofs may be readily found in the literature (for example, [DK18, Lemma 9.5]). This is because finitely generated groups are usually the objects of interest, and their Cayley graphs are locally finite. To the authors' knowledge, analogous statements without assuming conditions such as local compactness do not exist in the literature, so we include a proof here for the sake of completeness.

Given a vertex x and radius $r \geq 0$, the ball of radius r centered at x is

$$B(x, r) = \{y \in V(X) : d(x, y) \leq r\}.$$

Note that if B_1 and B_2 are bounded subsets of $V(X)$, and $B_1 \subset B_2$, then $|U(X, B_1)| \leq |U(X, B_2)|$. Since balls are bounded, and bounded sets are contained in balls, we have the following lemma.

Lemma 2.1. *Let $x \in V(X)$ be any vertex. Then*

$$e(X) = \sup_{r>0} |U(X, B(x, r))| = \sup_{\substack{r>0 \\ y \in X}} |U(X, B(y, r))|.$$

The next lemma will be used to show that the number of ends is invariant under quasi-isometry.

Lemma 2.2. *Let X and X' be connected graphs, and let $B = B(x, r)$ be a ball in X. If $f : X \to X'$ is a quasi-isometry, then there is an $R > 0$ such that the ball $B' = B(f(x), R)$ in Y satisfies $|U(X', B')| \geq |U(X, B)|$.*

Proof. Suppose $f : X \to X'$ and $g : X' \to X$ are (L, A)-quasi-isometries that are quasi-inverses of each other. Let $x' = f(x)$, and by possibly increasing A, we may assume without loss of generality that $g(x') = x$. Put $r' = Lr + L^2 + 2LA$ and $B' = B(x', r')$. The goal is to construct a surjection $U(X', B') \to U(X, B)$. First observe that whenever $y \notin B'$,

$$d(x, g(y)) = d(gf(x), g(y)) \geq L^{-1} d(f(x), y) - A > L^{-1} r' - A = r + L + A. \tag{2.1}$$

That is, g maps the complement of B' into the complement of $B(x, r + L + A)$. Let $C \in U(X', B')$, and pick any $y \in C$. Since $y \notin B'$, we know that $g(y) \notin B$. So $g(y)$ must be contained in some connected component D of $X \setminus B$. We now show that $g(C) \subset D$. Let $z \in C$ be adjacent to y. Then $g(z) \notin B$ by the inequality 2.1, and

$$d(g(y), g(z)) \leq Ld(y, z) + A = L + A.$$

Let γ be a shortest path in X between $g(y)$ and $g(z)$. If $g(z) \notin D$, then $g(z)$ is contained in a different component of $X \setminus B$. Hence γ must intersect B at some point, call it w, and we get

$$d(x, g(y)) \leq d(x, w) + d(w, g(y)) \leq r + d(g(z), g(y)) \leq r + L + A,$$

contrary to the inequality 2.1. So $g(z) \in D$, and since C is connected, it follows that $g(C) \subset D$. Since C is unbounded and g is a quasi-isometry, D must be unbounded. Therefore $D \in U(X, B)$ and we put $\Phi(C) = D$. Thus we have a well-defined map $\Phi : U(X', B') \to U(X, B)$ which we will show is surjective.

Let $D \in U(X, B)$ be arbitrary. Like before, we may take a ball $B_1 = B(x, R)$ of sufficiently large radius R such that whenever $y \notin B_1$, we have $d(x', f(y)) > r' + L + A$. Then by similar reasoning used before, f maps each element of $U(X, B_1)$ into an element of $U(X', B')$. Since D is an unbounded component of $X \setminus B$, and B_1 is just a bounded neighborhood of B, there must be a $D' \in U(X, B_1)$ with $D' \subset D$. Then like before, $f(D') \subset C$ for some $C \in U(X', B')$. Since f and g are quasi-inverse, and D' is unbounded, there is a point in C that g maps into $D' \subset D$. Hence $g(C) \subset D$, and therefore $\Phi(C) = D$. Thus Φ is surjective, and $|U(X', B')| \geq |U(X, B)|$. $\square$

Proposition 2.3. *If X and X' are connected graphs that are quasi-isometric, then $e(X) = e(X')$.*

Proof. By Lemma 2.2, for any ball B in X there is a ball B' in X' such that $|U(X', B')| \geq |U(X, B)|$. It follows that $e(X') \geq e(X)$. Replacing the roles of X and X' yields the reverse inequality. $\square$

Lastly, we establish a practical lemma.

Lemma 2.4. *Let X be a connected, unbounded, transitive graph, and fix a vertex o. Let $f : \mathbb{N} \to \mathbb{N}$ be any function with $f(n) > n$. Then $e(X) = 1$ if the following holds for each integer $R > 0$: Any $x_1, x_2 \in X$ with $d(x_1, o) = d(x_2, o) = f(R)$ can be connected by a path which is disjoint from $B(o, R)$.*

Proof. We prove the contrapositive statement. If $e(X) > 1$, then there exists an integer $R > 0$ such that $B = B(o, R)$ satisfies $|U(X, B)| \geq 2$. Let C_1 and C_2 be different unbounded components of $X \setminus B$. Since these are unbounded, one may choose for $i = 1, 2$, a vertex $x_i \in C_i$ with $d(x_i, o) = f(R)$. Since B separates C_1 and C_2, all paths between x_1 and x_2 must intersect B. $\square$

2.3. **Surfaces and the mapping class group.** A *surface* is a connected, orientable, 2-dimensional topological manifold without boundary. A surface Σ is of *finite type* if its fundamental group is finitely generated; otherwise Σ is of *infinite type*. A *subsurface* S of a surface Σ is a closed, connected subset of Σ which is itself a surface with boundary. Assume the boundary of a subsurface consists of a finite number of pairwise disjoint simple closed curves, and that none of these boundary curves bounds a disk or a punctured disk in Σ.

Denote by $\mathrm{Homeo}^+(\Sigma)$ the group of orientation-preserving homeomorphisms of Σ with the compact-open topology, and denote by $\mathrm{Homeo}_0(\Sigma)$ the connected component of the identity in $\mathrm{Homeo}^+(\Sigma)$. The *mapping class group* of Σ is defined as the group $\mathrm{Map}(\Sigma) = \mathrm{Homeo}^+(\Sigma)/\mathrm{Homeo}_0(\Sigma)$ of all isotopy classes of orientation-preserving homeomorphisms of Σ, and $\mathrm{Map}(\Sigma)$ is endowed with the quotient topology. When Σ is of infinite type, $\mathrm{Map}(\Sigma)$ is a Polish group which is not locally compact.

2.4. **The space of ends of a surface.** The *space of ends* $\mathrm{End}(\Sigma)$ of a surface Σ is the inverse limit of the system of components of complements of compact subsets of Σ. Intuitively, each end corresponds to a way of leaving every compact subset of Σ (see [Ric63] for details). Denote by $\mathrm{End}^g(\Sigma) \subseteq \mathrm{End}(\Sigma)$ the subspace of ends which are accumulated by genus, and denote by $\mathrm{genus}(\Sigma)$ the (possibly infinite) genus of Σ. By a theorem of Richards [Ric63], connected, orientable surfaces Σ are classified up to homeomorphism by the triple $(\mathrm{genus}(\Sigma), \mathrm{End}(\Sigma), \mathrm{End}^g(\Sigma))$. For a subsurface $S \subseteq \Sigma$, the space of ends of S is defined similarly and is denoted by $\mathrm{End}(S)$. The embedding of S in Σ gives a natural embedding of $\mathrm{End}(S)$ into $\mathrm{End}(\Sigma)$.

Assume that Σ is a surface of infinite type. One of the key tools used in [MR23] was a partial pre-order $\preceq$ on $\mathrm{End}(\Sigma)$. By definition, $y \preceq x$ if, for every clopen neighborhood U of x, there exists a clopen neighborhood V of y which is homeomorphic to a subset of U. This induces an equivalence relation on $\mathrm{End}(\Sigma)$, where x and y are *equivalent* ends if $x \preceq y$ and $y \preceq x$, and we denote by $E(x)$ the equivalence class of x. Denote by $\mathcal{M}(\mathrm{End}(\Sigma))$ the subset of $\mathrm{End}(\Sigma)$ of ends which are maximal with respect to $\preceq$, that is, the set of ends $x \in \mathrm{End}(\Sigma)$ such that $x \preceq y$ implies $y \in E(x)$. The elements of $\mathcal{M}(\mathrm{End}(\Sigma))$ are called *maximal ends*.

For an end $x \in \mathrm{End}(\Sigma)$, a clopen neighborhood U of x is *stable* if for any smaller neighborhood $U' \subset U$ of x, there is a homeomorphic copy of U contained in U'. (see [BDR, Proposition 3.2] for equivalent definitions of a stable neighborhood). The surface Σ is *stable* if every end $x \in \mathrm{End}(\Sigma)$ has a stable neighborhood.

2.5. **Geometry of big mapping class groups.** Let G be a Polish group. A subset $A \subset G$ is *coarsely bounded*, abbreviated CB, if it has finite diameter with respect to every compatible left-invariant metric on G. A group is *locally CB* if it has a CB neighborhood of the identity, and it is *CB generated* if it has a CB generating set. It is a fact that a CB generated group is necessarily locally CB [Ros22, Theorem 1.2]. From the point of view of large-scale geometry, CB sets in Polish groups are analogous to finite sets in discrete groups. In particular, a CB generated group admits a metric that is well-defined up to quasi-isometry.

Theorem 2.5. [Ros22]. *Let G be a CB generated Polish group. Then the identity map on G is a quasi-isometry between the word metrics associated to any two symmetric, CB generating sets.*

Thus, we can reasonably define the number of ends of CB generated Polish groups, and hence, CB generated big mapping class groups.

Definition 2.6. Let G be a CB generated Polish group, and fix a symmetric CB generating set S. The *number of ends* of G is the number of ends of the Cayley graph of G with respect to S.

By Theorem 2.5, any other choice of S yields a quasi-isometric Cayley graph which has the same number of ends by Proposition 2.3.

2.6. **Avenue surfaces and translatable surfaces.** Next we recall the definition of an avenue surface and a translatable surface. These notions were introduced in [HQR22] and [SC24], respectively, and we explain their precise relationship.

Definition 2.7 (Avenue surface)**.** An avenue surface is a connected orientable surface Σ which does not contain any non-displaceable subsurface of finite type, whose end space is tame, and whose mapping class group $\mathrm{Map}(\Sigma)$ is CB-generated but not CB.

The condition of tameness only requires that certain ends of Σ admit a stable neighborhood. Since we will eventually only consider stable surfaces, where every end admits a stable neighborhood, we refer the interested reader to [MR23] for a precise definition of tame and omit writing it here.

Definition 2.8 (Translatable surface)**.** Given a surface Σ, an end e of Σ, and a sequence α_n of simple closed curves on Σ, we write $\lim_{n\to\infty} \alpha_n = e$ if for each neighborhood V of e, all but finitely many α_n are contained in V (after isotopy). A homeomorphism $h \in \mathrm{Homeo}(\Sigma)$ is a *translation* if there are two distinct ends e_+ and e_- of Σ such that for any simple closed curve α on Σ, $\lim_{n\to\infty} h^n(\alpha) = e_+$ and $\lim_{n\to\infty} h^{-n}(\alpha) = e_-$. A surface Σ is *translatable* if it admits a translation.

By [SC24, Theorem 5.3], if a surface Σ has tame end space and $\mathrm{Map}(\Sigma)$ is CB generated and not CB, then Σ is an avenue surface if and only if it is translatable. In particular, every stable avenue surface is translatable.

In a translatable surface Σ, a simple closed curve α is *separating* if $\Sigma \setminus \alpha$ is disconnected and e_+ and e_- are ends of different components. (Note that this condition is stronger than the usual notion of separating which only requires $\Sigma \setminus \alpha$ to be disconnected.) If α is a separating curve, then the component of $\Sigma \setminus \alpha$ with e_+ (resp. e_-) as one of its ends is called the *right* (resp. *left*) *side* of α, and it is denoted by α_+ (resp. α_-). If β is a separating curve that lies on the right side of α, denote by $[\alpha, \beta]$ the subsurface $(\alpha_+ \cap \beta_-) \cup \alpha \cup \beta$ of Σ with boundary components α and β. If β lies on the left side of α instead, then by an abuse of notation, still use $[\alpha, \beta]$ to denote $[\beta, \alpha]$.

Next we recall a characterization of translatable surfaces from [SC24]. Let S be any surface with exactly two boundary components, and denote by $S^{\natural\mathbb{Z}}$ the surface obtained by arranging countably many copies of S like $\mathbb{Z}$ and then gluing consecutive boundary components together in the natural way. By construction, this surface is translatable. Conversely, every translatable surface admits such a decomposition.

Proposition 2.9. [SC24, Proposition 3.5] *If Σ is a translatable surface with translation h, and α is any separating curve, then there is a subsurface $S = [\alpha, h^n(\alpha)]$ for some n such that Σ is homeomorphic to $S^{\natural\mathbb{Z}}$.*

2.7. **The translatable curve graph.** We now recall the definition of the translatable curve graph from [SC24]. Let $\Sigma = S^{\natural\mathbb{Z}}$ be a translatable surface and assume that Σ is tame. By [SC24, Lemma 4.4] and [MR23, Proposition 4.7], the end space $\mathrm{End}(S)$ has a finite, positive number of equivalence classes of maximal ends, and each such equivalence class intersects $\mathrm{End}(S)$ at either a finite set or a Cantor set. So we may fix, once and for all, a set $\mathcal{R} = \{f_1, \ldots, f_N, c_1, \ldots, c_M\}$ of representatives of the equivalence classes of maximal ends of S such that each $E(f_i) \cap \mathrm{End}(S)$ is finite and each $E(c_i) \cap \mathrm{End}(S)$ is a Cantor set. Note that in the whole end space $\mathrm{End}(\Sigma)$, since $\Sigma = S^{\natural\mathbb{Z}}$, each $E(f_i)$ is homeomorphic to $\mathbb{Z}$ and each $E(c_i)$ is homeomorphic to the Cantor set with two points removed. We say an end of Σ is of *discrete type* if it is equivalent to some f_i. Observe that Σ has at least one end of discrete type if and only if $N > 0$.

Pick mutually disjoint stable neighborhoods V_{f_i} and V_{c_i} of the elements of $\{f_1, \ldots, f_N, c_1, \ldots, c_M\}$. By the definition of stable, each V_{f_i} contains exactly one discrete-type end. For each $i = 1, \ldots, N$, let T_i be a subsurface of S with two boundary circles and end space homeomorphic to $V_{f_i} \sqcup \bigsqcup_{j=1}^{M} V_{c_j}$. If f_i or any of the c_j is accumulated by genus, then T_i necessarily has infinite genus; otherwise, we require T_i to have genus 0. If S has finite positive genus, then let T_{N+1} be a subsurface with two boundary circles, genus 1, and end space homeomorphic to $\bigsqcup_{j=1}^{M} V_{c_j}$. Put $\mathcal{S} = \{T_1, \ldots, T_N, T_{N+1}\}$ (including T_{N+1} only if it is defined, of course). The *translatable curve graph* $\mathcal{TC}(\Sigma)$ of Σ is the graph whose vertices are the isotopy classes of separating curves on Σ, with an edge between two curves α and β if they have disjoint representatives and $[\alpha, \beta]$ is homeomorphic to a subsurface in $\mathcal{S}$. By [SC24, Lemma 4.7], $\mathcal{TC}(\Sigma)$ is connected, and we denote its path metric by d. Observe that $\mathrm{Map}(\Sigma)$ naturally acts by isometries on $\mathcal{TC}(\Sigma)$.

There is an edge case: if $N = 0$ and the genus of S is either 0 or infinite, then the above construction gives $\mathcal{S} = \emptyset$. This is not desirable, so in this case we put $\mathcal{S} = \{S\}$. In some sense this choice is consistent with the construction above because S is indeed a subsurface with two boundary circles and end space homeomorphic to $\bigsqcup V_{c_i}$. More importantly, in this case, $\mathcal{TC}(\Sigma)$ has diameter 2 and $\mathrm{Map}(\Sigma)$ is CB, which means that both are quasi-isometric to a point (see [SC24] for details). If Σ is a stable avenue surface, the $\mathrm{Map}(\Sigma)$ is not CB, and so $N = 0$ implies that S has finite positive genus.

Since every stable avenue surface is a translatable surface with tame end space, the following result allows us to use the translatable curve graph to understand the large-scale geometry of their mapping class groups.

Theorem 2.10. [SC24, Theorem 4.9] *If Σ is a stable avenue surface, then* $\mathrm{Map}(\Sigma)$ *is quasi-isometric to $\mathcal{TC}(\Sigma)$.*

3. Example: bi-infinite flute

In this section we illustrate the strategy of the proof of the main theorem with an example that captures many of the key ideas. Let Σ be the *bi-infinite flute*, i.e. the genus 0 surface with two maximal ends, e_+ and e_-, such that the other ends, represented as punctures, form a single equivalence class which accumulates to both e_+ and e_-. Equivalently, $\Sigma = S^{\natural \mathbb{Z}}$, where S is the annulus with one puncture. In this case, the translatable curve graph $\mathcal{TC}(\Sigma)$ is the graph whose vertices are (isotopy classes of) separating curves, with an edge between two curves α and β if they have disjoint representatives and $[\alpha, \beta]$ is homeomorphic to an annulus with one puncture. From now on, denote the translatable curve graph simply by Γ. We first introduce two operations that induce maps on separating curves in Σ, or equivalently, vertices in Γ. These operations will be referred to as "forgetting a puncture" and "lassoing a puncture".

3.1. **Forgetting a puncture.** Identify Σ with the space X which is defined to be the strip $\mathbb{R} \times [0, 1]$ modulo the equivalence relation $(x, 0) \sim (x, 1)$, with marked points $(n, 0)$ for each $n \in \mathbb{Z}$. Suppose we erase the marking at a point $(n, 0)$ and call the new space X'. As marked spaces, X' is isomorphic to X, and one isomorphism is given by the homeomorphism $X' \to X$ which is the identity map on $(-\infty, n-1] \times [0, 1]$, a horizontal contraction on $[n-1, n+1] \times [0, 1]$, and the unit left shift on $[n+1, \infty) \times [0, 1]$. Call this homeomorphism ϕ_n and denote by $\iota : X \to X'$ the map which forgets the marking at $(n, 0)$. Then $\Phi_n = \phi_n \circ \iota : X \to X$ is a homeomorphism and it maps separating curves to separating curves. One can similarly define a homeomorphism $\psi_n : X' \to X$ using the identity map on $[n+1, \infty) \times [0, 1]$ and the unit right shift on $(-\infty, n-1] \times [0, 1]$. We freely allow Φ_n to mean either $\phi_n \circ \iota$ or $\psi_n \circ \iota$ depending on which is more appropriate for a given situation.

Definition 3.1. The operation of *forgetting the puncture* at $(n, 0)$ is the map $V(\Gamma) \to V(\Gamma)$ defined by sending each separating curve in $\Sigma \cong X$ to its image under Φ_n.

In the next lemma, we observe that forgetting a puncture maps each pair of adjacent vertices in Γ either to a pair of adjacent vertices or to the same vertex.

Lemma 3.2. *If α and β are neighbors in Γ and α' and β' are their images after forgetting a puncture, then either $\alpha' = \beta'$ or α' and β' are neighbors.*

Proof. Assume that $[\alpha, \beta]$ is an annulus with puncture p. If p is the forgotten puncture, then α' and β' are isotopic and therefore equal as vertices in Γ. Otherwise, $[\alpha', \beta']$ is still an annulus with puncture p, and therefore α' and β' are neighbors in Γ. ☐

This implies that forgetting a puncture maps paths to (possibly shorter) paths.

Lemma 3.3. *Forgetting a puncture sends the vertices on a given path to the vertices on a path which is not longer than the original path.*

Proof. Proceed by induction on the length n of the given path. The case $n = 1$ follows from Lemma 3.2. So assume the statement holds when $n = k - 1$, and consider a path $(\alpha_0, \alpha_1, \ldots, \alpha_k)$ of length k. For each $i = 0, \ldots, k$, denote by α_i' the image of α_i after forgetting a puncture. Since $(\alpha_1, \ldots, \alpha_k)$ is a path of length $k - 1$, the induction hypothesis implies that $(\alpha_1', \ldots, \alpha_k')$ (after removing consecutive repeated vertices) is a path of length at most $k - 1$. Moreover, Lemma 3.2 implies that either $\alpha_0' = \alpha_1'$ or α_0' and α_1' are neighbors. Hence, $(\alpha_0', \alpha_1', \ldots, \alpha_k')$ (after removing consecutive repeated vertices) is a path of length at most k. □

3.2. **Lassoing a puncture.** Let α be a separating curve, and let ℓ be an arc in Σ which intersects α and has only one endpoint equal to a puncture, where the puncture is viewed as a marked point. Denote by ℓ' the subarc of ℓ obtained by traveling along ℓ from the marked endpoint until the first intersection point with α. In other words, ℓ' is the component of $\ell \setminus \alpha$ containing the marked endpoint.

Definition 3.4. Given α and ℓ as above, the *lasso curve* defined by α and ℓ, denoted by $\alpha(\ell)$, is the simple closed curve obtained by tracing along α and ℓ'. In this case, the arc ℓ is called a *lasso*.

To give a more precise definition, consider a small tubular neighborhood of $\alpha \cup \ell'$. This subsurface has two boundary components, one of which is isotopic to α. Then $\alpha(\ell)$ is equal to the other boundary component.

By construction, $\alpha(\ell)$ is a separating curve, which is adjacent to α in Γ. In fact, every neighbor of α can be realized as a lasso curve. Indeed, if α and β are neighbors, then $[\alpha, \beta]$ is an annulus with one puncture p. So let ℓ be an arc in $[\alpha, \beta]$ from α to p, and note that $\alpha(\ell)$ is isotopic to β because $[\alpha(\ell), \beta]$ is an annulus.

Definition 3.5. Given a set of curves α_i in Σ, a connected subsurface K of Σ is a *carrier* of the α_i if K contains each α_i and has exactly two boundary components, each of which is a separating curve in Σ.

For example, every finite set of closed curves admits a carrier which is a "finite flute", that is, a union of finitely many consecutive copies of S in $\Sigma = S^{\natural\mathbb{Z}}$. The next lemma says that given two curves α and β, taking a step in Γ from α by lassoing a distant puncture is not a step towards β.

Lemma 3.6. *Let α and β be distinct curves in a carrier K. Let ℓ be a lasso from α to a puncture $p \notin K$. Then in Γ, the vertex $\alpha(\ell)$ does not lie on any geodesic path between α and β. In particular, $d(\alpha(\ell), \beta) \geq d(\alpha, \beta)$.*

Proof. Any path from α to β which contains $\alpha(\ell)$ can be made shorter by replacing the initial subpath from α to $\alpha(\ell)$ with just the edge between α and $\alpha(\ell)$. So let P be any path in Γ from α to β which takes its first step at $\alpha(\ell)$. We claim that P is not a geodesic between α and β. By Lemma 3.3, forgetting the puncture p transforms P into another path P' which is at most as long as P. Since $\alpha, \beta \subset K$ and $p \notin K$, forgetting p maps α to α and β to β. So P is still a path from α to β. On the other hand, forgetting p maps $\alpha(\ell)$ to α. So P' is strictly shorter than P.

If $d(\alpha(\ell), \beta) < d(\alpha, \beta)$, then $d(\alpha(\ell), \beta) = d(\alpha, \beta) - 1$ because α and $\alpha(\ell)$ are neighbors. In this case, a geodesic between $\alpha(\ell)$ and β together with the edge between α and $\alpha(\ell)$ would give a geodesic path between α and β containing $\alpha(\ell)$. It was shown that no such geodesic exists; thus $d(\alpha(\ell), \beta) \geq d(\alpha, \beta)$. □

3.3. **Lasso paths.** Given a single curve α and a lasso ℓ, we defined the lasso curve $\alpha(\ell)$. We now perform a similar construction given a set of mutually disjoint curves.

Definition 3.7. A sequence of separating curves $\alpha_0, \ldots, \alpha_n$ is *straight* if α_i lies on the right side of α_{i-1} for each $i = 1, \ldots, n$. It is a *straight path* if $(\alpha_0, \ldots, \alpha_n)$ is a path in Γ.

Let $\alpha_0, \ldots, \alpha_n$ be a straight sequence of separating curves, and fix a carrier K. Let ℓ be an arc from K_- to a puncture $p \in K_+$ such that ℓ intersects each α_i exactly once. Such an arc may be inductively constructed since $[\alpha_{i-1}, \alpha_i]$ is connected and its boundary components are α_{i-1} and α_i. Construct the lasso curves $\alpha_i(\ell)$ one by one, in order of decreasing index starting from $i = n$, so that $\alpha_{i-1}(\ell)$ is disjoint from $\alpha_i(\ell)$. This can always be done by drawing the "lasso" portion of $\alpha_{i-1}(\ell)$ in a small enough neighborhood of ℓ. If instead, ℓ is an arc from K_+ to a puncture $p \in K_-$ which intersects each α_i exactly once, then disjoint lasso curves $\alpha_i(\ell)$ may similarly be drawn. In both constructions, $\alpha_i(\ell)$ lies on the right side of $\alpha_{i-1}(\ell)$ for each $i = 1, \ldots, n$. So the following lemma is true.

Lemma 3.8. *A straight sequence of separating curves remains straight after lassoing as above.*

Moreover, lassoing a straight path again yields a straight path.

Lemma 3.9. *If $(\alpha_0, \dots, \alpha_n)$ is a straight path in Γ and ℓ is a lasso as above, then $(\alpha_0(\ell), \dots, \alpha_n(\ell))$ is also a straight path in Γ.*

Proof. Since $(\alpha_0, \dots, \alpha_n)$ is a path, each $[\alpha_{i-1}, \alpha_i]$ is an annulus with one puncture. By the construction of the lasso curves $\alpha_i(\ell)$ and $\alpha_{i-1}(\ell)$ above, $[\alpha_{i-1}(\ell), \alpha_i(\ell)]$ is also an annulus with the same puncture as $[\alpha_{i-1}, \alpha_i]$. So $\alpha_{i-1}(\ell)$ and $\alpha_i(\ell)$ are neighbors in Γ. □

Definition 3.10. Given a straight path $(\alpha_0, \dots, \alpha_n)$ and a lasso ℓ as above, the *lasso path* defined by the α_i and ℓ is the straight path $(\alpha_0(\ell), \dots, \alpha_n(\ell))$.

3.4. **Flux.** The last ingredient needed for this section is the notion of flux. This will provide a lower bound for the graph distance d in Γ. For disjoint separating curves α and β, define $F_0(\alpha, \beta)$ to be the number of punctures in $[\alpha, \beta]$. Note that $F_0(\alpha, \beta) = 0$ if and only if $[\alpha, \beta]$ is an annulus if and only if α and β are isotopic. To define the flux between pairs of intersecting curves, the next two lemmas are needed.

Lemma 3.11. *If α, β, γ is a straight sequence of separating curves, then $F_0(\alpha, \gamma) = F_0(\alpha, \beta) + F_0(\beta, \gamma)$.*

Proof. By definition, β lies on the right side of α and γ lies on the right side of β. Then $[\alpha, \gamma]$ is equal to the gluing of $[\alpha, \beta]$ and $[\beta, \gamma]$ along γ, and the equation follows. □

For separating curves α and β, if γ is a separating curve in either $\alpha_+ \cap \beta_+$ or $\alpha_- \cap \beta_-$, define

$$F(\alpha, \beta; \gamma) = |F_0(\alpha, \gamma) - F_0(\beta, \gamma)|$$

The next lemma shows that the value of $F(\alpha, \beta; \gamma)$ does not depend on the choice of γ.

Lemma 3.12. *Let α and β be separating curves. If γ and γ' are any two separating curves, each of which lies in either $\alpha_+ \cap \beta_+$ or $\alpha_- \cap \beta_-$, then $F(\alpha, \beta; \gamma) = F(\alpha, \beta; \gamma')$.*

Proof. Assume without loss of generality that $\gamma \subset \alpha_+ \cap \beta_+$; the case $\gamma \subset \alpha_- \cap \beta_-$ is handled similarly. Then there are two cases corresponding to whether γ' lies in $\alpha_+ \cap \beta_+$ or $\alpha_- \cap \beta_-$. First suppose that γ' also lies in $\alpha_+ \cap \beta_+$. Let γ'' be a separating curve in $\gamma_+ \cap \gamma'_+$. Then by Lemma 3.11,

$$\begin{aligned} F(\alpha, \beta; \gamma) = |F_0(\alpha, \gamma) - F_0(\beta, \gamma)| &= |F_0(\alpha, \gamma) + F_0(\gamma, \gamma'') - F_0(\beta, \gamma) - F_0(\gamma, \gamma'')| \\ &= |F_0(\alpha, \gamma'') - F_0(\beta, \gamma'')| \\ &= F(\alpha, \beta; \gamma''). \end{aligned}$$

Replacing γ with γ' in the above calculation also yields $F(\alpha, \beta; \gamma') = F(\alpha, \beta; \gamma'')$. Hence, $F(\alpha, \beta; \gamma) = F(\alpha, \beta; \gamma')$. Now suppose that γ' lies in $\alpha_- \cap \beta_-$. By Lemma 3.11,

$$F_0(\gamma', \gamma) = F_0(\gamma', \alpha) + F_0(\alpha, \gamma) \qquad \text{and} \qquad F_0(\gamma', \gamma) = F_0(\gamma', \beta) + F_0(\beta, \gamma)$$

Hence,

$$\begin{aligned} F(\alpha, \beta; \gamma) = |F_0(\alpha, \gamma) - F_0(\beta, \gamma)| &= |F_0(\gamma', \gamma) - F_0(\gamma', \alpha) - F_0(\gamma', \gamma) + F_0(\gamma', \beta)| \\ &= |F_0(\gamma', \alpha) - F_0(\gamma', \beta)| \\ &= F(\alpha, \beta; \gamma'). \end{aligned}$$

□

This allows for the following definition.

Definition 3.13. Let α and β be separating curves. The *flux* of α and β, denoted by $F(\alpha, \beta)$, is defined to be $F(\alpha, \beta; \gamma)$ for some (any) choice of separating curve γ in $\alpha_+ \cap \beta_+$ or $\alpha_- \cap \beta_-$.

Indeed, this definition agrees with F_0 when α and β are disjoint.

Lemma 3.14. *If α and β are disjoint separating curves, then $F(\alpha, \beta) = F_0(\alpha, \beta)$.*

Proof. Assume without loss of generality that β lies on the right side of α. Let γ be a separating curve in $\alpha_+ \cap \beta_+ = \beta_+$. Then by Lemma 3.11,

$$F(\alpha, \beta) = |F_0(\alpha, \gamma) - F_0(\beta, \gamma)| = |F_0(\alpha, \beta) + F_0(\beta, \gamma) - F_0(\beta, \gamma)| = F_0(\alpha, \beta).$$

□

Observe that if α and β are neighbors in Γ, then $[\alpha, \beta]$ has one puncture and so $F(\alpha, \beta) = F_0(\alpha, \beta) = 1$. The next lemma implies that F is a pseudometric on Γ.

Lemma 3.15. *All separating curves α, β, γ satisfy $F(\alpha, \beta) \le F(\alpha, \gamma) + F(\gamma, \beta)$.*

Proof. Let η be a separating curve in $\alpha_+ \cap \beta_+ \cap \gamma_+$. Then

$$\begin{aligned} F(\alpha, \beta) = |F_0(\alpha, \eta) - F_0(\beta, \eta)| &= |F_0(\alpha, \eta) - F_0(\gamma, \eta) + F_0(\gamma, \eta) - F_0(\beta, \eta)| \\ &\le |F_0(\alpha, \eta) - F_0(\gamma, \eta)| + |F_0(\gamma, \eta) - F_0(\beta, \eta)| \\ &= F(\alpha, \gamma) + F(\gamma, \beta). \end{aligned}$$

□

However, it can happen that $F(\alpha, \beta) = 0$ for distinct α, β. In fact, the Hamming distance (Definition 4.8) can be used to find α and β with $F(\alpha, \beta) = 0$ such that $d(\alpha, \beta)$ is arbitrarily large.

The notion of flux is useful yet because it provides a lower bound on the graph distance d in Γ.

Lemma 3.16. *All separating curves α and β satisfy $d(\alpha, \beta) \ge F(\alpha, \beta)$.*

Proof. Suppose $(\gamma_0 = \alpha, \gamma_1, \ldots, \gamma_n = \beta)$ is a geodesic path in Γ between α and β. Since F is a pseudometric by Lemma 3.15,

$$F(\alpha, \beta) \le \sum_{i=0}^{n-1} F(\gamma_i, \gamma_{i+1}) = n = d(\alpha, \beta).$$

□

It can now be shown that straight paths are geodesic.

Lemma 3.17. *If $\alpha_0, \ldots, \alpha_n$ is a straight path, then $d(\alpha_0, \alpha_n) = n$.*

Proof. Clearly, $d(\alpha_0, \alpha_n) \le n$. On the other hand, it follows from an inductive argument using Lemma 3.11 that $F(\alpha_0, \alpha_n) = n$. Then Lemma 3.16 implies $d(\alpha_0, \alpha_n) \ge n$. □

In general, flux only gives a lower bound on the graph distance. For disjoint separating curves, however, the flux is equal to the graph distance.

Lemma 3.18. *If α and β are disjoint separating curves, then there exists a straight path from α to β of length $F(\alpha, \beta)$. In particular, $d(\alpha, \beta) = F(\alpha, \beta)$.*

Proof. Since α and β are disjoint, $F(\alpha, \beta) = F_0(\alpha, \beta)$ is equal to the number of punctures in $[\alpha, \beta]$. Denote these punctures by $p_1, \ldots, p_n$. Put $\alpha_0 = \alpha$. For $i = 1, \ldots, n$, inductively define α_i to be the lasso curve $\alpha_{i-1}(\ell_i)$, where ℓ_i is an arc from α_{i-1} to p_i contained in $[\alpha_{i-1}, \beta]$. By construction, $[\alpha_{i-1}, \alpha_i]$ is an annulus with one puncture, and so $d(\alpha_{i-1}, \alpha_i) = 1$. Moreover, α_n is isotopic to β because $[\alpha_n, \beta]$ is just an annulus. Hence, $\alpha_0, \ldots, \alpha_n$ is a straight path from α to β of length $F(\alpha, \beta)$. By Lemma 3.17, $d(\alpha, \beta) = F(\alpha, \beta)$. □

Lastly, flux gives a notion of left and right for pairs of separating curves which are not disjoint.

Definition 3.19. For separating curves α and β, we say that β is *flux-right of* α if $F_0(\alpha, \eta) \ge F_0(\beta, \eta)$ holds for every separating curve η in $\alpha_+ \cap \beta_+$. We say that β is *flux-left of* α if the inequality holds for every separating curve η in $\alpha_- \cap \beta_-$.

We collect some basic facts about this notion. Lemma 3.11 is used throughout the proof without explicit mention.

Lemma 3.20. *Let α and β be separating curves. The following statements are true.*

(1) If β lies on the right side of α, then β is flux-right of α and α is flux-left of β.
(2) β must be either flux-right or flux-left of α.
(3) β is flux-right of α if and only if α is flux-left of β.
(4) If γ is a separating curve such that β is flux-right of α and γ is flux-right of β, then γ is flux-right of α.
(5) β is both flux-right and flux-left of α if and only if $F(\alpha, \beta) = 0$.
(6) β is flux-right of α if and only if $F(\alpha, \gamma) > F(\alpha, \beta)$ holds for every separating curve $\gamma \subset \beta_+$ not isotopic to β, and β is flux-left of α if and only if the inequality holds for every separating curve $\gamma \subset \beta_-$ not isotopic to β.

Proof. (1) For any separating curve η in $\alpha_+ \cap \beta_+ = \beta_+$,

$$F_0(\alpha, \eta) = F_0(\alpha, \beta) + F_0(\beta, \eta) \geq F_0(\beta, \eta).$$

Hence, β is flux-right of α, and similarly, α is flux-left of β.

(2) Suppose β is not flux-left of α, and let γ be a separating curve in $\alpha_- \cap \beta_-$ such that $F_0(\alpha, \gamma) < F_0(\beta, \gamma)$. Let η be any separating curve in $\alpha_+ \cap \beta_+$. Then

$$F_0(\gamma, \alpha) + F_0(\alpha, \eta) = F_0(\gamma, \eta) = F_0(\gamma, \beta) + F_0(\beta, \eta).$$

Since $F_0(\alpha, \gamma) < F_0(\beta, \gamma)$, it follows that $F_0(\alpha, \eta) > F_0(\beta, \eta)$. Hence, β is flux-right of α.

(3) Suppose β is flux-right of α, and let η be a separating curve in $\alpha_+ \cap \beta_+$. Then $F_0(\alpha, \eta) \geq F_0(\beta, \eta)$. Let γ be any separating curve in $\alpha_- \cap \beta_-$. Then

$$F_0(\gamma, \alpha) + F_0(\alpha, \eta) = F_0(\gamma, \eta) = F_0(\gamma, \beta) + F_0(\beta, \eta)$$

Since $F_0(\alpha, \eta) \geq F_0(\beta, \eta)$, it follows that $F_0(\gamma, \alpha) \leq F_0(\gamma, \beta)$. Hence, α is flux-left of β. The reverse implication is proved similarly.

(4) Let η be any separating curve in $\alpha_+ \cap \gamma_+$, and let η' be a separating curve in $\beta_+ \cap \eta_+$. By the definition of flux-right, $F_0(\alpha, \eta') \geq F_0(\beta, \eta') \geq F_0(\gamma, \eta')$. Then

$$F_0(\alpha, \eta) = F_0(\alpha, \eta') - F_0(\eta, \eta') \geq F_0(\gamma, \eta') - F_0(\eta, \eta') = F_0(\gamma, \eta).$$

Hence, γ is flux-right of α.

(5) Suppose $F(\alpha, \beta) = 0$, and let η be any separating curve in $\alpha_+ \cap \beta_+$. Then

$$0 = F(\alpha, \beta) = \left| F_0(\alpha, \eta) - F_0(\beta, \eta) \right|.$$

Hence, $F_0(\alpha, \eta) = F_0(\beta, \eta)$. So β is flux-right of α, and similarly, β is shown to be flux-left of α.

Conversely, suppose that β is both flux-right and flux-left of α. By statement (3), α is flux-right of β. Let η be a separating curve in $\alpha_+ \cap \beta_+$. Then $F_0(\alpha, \eta) \geq F_0(\beta, \eta)$ and $F_0(\beta, \eta) \geq F_0(\alpha, \eta)$. Hence,

$$F(\alpha, \beta) = |F_0(\alpha, \eta) - F_0(\beta, \eta)| = 0.$$

(6) Suppose β is flux-right of α. Let γ be any separating curve in β_+ not isotopic to β, and let η be a separating curve in $\alpha_+ \cap \gamma_+$. Then $F_0(\alpha, \eta) \geq F_0(\beta, \eta)$. Since γ is not isotopic to β, $F_0(\beta, \gamma) > 0$. So,

$$F_0(\beta, \eta) = F_0(\beta, \gamma) + F_0(\gamma, \eta) > F_0(\gamma, \eta).$$

Hence,

$$F(\alpha, \gamma) = F_0(\alpha, \eta) - F_0(\gamma, \eta) > F_0(\alpha, \eta) - F_0(\beta, \eta) = F(\alpha, \beta).$$

Conversely, suppose there exists a separating curve η in $\alpha_+ \cap \beta_+$ such that $F_0(\alpha, \eta) < F_0(\beta, \eta)$. Then $[\beta, \eta]$ contains at least one puncture p. Let ℓ be an arc in $[\beta, \eta]$ from β to p, and put $\gamma = \beta(\ell)$. Then γ is a separating curve in β_+ not isotopic to β, and $F_0(\beta, \eta) = F_0(\gamma, \eta) + 1$. Since $F_0(\alpha, \eta) < F_0(\beta, \eta)$,

$$F(\alpha, \gamma) = |F_0(\alpha, \eta) - F_0(\gamma, \eta)| = |F_0(\alpha, \eta) - F_0(\beta, \eta) + 1| = F(\alpha, \beta) - 1.$$

The second statement is proved similarly. □

3.5. **Main theorem (special case).** Now we are ready to prove the main theorem in the case of the bi-infinite flute.

Theorem 3.21. *The mapping class group of the bi-infinite flute is one-ended.*

Proof. By Theorem 2.10, it suffices to show that the translatable curve graph Γ is one-ended. Fix a vertex o of Γ and let $R > 0$ be any integer. To prove that Γ is one-ended, it suffices by Lemma 2.4 to show that for any $\alpha, \beta \in \Gamma$ with $d(\alpha, o) = d(\beta, o) = 3R$, there exists a path from α to β which is disjoint from $B = B(o, R)$.

For any separating curve η, there is a mapping class in $\mathrm{Map}(\Sigma)$ which sends β to η [SC24, Lemma 3.6]. Therefore, since $\mathrm{Map}(\Sigma)$ acts isometrically on Γ, we may assume without loss of generality that β is a boundary curve of some copy of S in $\Sigma = S^{\natural\mathbb{Z}}$. By Lemma 3.20 (2), β is either flux-right or flux-left of o. Assume without loss of generality that β is flux-right of o. Recall that Σ admits a translation h such that $h(\beta)$ lies on the right side of β. Then Lemma 3.20 (6) implies $F(o, h(\beta)) > F(o, \beta)$. Moreover, $h(\beta)$ is flux-right of β by Lemma 3.20 (1), and so Lemma 3.20 (4) implies that $h(\beta)$ is flux-right of o. It follows from induction that $F(o, h^k(\beta)) \geq F(o, \beta) + k$ for all $k \in \mathbb{N}$. Since h is a translation, we may fix a large enough k so that $h^k(\beta)$ lies on the right side of α. Put $\beta' = h^k(\beta)$. Then $(\beta, h(\beta), \ldots, h^k(\beta))$ is a path from β to β'. If this path has length at most R, then it must be disjoint from B because $d(\beta, o) = 3R$.

Otherwise if the length is greater than R, then the first $R+1$ vertices on the path lie outside of B for the same reason as before, and the remaining vertices also lie outside of B because for all $R < i \leq k$,

$$d(o, h^i(\beta)) \geq F(o, h^i(\beta)) \geq F(o, \beta) + i > R.$$

So β and β' are connected by a path which is disjoint from B. If instead β is flux-left of o, then a similar argument gives a path $(\beta, h^{-1}(\beta), \ldots, h^{-k}(\beta))$ from β to some $h^{-k}(\beta)$ which is disjoint from α. It now remains to construct a path from α to β' which disjoint from B.

Since β' lies on the right side of α, Lemma 3.18 gives a straight path $(\gamma_0 = \alpha, \gamma_1, \ldots, \gamma_n = \beta')$ from α to β'. We now describe a procedure to be iterated several times. Fix a carrier K of o and the γ_i. Let ℓ be an arc from K_- to a puncture $p \in K_+$ such that ℓ intersects each γ_i exactly once, and construct the lasso path $\gamma_0(\ell), \ldots, \gamma_n(\ell)$. As lasso curves, each $\gamma_i(\ell_i)$ is adjacent to γ_i in Γ. In particular, α and $\alpha(\ell)$ are connected by an edge, and so are β' and $\beta'(\ell)$. Since K is a carrier containing o, α, β, Lemma 3.6 implies

$$d(\alpha(\ell), o) \geq d(\alpha, o) > R \qquad \text{and} \qquad d(\beta'(\ell), o) \geq d(\beta', o) > R.$$

So $\alpha(\ell), \beta'(\ell) \notin B$. Since the lasso path is again a straight path, the procedure described above may be iterated indefinitely. Indeed, after fixing a carrier K' of o and the $\gamma_i(\ell)$, a puncture $p' \in K'_+$, and an arc from K'_- to p' which intersects each $\gamma_i(\ell)$ exactly once, a new lasso path may again be constructed, the endpoints of which are not in B and are adjacent to the endpoints of the previous path. Repeat the procedure for a total of $D + 2R$ iterations, where $D = \max_i F(o, \gamma_i)$, and denote by $(\gamma'_0, \ldots, \gamma'_n)$ the final path obtained after the last iteration. Now, observe that this process also yields a path from α to γ'_0 and a path from β' to γ'_n consisting of the endpoints of the lasso paths constructed in each iteration. Applying Lemma 3.6 at each iteration shows that both of these paths are disjoint from B. Once it is shown that $(\gamma'_0, \ldots, \gamma'_n)$ is disjoint from B, the concatenation of the three paths then gives a path from α to β' which is disjoint from B.

Let $i \in \{0, \ldots, n\}$. By construction, γ_i and γ'_i are disjoint and $[\gamma_i, \gamma'_i]$ is an annulus with $D + 2R$ punctures. In particular, $F_0(\gamma_i, \gamma'_i) = D + 2R$. By definition, $D \geq F(o, \gamma_i)$. Then by Lemma 3.15,

$$F(o, \gamma'_i) \geq F(\gamma_i, \gamma'_i) - F(o, \gamma_i) \geq (D + 2R) - D = 2R.$$

So $d(o, \gamma'_i) \geq F(o, \gamma'_i) \geq 2R$, which implies $\gamma'_i \notin B$. Thus $(\gamma'_0, \ldots, \gamma'_n)$ is disjoint from B. □

4. Stable avenue surfaces

Now we consider the general setting when Σ is a stable avenue surface. Again, we denote by Γ the translatable curve graph $\mathcal{TC}(\Sigma)$ of Σ. First we need to establish some notation. If a simple closed curve η that does not separate e_+ and e_-, but $\Sigma \setminus \eta$ does have two components, then one of the components must have neither e_+ nor e_- as an end. The union of this component with η is a subsurface of Σ with boundary η, and it is denoted by C_η. If a connected surface T has at least one boundary circle, then denote by $\widehat{T}$ the surface obtained from T by capping a disk onto a boundary circle.

4.1. General lassoing and flux.

Let α be a separating curve, and let V be a clopen subset of ends contained either in $\mathrm{End}(\alpha_+)$ or $\mathrm{End}(\alpha_-)$. By the proof of [SC24, Lemma 4.6], there exists a simple closed curve η disjoint from α which bounds V. In particular, η does not separate e_+ and e_-, and $\mathrm{End}(C_\eta) = V$. Moreover, if S has finite positive genus, then for all $g \geq 0$, η may be chosen so that C_η has genus g. Let λ be an arc which has one endpoint on η, is disjoint from η otherwise, and intersects α. Denote by λ' the subarc of λ obtained by traveling along λ from its endpoint on η until the first intersection point with α. In other words, λ' is the component of $\lambda \setminus \alpha$ containing the endpoint on η.

Definition 4.1. Given α, η, and λ as above, the *lasso curve* defined by α, η, and λ, denoted by $\alpha(\eta, \lambda)$, is the simple closed curve obtained by tracing along α, λ, and η. The pair (η, λ) and the union $\eta \cup \lambda$ are both referred to as the *lasso* of $\alpha(\eta, \lambda)$.

Equivalently, $\alpha(\eta, \lambda)$ is the boundary curve of a small tubular neighborhood of $\alpha \cup \lambda' \cup \eta$ which is not isotopic to either α or η. Observe that by construction, the lasso curve $\alpha(\eta, \lambda)$ is a separating curve disjoint from α, and the end space of $[\alpha, \alpha(\eta, \lambda)]$ is equal to $\mathrm{End}(C_\eta)$. Then the following lemma follows from the definition of the translatable curve graph.

Lemma 4.2. *Let α be a separating curve and let $\alpha(\eta, \lambda)$ be a lasso curve defined by α and some η and λ. Then α and $\alpha(\eta, \lambda)$ are neighbors in Γ if either of the following conditions is met.*

- *C_η is homeomorphic to $\widehat{T}_i$ for some $1 \leq i \leq n$,*
- *C_η is homeomorphic to $\widehat{T}_{n+1}$.*

Let $(\alpha_0, \dots, \alpha_n)$ be a straight path in Γ. Let η be a curve contained in either α_{n+} or α_{0-} which bounds a clopen subset of ends, and let λ be an arc which has one endpoint on η, is disjoint from η otherwise, and intersects each α_i exactly once. Then by the same construction and arguments used in the previous section, the *lasso path* defined by the α_i, η, and ℓ is the straight path $(\alpha_0(\eta, \lambda), \dots, \alpha_n(\eta, \lambda))$.

Next we extend the notion of flux for the more general surface Σ. For disjoint separating curves α and β, define $p_0(\alpha, \beta)$ to be the number of discrete-type ends of $[\alpha, \beta]$, and define

$$g_0(\alpha, \beta) = \begin{cases} \text{genus of } [\alpha, \beta], & \text{if } S \text{ has finite genus.} \\ 0, & \text{if } S \text{ has infinite genus.} \end{cases}$$

Then put $F_0(\alpha, \beta) = p_0(\alpha, \beta) + g_0(\alpha, \beta)$. Now remove the assumption that α and β are disjoint. For separating curves γ in $\alpha_+ \cap \beta_+$ or $\alpha_- \cap \beta_-$, define $p(\alpha, \beta; \gamma) = |p_0(\alpha, \gamma) - p_0(\beta, \gamma)|$ and $g(\alpha, \beta; \gamma) = |g_0(\alpha, \gamma) - g_0(\beta, \gamma)|$. The statements and proofs of Lemma 3.11 and Lemma 3.12 all hold for p_0 and g_0. So define $p(\alpha, \beta) = p(\alpha, \beta; \gamma)$ and $g(\alpha, \beta) = g(\alpha, \beta; \gamma)$ for some (any) choice of separating curve γ in $\alpha_+ \cap \beta_+$ or $\alpha_- \cap \beta_-$. The *flux* of separating curves α and β is then defined by

$$F(\alpha, \beta) = p(\alpha, \beta) + g(\alpha, \beta).$$

Again, whenever α and β are disjoint we have $p(\alpha, \beta) = p_0(\alpha, \beta)$ and $g(\alpha, \beta) = g_0(\alpha, \beta)$, and therefore, $F(\alpha, \beta) = F_0(\alpha, \beta)$.

Lemma 4.3. *For all separating curves α and β, if $d(\alpha, \beta) = 1$ then $F(\alpha, \beta) = 1$.*

Proof. By definition of the translatable curve graph Γ, $d(\alpha, \beta) = 1$ means that α and β are disjoint and $[\alpha, \beta]$ is homeomorphic to some $T \in \mathcal{S}$. If $T = T_i$ for some $i = 1, \dots, n$, then T has exactly one discrete-type end and its genus is either 0 or infinite. So $p_0(\alpha, \beta) = 1$ and $g_0(\alpha, \beta) = 0$. If $T = T_{n+1}$, then T has no discrete-type ends and its genus is 1. So $p_0(\alpha, \beta) = 0$ and $g_0(\alpha, \beta) = 1$. In either case, $F(\alpha, \beta) = F_0(\alpha, \beta) = 1$. □

Then the same arguments in the previous section may be used to show that F is a pseudometric on Γ which bounds d from below. Also, it is still true in this setting that straight paths are geodesic. On the other hand, Lemma 3.18 does not immediately extend to the more general setting. Indeed, even if α and β are disjoint, then it is possible that $d(\alpha, \beta) > F_0(\alpha, \beta)$. This is due to the subtle definition of Γ. For example, suppose Σ has discrete-type ends, but $[\alpha, \beta]$ has genus 1 and no ends. Then $F_0(\alpha, \beta) = 1$, but $d(\alpha, \beta) \neq 1$ because $[\alpha, \beta]$ is not homeomorphic to T_{n+1}. However, next we give a sufficient condition on $[\alpha, \beta]$ to guarantee that $d(\alpha, \beta) = F(\alpha, \beta)$.

4.2. **Full subsurfaces.** Recall from Section 2.7 the fixed set $\mathcal{R} = \{f_1, \dots, f_N, c_1, \dots, c_M\}$ of representatives of the equivalence classes of maximal ends of S used to define the translatable curve graph.

Definition 4.4. A subsurface $T \subset \Sigma$ is *full* if $\operatorname{End}(T)$ contains an element of $E(f_i)$ for each $i = 1, \dots, N$ and an element of $E(c_j)$ for each $j = 1, \dots, M$. If $N = 0$, then S has finite positive genus and we further require T to have positive genus.

Note that the condition above is readily satisfied. For example, suppose α and β are separating curves such that β lies on the right side of α. Even if $[\alpha, \beta]$ is not itself full, $[\alpha, h(\beta)]$ must be full because it contains a subsurface homeomorphic to S. The proof of the next lemma resembles the proof that Γ is connected [SC24, Lemma 4.7].

Lemma 4.5. *Let α and β be disjoint separating curves. If $[\alpha, \beta]$ is full, then there exists a straight path from α to β of length $F(\alpha, \beta)$. In particular, $d(\alpha, \beta) = F(\alpha, \beta)$.*

Proof. Assume without loss of generality that β lies on the right side of α. By [SC24, Lemma 4.5], the end space of $[\alpha, \beta]$ can be expressed as a disjoint union $\bigsqcup_{i=1}^k U_i$ where each U_i is a stable neighborhood of a maximal end equivalent to one of the representatives in $\mathcal{R} = \{f_1, \dots, f_N, c_1, \dots, c_M\}$ chosen above. Moreover, it follows from stability and maximality that each U_i intersects $E(r)$ for exactly one $r \in R$. On the other hand, for each representative $r \in R$, at least one of the U_i intersects $E(r)$ because $[\alpha, \beta]$ is full. In particular, $k \geq N + M$. Put $p = p(\alpha, \beta)$, and note that $N \leq p \leq k$. Denote by $\{V_i\}_{i=1}^p$ the set of U_i which are stable neighborhoods of discrete-type ends. The remaining U_i (if any exist) are stable neighborhoods of Cantor-type ends. Note that if U_i and $U_{i'}$ both intersect some $E(c_j)$, then the disjoint union $U_i \sqcup U_{i'}$ is again a stable neighborhood which intersects $E(c_j)$. So the remaining U_i may be combined to form a set $\{W_i\}_{i=1}^M$ where for each $i = 1, \ldots, M$, W_i is a stable neighborhood of an end equivalent to c_i. In summary, we have

$$\bigsqcup_{i=1}^k U_i = \left(\bigsqcup_{i=1}^{\ell} V_i\right) \sqcup \left(\bigsqcup_{i=1}^{M} W_i\right).$$

Put $F = F_0(\alpha, \beta)$ and $g = g_0(\alpha, \beta)$. Since $[\alpha, \beta]$ is full, $F = p + g > 0$. For each $1 \leq i \leq M$, $E(c_i) \cap W_i$ is a Cantor set. So W_i can be expressed as $\bigsqcup_{j=1}^F W_{i,j}$ where each $W_{i,j}$ is a stable neighborhood of an end in $E(c_i)$, and is thus homeomorphic to W_i. For $1 \leq j \leq p$, put $Z_j = V_j \sqcup \bigsqcup_{i=1}^M W_{i,j}$, and for $p+1 \leq j \leq F$, put $Z_j = \bigsqcup_{i=1}^M W_{i,j}$. Then the end space of $[\alpha, \beta]$ is equal to $\bigsqcup_{j=1}^F Z_j$ and each Z_j is homeomorphic to $\mathrm{End}(T)$ for some $T \in \mathcal{S}$.

Put $\alpha_0 = \alpha$. For $j = 1, \ldots, p$, inductively define α_j to be the lasso curve $\alpha_{j-1}(\eta_j, \lambda_j)$, where the lasso $\eta_j \cup \lambda_j$ is contained in $[\alpha_{j-1}, \beta]$, the end space of C_{η_j} is equal to Z_j, and

$$\mathrm{genus}(C_{\eta_j}) = \begin{cases} 0 \text{ or } \infty, & 1 \leq j \leq p, \\ 1, & p < j \leq F. \end{cases}$$

By construction, for $1 \leq j \leq p$, $[\alpha_{j-1}, \alpha]$ is homeomorphic to $T_i \in \mathcal{S}$ for some $i = 1, \ldots, n$, and for $p < j \leq F$, $[\alpha_{j-1}, \alpha_j]$ is homeomorphic to T_{n+1}. Hence, α_{j-1} and α_j are neighbors in Γ for all $j = 1, \ldots, F$. Moreover, each α_j lies on the right side of α_{j-1} by construction, and α_F is isotopic to β because $[\alpha_F, \beta]$ is an annulus. Therefore, $(\alpha_0 = \alpha, \ldots, \alpha_F = \beta)$ is a straight path from α to β of length $F(\alpha, \beta)$. Since straight paths are geodesic, $d(\alpha, \beta) = F(\alpha, \beta)$. ☐

In fact, if $[\alpha, \beta]$ is full, then every geodesic in Γ from α to β must be a straight path.

Lemma 4.6. *Let α and β be disjoint separating curves such that β lies on the right side of α and $[\alpha, \beta]$ is full. If $(\gamma_0 = \alpha, \gamma_1, \ldots, \gamma_n = \beta)$ is any geodesic path in Γ between α and β, then for each $i = 1, \ldots, n$, γ_i lies on the right side of γ_{i-1}. In particular, γ_i lies in $[\alpha, \beta]$ for all $i = 0, \ldots, n$.*

Proof. Suppose for contradiction that the conclusion fails, and let j be the smallest index for which γ_{j+1} lies on the left side of γ_j. Since γ_i lies on the right side of γ_{i-1} for each $i = 1, \ldots, j$, it follows from an inductive argument that $F(\alpha, \gamma_j) = j$. Let η be a separating curve in $\alpha_- \cap \gamma_{j+1\,-}$. Then

$$\begin{aligned} F_0(\eta, \beta) &= F_0(\eta, \alpha) + F_0(\alpha, \beta) \\ F_0(\eta, \gamma_j) &= F_0(\eta, \alpha) + F_0(\alpha, \gamma_j) \\ F_0(\eta, \gamma_j) &= F_0(\eta, \gamma_{j+1}) + F_0(\gamma_{j+1}, \gamma_j). \end{aligned}$$

Lemma 4.5 implies $F_0(\alpha, \beta) = d(\alpha, \beta) = n$, and Lemma 4.3 implies $F_0(\gamma_{j+1}, \gamma_j) = 1$. So

$$\begin{aligned} F(\beta, \gamma_{j+1}) &= |F_0(\beta, \eta) - F_0(\gamma_{j+1}, \eta)| \\ &= |F_0(\eta, \alpha) + F_0(\alpha, \beta) - F_0(\eta, \gamma_j) + F_0(\gamma_{j+1}, \gamma_j)| \\ &= |F_0(\eta, \alpha) + n - (F_0(\eta, \alpha) + F_0(\alpha, \gamma_j)) + 1)| \\ &= n - j + 1. \end{aligned}$$

Hence, $d(\beta, \gamma_{j+1}) \geq F(\beta, \gamma_{j+1}) = n - j + 1$. But recall that $(\gamma_0 = \alpha, \gamma_1, \ldots, \gamma_n = \beta)$ is geodesic path between α and β. In particular, $d(\beta, \gamma_{j+1}) = n - (j+1) = n - j - 1$. So we have a contradiction. ☐

Lemma 4.7. *Let α and β be disjoint separating curves such that $[\alpha, \beta]$ is full. If $\alpha(\eta, \lambda)$ is a lasso curve adjacent to α in Γ, and $\alpha(\eta, \lambda)$ is not contained in $[\alpha, \beta]$, then $d(\alpha(\eta, \lambda), \beta) \geq d(\alpha, \beta)$.*

Proof. We prove the contrapositive statement. Let $n = d(\alpha, \beta)$ and suppose $d(\alpha(\eta, \lambda), \beta) < n$. Since $\alpha(\eta, \lambda)$ and α are neighbors in Γ, we must have $d(\alpha(\eta, \lambda), \beta) = n - 1$. Take a geodesic path of length $n - 1$ between $\alpha(\eta, \lambda)$ and β, and include (the edge between $\alpha(\eta, \lambda)$ and) α and to obtain a path of length n between α and β. Then this is a geodesic path between α and β which contains $\alpha(\eta, \lambda)$. By Lemma 4.6, $\alpha(\eta, \lambda)$ is contained in $[\alpha, \beta]$. ☐

The above lemma is reminiscent of Lemma 3.6, but crucially, it is not as powerful because it requires α and β to be disjoint. Lastly, we define the notions of *flux-right* and *flux-left* in the exact same way as in the previous section, and the statements and proofs of Lemma 3.20 all still hold in the more general setting.

4.3. **Hamming distance.** In the previous case of the bi-infinite flute, one of the tools we used was the operation of forgetting a puncture. However, it does not seem to be easy to extend this idea to the more general setting. How does one precisely define "forgetting a discrete-type end" or "forgetting a genus" as operations which map separating curves to separating curves? Thus, to compensate for the absence of this tool, we introduce another pseudometric on Γ which provides a better lower bound on d than flux does. For a subsurface $T \subset \Sigma$, denote by $\mathrm{End}_d(T)$ the subset of discrete-type ends in $\mathrm{End}(T)$. For each separating curve α, put $P(\alpha) = \mathrm{End}_d(\alpha_+)$.

Definition 4.8. The *Hamming distance* between separating curves α and β, denoted by $H(\alpha, \beta)$, is defined by

$$H(\alpha, \beta) = |P(\alpha) \triangle P(\beta)| + g(\alpha, \beta).$$

To get some intuition behind this definition, suppose that S has zero or infinite genus, so that $g(\alpha, \beta)$ is always 0. Each separating curve α partitions the set of discrete-type ends of Σ, which is identified with $\mathbb{Z}$, into two sets, thereby associating to α a bi-infinite binary sequence which stabilizes at 0 in one direction and at 1 in the other direction. Then $H(\alpha, \beta) = |P(\alpha) \triangle P(\beta)|$ is equal to the classical Hamming distance between the two sequences associated to α and β.

Lemma 4.9. *For all separating curves α and β, if $d(\alpha, \beta) = 1$ then $H(\alpha, \beta) = 1$.*

Proof. By definition, $d(\alpha, \beta) = 1$ means that α and β are disjoint and $[\alpha, \beta]$ is homeomorphic to some $T \in \mathcal{S}$. If $T = T_i$ for some $i = 1, \ldots, n$, then T has exactly one discrete-type end f and its genus is either 0 or infinite. Assuming α lies on the left side of β, this means that $P(\alpha) = P(\beta) \sqcup \{f\}$. So $P(\alpha) \triangle P(\beta) = \{f\}$ and $g_0(\alpha, \beta) = 0$. If $T = T_{n+1}$, then T has no discrete-type ends and its genus is 1. So $P(\alpha) = P(\beta)$ and $g_0(\alpha, \beta) = 1$. In either case, $H(\alpha, \beta) = 1$. □

Moreover, H is indeed a pseudometric.

Lemma 4.10. *For all separating curves α, β, γ in Σ, we have $H(\alpha, \beta) \leq H(\alpha, \gamma) + H(\gamma, \beta)$.*

Proof. It suffices to show that the inequality holds separately for $|P(\alpha) \triangle P(\beta)|$ and $g(\alpha, \beta)$. Indeed,

$$|P(\alpha) \triangle P(\beta)| \leq |P(\alpha) \triangle P(\gamma) \cup P(\gamma) \triangle P(\beta)| \leq |P(\alpha) \triangle P(\gamma)| + |P(\gamma) \triangle P(\beta)|$$

and letting η be a separating curve in $\alpha_+ \cap \beta_+ \cap \gamma_+$,

$$g(\alpha, \beta) = |g_0(\alpha, \eta) - g_0(\beta, \eta)| = |g_0(\alpha, \eta) - g_0(\gamma, \eta) + g_0(\gamma, \eta) - g_0(\beta, \eta)| \leq g(\alpha, \gamma) + g(\gamma, \beta).$$

□

Furthermore, the Hamming distance gives a tighter lower bound on the graph distance d in Γ.

Lemma 4.11. *For all separating curves α and β, we have $d(\alpha, \beta) \geq H(\alpha, \beta) \geq F(\alpha, \beta)$.*

Proof. The first inequality follows from the same argument used in the proof of Lemma 3.16. To get the second inequality, let γ be a separating curve in $\alpha_+ \cap \beta_+$. By definition, $p_0(\alpha, \gamma) = |\mathrm{End}_d([\alpha, \gamma])|$ and $p_0(\beta, \gamma) = |\mathrm{End}_d([\beta, \gamma])|$. Since γ_+ is contained in both α_+ and β_+,

$$P(\alpha) \triangle P(\beta) = \mathrm{End}_d([\alpha, \gamma]) \triangle \mathrm{End}_d([\beta, \gamma]).$$

Hence,

$$\begin{aligned}
H(\alpha, \beta) &= |P(\alpha) \triangle P(\beta)| + g(\alpha, \beta) \\
&= |\mathrm{End}_d([\alpha, \gamma]) \triangle \mathrm{End}_d([\beta, \gamma])| + g(\alpha, \beta) \\
&\geq ||\mathrm{End}_d([\alpha, \gamma])| - |\mathrm{End}_d([\beta, \gamma])|| + g(\alpha, \beta) \\
&= |p_0(\alpha, \gamma) - p_0(\beta, \gamma)| + g(\alpha, \beta) \\
&= p(\alpha, \beta) + g(\alpha, \beta) \\
&= F(\alpha, \beta).
\end{aligned}$$

□

We now aim to understand how lassoing affects the Hamming distance between curves.

Lemma 4.12. *Let α be a separating curve. If $\alpha(\eta, \lambda)$ is a lasso curve defined by α and some η and λ, then $P(\alpha) \triangle P(\alpha(\eta, \lambda)) = \mathrm{End}_d(C_\eta)$.*

Proof. First suppose η lies on the right side of α, so that $\mathrm{End}(C_\eta) \subset \mathrm{End}(\alpha_+)$. It follows from the construction of $\alpha(\eta, \lambda)$ that after lassoing, η now lies on the left side of $\alpha(\eta, \lambda)$, and $\mathrm{End}(\alpha(\eta, \lambda)_+) = \mathrm{End}(\alpha_+) \setminus \mathrm{End}(C_\eta)$. Similarly, if η lies on the left side of α, then $\mathrm{End}(C_\eta)$ and $\mathrm{End}(\alpha_+)$ are disjoint, and $\mathrm{End}(\alpha(\eta, \lambda)_+) = \mathrm{End}(\alpha_+) \sqcup \mathrm{End}(C_\eta)$. Thus, in both cases, $\mathrm{End}(\alpha_+) \triangle \mathrm{End}(\alpha(\eta, \lambda)_+) = \mathrm{End}(C_\eta)$. Considering only the discrete-type ends in these sets then gives the result. $\square$

We observe in the next lemma that given a pair of curves, applying the lassoing procedure to just one of the curves, with respect to a discrete-type end that is on the same side of both curves, increases the Hamming distance between the pair.

Lemma 4.13. *Let α and β be separating curves. Suppose $\alpha(\eta, \lambda)$ is a lasso curve such that η lies on the same side of α and β, and C_η has genus 0 if S has finite genus. Then $H(\alpha(\eta, \lambda), \beta) = H(\alpha, \beta) + |\mathrm{End}_d(C_\eta)|$.*

Proof. If S has infinite genus, then $g(\alpha(\eta, \lambda), \beta) = 0 = g(\alpha, \beta)$. On the other hand, if S has finite genus, then C_η has genus 0 by the hypothesis, and it follows that $g(\alpha(\eta, \lambda), \beta) = g(\alpha, \beta)$.

By the proof of Lemma 4.12, $P(\alpha(\eta, \lambda)) = P(\alpha) \setminus \mathrm{End}_d(C_\eta)$ if η lies on the right side of α, and $P(\alpha(\eta, \lambda)) = P(\alpha) \sqcup \mathrm{End}_d(C_\eta)$ if η lies on the left side of α. In either case, whether $\eta \subset \alpha_+ \cap \beta_+$ or $\eta \subset \alpha_- \cap \beta_-$,

$$P(\alpha(\eta, \lambda)) \triangle P(\beta) = (P(\alpha) \triangle P(\beta)) \sqcup \mathrm{End}_d(C_\eta).$$

Altogether, we get

$$\begin{aligned} H(\alpha(\eta, \lambda), \beta) &= |P(\alpha(\eta, \lambda)) \triangle P(\beta)| + g(\alpha(\eta, \lambda), \beta) \\ &= |P(\alpha) \triangle P(\beta)| + |\mathrm{End}_d(C_\eta)| + g(\alpha, \beta) \\ &= H(\alpha, \beta) + |\mathrm{End}_d(C_\eta)|. \end{aligned}$$

$\square$

4.4. **Main theorem.** Now we are ready to prove the main theorem.

Theorem 4.14. *If Σ is a stable avenue surface with at least one discrete-type end, then* $\mathrm{Map}(\Sigma)$ *is one-ended.*

Proof. By Theorem 2.10, it suffices to prove that Γ is one-ended. Fix a vertex $o \in \Gamma$ and let $R > 0$ be any integer. To prove that Γ is one-ended, it suffices by Lemma 2.4 to show that for any $\alpha, \beta \in \Gamma$ with $d(\alpha, o) = d(\beta, o) = 6R$, there exists a path from α to β which is disjoint from $B = B(o, R)$.

Like in the proof of Theorem 3.21, the action of $\mathrm{Map}(\Sigma)$ on Γ allows us to assume that β is the boundary curve of some copy of S. Then we may use the translation h to push β as far towards e_+ as we want. In total, we may assume without loss of generality that β lies on the right side of α such that $[\alpha, \beta]$ is full and $d(\beta, o) \geq 3R$. Then Lemma 4.5 gives a straight path $(\gamma_0 = \alpha, \gamma_1, \dots, \gamma_k = \beta)$ from α and β. Since Σ has at least one class of discrete-type ends, there exists a lasso (η, λ) such that $\eta \subset \beta_+ \cap o_+$, C_η is homeomorphic to $\widehat{T}_i$ for some $1 \leq i \leq n$, and λ intersects each γ_i exactly once. Consider the lasso path $(\gamma_0(\eta, \lambda), \dots, \gamma_k(\eta, \lambda))$. Note that α and $\gamma_0(\eta, \lambda)$ are neighbors in Γ, and so are β and $\gamma_k(\eta, \lambda)$. Also, for each $i = 0, \dots, k$, Lemma 4.13 implies that $H(\gamma_i(\eta, \lambda), o) = H(\gamma_i, o) + 1$. Since the lasso path is again a straight path, this procedure may be iterated indefinitely, and each iteration produces a new lasso path whose vertices have a Hamming distance from o increased by 1. Repeat the procedure for a total of $2R$ iterations and denote by $(\gamma'_0, \dots, \gamma'_k)$ the final path obtained after the last iteration. Then for each $i = 0, \dots, k$,

$$d(\gamma'_i, o) \geq H(\gamma'_i, o) = H(\gamma_i, o) + 2R > R.$$

So $(\gamma'_0, \dots, \gamma'_k)$ is disjoint from B. Furthermore, the endpoints of the lasso paths constructed in each iteration forms a path from α to γ'_0 and a path from β to γ'_k. The first $R+1$ vertices along these paths have a distance of at least $2R$ from o because $d(\alpha, o), d(\beta, o) \geq 3R$. The remaining vertices along these paths have a distance greater than R from o because the Hamming distance is greater than R. So these two paths are also disjoint from B. The concatenation of all three paths then is a path from α to β which is disjoint from B. $\square$

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