

THE QUASI-REDIRECTING BOUNDARY

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ABSTRACT. We generalize the notion of Gromov boundary to a larger class of metric spaces beyond Gromov hyperbolic spaces. Points in this boundary are classes of quasi-geodesic rays and the space is equipped with a topology that is naturally invariant under quasi-isometries. It turns out that this boundary is compatible with other notions of boundary in many ways; it contains the sublinearly Morse boundary as a topological subspace and it matches the Bowditch boundary of relative hyperbolic spaces when the peripheral subgroups have no intrinsic hyperbolicity. We also give a complete description of the boundary of the Croke-Kleiner group where the quasi-redirecting boundary reveals a new class of QI-invariant, Morse-like quasi-geodesics.

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1. INTRODUCTION

Our goal in this paper is to organize and understand the space of quasi-geodesic rays in a given metric space. A quasi-geodesic ray represents a possible direction towards infinity hence the space of quasi-geodesic rays could be thought of as the boundary at infinity of a metric space X . However, different quasi-geodesics may represent the same direction. This is in direct analogy with the Gromov boundary defined for a Gromov hyperbolic metric space where two quasi-geodesics represent the same point in the Gromov boundary when they fellow travel each other. Our approach is to start from first principles and choose definitions that are intuitively natural and immediately invariant under a quasi-isometry. More precisely, we would like a notion of boundary where

- (1) Points in the boundary are equivalence classes of quasi-geodesic rays.

- (2) The definition of an equivalence class and the topology rely only on the coarse geometry of X and hence are invariant under quasi-isometry.
- (3) The boundary is as large as possible.

To start, we need to decide when two quasi-geodesic rays α and β represent the same direction in the metric space X . Our intuitive answer is that, if there are quasi-geodesic rays with uniform constants that travel along α for arbitrary distances and then change course and eventually coincide with β then traveling in the direction of α does not move one away from the direction defined by β and hence α and β do not represent distinct directions. In this case, we say α can be quasi-redirected to β and write $\alpha \preceq \beta$ (see Definitions 3.1 and 3.3 for precise definitions). However, this turns out to not be symmetric in general and $\alpha \preceq \beta$ does not always imply $\beta \preceq \alpha$. We let $P(X)$ be a set of equivalence classes of this relation. Then \preceq induces a partial order in $P(X)$.

Proposition A. *A quasi-isometry between metric spaces X and Y induces a bijection from $P(X)$ to $P(Y)$ that preserves the partial order.*

We think of $P(X)$ as the set of directions in X . One could force the relation to be symmetric. However, this runs counter to the idea of making the boundary as large as possible. In fact, as we shall see, the asymmetry highlights interesting features of shape of the metric space X at infinity which is recorded in the set $P(X)$.

To form a boundary at infinity, we need to put a topology on the set of all directions. However, this cannot be done in the setting of general proper metric spaces as they can be quite untamed. Our main motivation is always to study finitely generated groups or spaces quasi-isometric to them. Thus we put some technical assumptions on the metric space X to allow for a cone-like topology to be defined on $P(X)$. These are marked as Assumption 0, 1 and 2 and we make it clear throughout the paper which assumptions are used where. Assumption 0 holds for all finitely generated groups. To our knowledge, we do not know of a finitely generated group whose Cayley graph does not satisfy Assumptions 1 and 2 and we check the validity of these assumptions for a large class of groups.

Theorem B. *Let X be a proper, quasi-geodesic metric space satisfying Assumptions 0, 1 and 2. Then the space of directions in X can be organized into a topological space which we denote by ∂X . A quasi-isometry from a metric space X to a metric space Y induces a homeomorphism between ∂X to ∂Y .*

The quasi-isometry invariance allows us to write ∂G for all finitely generated groups G . We also check the compatibility of this boundary with other generalization of the Gromov boundary. The notion of the sublinearly Morse boundary was developed in [QRT22, QRT23]. This is a family of boundaries $\partial_\kappa X$ for every sublinear function κ . A point in the sublinearly Morse boundary is a class of quasi-geodesic rays that resemble Morse geodesics where the Morse gauge is allowed to tend to infinity sublinearly with the radius. Two quasi-geodesics are in the same class if they fellow travel each other sublinearly. The sublinearly Morse boundary was shown to be large enough to be used as a topological model for the Poisson boundary in many settings (see [QRT22, QRT23]) as well large enough for other measures of genericity [GQR23]. We show that the quasi-redirecting boundary is an enlargement of the sublinearly Morse boundary.

Theorem C. *Let X be a proper, geodesic metric space satisfying Assumptions 0, 1 and 2 and let κ be a sublinear function. Then, for every κ -Morse quasi-geodesic α , the class of*

quasi-geodesics that sublinearly fellow travel α is the same as the quasi-redirecting class of α and hence $\partial_\kappa X \subset \partial X$. In fact, $\partial_\kappa X$ is a topological subspace of ∂X .

The benefit of this enlargement is that, unlike $\partial_\kappa X$, ∂X is often compact. Also, there are quasi-redirecting classes which exhibit Morse-like properties that are not sublinearly Morse (see Section 11) and hence ∂X encodes strictly more information than the sublinearly Morse boundary. The cost of the enlargement is that ∂X is not always metrizable (in our usage, compact does not imply metrizable).

Question D. *Let X be a Cayley graph of a finitely generated group. Is ∂X always defined? Is ∂X always compact?*

Relatively hyperbolic groups. When X exhibits no hyperbolicity, ∂X is trivial (has only one point). This includes spaces with a product structure (Proposition 4.2) and the Cayley graphs of Baumslag–Solitar groups [McM]. We refer to such groups and spaces as *mono-directional*.

The main class of examples we consider is the class of proper geodesic metric spaces that are asymptotically tree-graded with respect to mono-directional sets (ATM spaces). Asymptotically tree-graded spaces were first introduced by Drutu–Sapir in [DS05] and systematically studied also in [DS05, DS08] and [Sis12], among others. The idea also appears in [HK05] in similar setting.

Theorem E. *Let X be an ATM space. Then:*

- (I) *The space X satisfies Assumptions 0, 1, and 2, thus ∂X is defined. In fact, \preceq is a symmetric relation on $P(X)$.*
- (II) *The boundary ∂X is compact, metrizable and second countable.*

These spaces are metric analogues of relative hyperbolic groups. In fact, when X is a Cayley graph of a relative hyperbolic group, this theorem gives an alternative description of the Bowditch boundary that is purely based on the coarse geometry of X , and does not use the algebraic structure of relative hyperbolic groups.

Theorem F. *Let G be a relatively hyperbolic group equipped with the word metric associated to a finite generating set such that the peripheral subgroups are mono-directional. Then ∂G is homeomorphic to the Bowditch boundary of G .*

Hence, when X is an ATM space, we can think of ∂X as the Bowditch boundary of X which is not defined without a group action. Also, using the quasi-redirecting boundary, we see how quasi-isometries between relatively hyperbolic groups naturally induce homeomorphisms between the Bowditch boundaries. (This statement also follows from [BDM09, Theorem 4.8]).

Corollary G. *Let G and G' be relatively hyperbolic groups equipped with the word metric associated to a finite generating set such that the peripheral subgroups are mono-directional. Then any quasi-isometry $\Phi: G \rightarrow G'$ induces a homeomorphism between the Bowditch boundaries of G and G' .*

It would be interesting to know if the metrizability of ∂X is a characterizing property of ATM spaces.

Question H. *Let X be a geodesic metric space where assumptions 0, 1 and 2 hold. Assume ∂X is metrizable and X has a cocompact action by a finitely generated group G . Does that imply that G is a relative hyperbolic group with respect to mono-directional peripheral subgroups?*

The Croke-Kleiner group. A good example of a non-positively curved group that is not relatively hyperbolic is the Croke-Kleiner group,

$$G = \langle a, b, c, d \mid [a, b], [b, c], [c, d] \rangle,$$

which we study in detail in Section 11. The Croke-Kleiner group is a well-known obstruction to attempts to generalize the Gromov boundary to visual boundaries in non-hyperbolic settings [CK02]. Therefore, it is useful to analyze the group in our current generalization of the Gromov boundary. It turns out that, certain (but not all) directions in the boundary of the Bass-Serre tree behave like Morse geodesics in a weak sense. As expected, this set contains the sublinearly Morse directions, but in this case, it is strictly a larger set. The quasi-redirecting boundary is a one point compactification of a set of *Morse-like* directions.

Theorem I. *Let X be the universal cover of the Salvetti complex of the Croke-Kleiner group G . Then X satisfies Assumptions 0, 1 and 2, thus $\partial G = \partial X$ is defined. The relation \preceq is not symmetric; $P(G)$ has one maximal element and other elements (the minimal elements) are not comparable. The set of minimal elements is a strict enlargement of the sublinearly Morse boundary.*

Our study of the Croke-Kleiner group offers a template to analyze quasi-redirecting boundaries of irreducible right-angled Artin groups, CK-admissible groups, mapping class groups and hierarchically hyperbolic groups where we expect the picture to be similar.

History. Hyperbolic groups and their boundaries were first introduced by Gromov [Gro87]. This notion was generalized to many other settings where the group is not hyperbolic but it has some weaker hyperbolic-like properties, notably, CAT(0) groups [Gro87, Ger94], relative hyperbolic groups [Bow12, Far98], the mapping class group [MM00], acylindrical hyperbolic groups [Osi16] and hierarchically hyperbolic groups [BHS17]. The hyperbolicity in these groups is captured in various boundaries, namely the visual boundaries for CAT(0) spaces [Gro91], the Bowditch boundary for relatively hyperbolic groups [Bow12], Thurston boundary of Teichmüller spaces [FLP12], Furstenberg boundary of the symmetric spaces [Fur63], the Floyd boundary [Flo80] and horofunction boundaries of geodesic metric spaces [Gro81] to name a few.

There has also been many attempts to define a natural boundary that is invariant under quasi-isometries. In 2013, the *contracting boundary* of CAT(0) spaces was constructed by Charney and Sultan [CS15], and is shown to be a first quasi-isometrically invariant geometric boundary in non-hyperbolic settings. The construction was generalized to proper geodesic spaces by Cordes in [Cor18] The Morse boundaries are equipped with a *direct limit topology* and are invariant under quasi-isometries. However, this space does not have good topological properties; for example, it is not first countable. Cashen-Mackay [CM19], following the work of Arzhantseva-Cashen-Gruber-Hume [ACGH17], defined a different topology on the Morse boundary. These turn out to be topological subspaces a larger space, namely, the sublinearly Morse boundaries [QRT22, QRT23]. Aside from being QI-invariant and metrizable, sublinearly Morse boundaries turn out to be generic in many senses. In the case of right-angled Artin groups, [QRT22] also shows that κ -Morse boundaries realize Poisson boundaries for $\kappa(t) = \log t$. For mapping class groups, Kaimanovich-Masur showed that uniquely ergodic projective measured foliations with the corresponding harmonic measure can be identified with the Poisson boundary of random walks; Qing-Rafi-Tiozzo [QRT23] showed that, when $\kappa = \log t$, the κ -boundary of the Cayley graph of the mapping class group can be identified

with the Poisson boundary of the associated random walks. Meanwhile, genericity of a more geometric flavor is also exhibited for sublinearly Morse boundaries. In [GQR23], genericity of sublinearly Morse directions under Patterson Sullivan measure was shown to hold in a more general context of actions admitting a strongly contracting element. In fact, the results in [GQR23] concerning stationary measures were recently obtained in this more general setting by Inhyeok Choi [Choi22], who in place of our ergodic theoretic and boundary techniques uses a pivoting technique developed by Gouëzel [Gou22]. Meanwhile, a Patterson-Sullivan theory on a certain quotient of the horofunction boundary for spaces admitting non-elementary group actions with contracting elements was recently obtained by Coulon [Cou22] and Yang [Yan22]. Following [Yan22], genericity of sublinearly Morse directions on the horofunction boundary was recently shown for all proper statistically convex-cocompact actions on proper metric spaces [QY24]. Furthermore, sublinearly Morse directions are invariant even under sublinear biLipschitz equivalence between metric spaces [PQ]. A compact metrizable boundary was introduced by Dydac and Rashed in [DR22] using C^* -algebra. However, this boundary seems to be small in many settings and does not contain the sublinearly Morse boundary as a topological subspace.

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2. PRELIMINARIES

In this section we recall some basic definitions and set up a few notations (see Notation 2.4). We also present a few old and new surgery constructions between quasi-geodesics as many arguments in this paper involve constructing quasi-geodesics with controlled constants. We also discuss Assumption 0 which is the most basic assumption on all the metric space we will be considering.

Definition 2.1 (Quasi Isometric embedding). Let (X, d_X) and (Y, d_Y) be metric spaces. For constants $q \geq 1$ and $Q \geq 0$, we say a map $\Phi: X \rightarrow Y$ is a (q, Q) -quasi-isometric embedding if, for all $x_1, x_2 \in X$

$$\frac{1}{q}d_X(x_1, x_2) - Q \leq d_Y(\Phi(x_1), \Phi(x_2)) \leq qd_X(x_1, x_2) + Q.$$

If, in addition, every point in Y lies in the Q -neighbourhood of the image of Φ , then Φ is called a (q, Q) -quasi-isometry. This is equivalent to saying that Φ has a *quasi-inverse*. That is, there exist constants $q', Q' > 0$ and a (q', Q') -quasi-isometric embedding $\Psi: Y \rightarrow X$ such that,

$$\forall x \in X \quad d_X(x, \Psi\Phi(x)) \leq Q' \quad \text{and} \quad \forall y \in Y \quad d_Y(y, \Phi\Psi(y)) \leq Q'.$$

When such a map Φ exists, we say (X, d_X) and (Y, d_Y) are *quasi-isometric*.

Definition 2.2 (Quasi-Geodesics). A *quasi-geodesic* in a metric space X is a quasi-isometric embedding $\alpha: I \rightarrow X$ where $I \subset \mathbb{R}$ is an (possibly infinite) interval. That is, $\alpha: I \rightarrow X$ is a

(q, Q) -quasi-geodesic if, for all $s, t \in I$, we have

$$\frac{|t - s|}{q} - Q \leq d_X(\alpha(s), \alpha(t)) \leq q \cdot |s - t|.$$

Furthermore, without loss of generality (see Lemma 2.3), in this paper we always assume α is $(2q + 2Q)$ -Lipschitz, in particular, α is continuous.

The assumption that α is Lipschitz is needed so we can apply the Arzelà-Ascoli theorem to a sequence of quasi-geodesics to obtain a limiting quasi-geodesic. However, this assumption can always be achieved by increasing the constants of the quasi-geodesic:

Lemma 2.3 (Taming of the quasi-geodesics). *Let X be a geodesic metric space and let $I \subset \mathbb{R}$ be an interval of length bigger than 1. Given a (q, Q) -quasi-isometric embedding $\alpha: I \rightarrow X$, one can find a (q', Q') -quasi-geodesic $\alpha': I \rightarrow X$, with q' and Q' depending only on q and Q , that is $2(q + Q)$ -Lipschitz and fellow travels α . In fact, for $t \in I$, we have*

$$d_X(\alpha(t), \alpha'(t)) \leq 2(q + Q).$$

Proof. We assume I is compact, the proof for other cases is similar. Let k be an integer larger than the length of I and choose times $t_0 \leq t_1 \leq \dots \leq t_k$ such that $\frac{1}{2} \leq |t_{i+1} - t_i| \leq 1$ and $I = [t_0, t_k]$. For $i = 0, \dots, k$, define $\alpha'(t) = \alpha(t)$. Then define $\alpha'[t_i, t_{i+1}]$ to be a geodesic segments connecting $\alpha(t_i)$ to $\alpha(t_{i+1})$. The length of each of these geodesic segments is at most $(q + Q)$ and the length of $[t_i, t_{i+1}]$ is at least $\frac{1}{2}$. Hence α' is $2(q + Q)$ -Lipschitz. For every $t \in I$, there is t_i such that $|t - t_i| \leq \frac{1}{2}$. Hence

$$d_X(\alpha(t), \alpha'(t)) \leq d_X(\alpha(t), \alpha(t_i)) + d_X(\alpha'(t_i), \alpha'(t)) \leq \frac{q}{2} + Q + \frac{q + Q}{2} \leq q + 2Q.$$

To see the lower bound for α' , let $t, s \in I$ and let t_i and t_j be such that $|t - t_i| \leq \frac{1}{2}$ and $|s - t_j| \leq \frac{1}{2}$. Then

$$\begin{aligned} d_X(\alpha'(t), \alpha'(s)) &\geq d_X(\alpha'(t_i), \alpha'(t_j)) - d_X(\alpha'(t), \alpha'(t_i)) - d_X(\alpha'(s), \alpha'(t_j)) \\ &\geq \frac{|t_i - t_j|}{q} - Q - \frac{(q + Q)}{2} - \frac{(q + Q)}{2} \\ &\geq \frac{|s - t| - 1}{q} - q - 2Q. \end{aligned}$$

That is, the Lemma holds for

$$q' = q + Q \quad \text{and} \quad Q' = q + \frac{1}{q} + 2Q. \quad \square$$

Notation 2.4. To simplify notation, we use $\mathfrak{q} = (q, Q) \in [1, \infty) \times [0, \infty)$ to indicate a pair of constants. That is, we say $\Phi: X \rightarrow Y$ is a \mathfrak{q} -quasi-isometry. We also say α is \mathfrak{q} -quasi-geodesic, which can be a ray and/or a segment depending on the context. Furthermore, we fix a base point \mathfrak{o} in the metric space X . By a \mathfrak{q} -ray we mean a \mathfrak{q} -quasi-geodesic ray $\alpha: [0, \infty) \rightarrow X$ such that $\alpha(0) = \mathfrak{o}$. For an interval $[s, t] \subset [0, \infty)$, we denote the restriction of α to the time interval $[s, t]$ by $\alpha[s, t]$ (simplified from $\alpha([s, t])$). However, if points $x, y \in X$ on the image of α are given, we denote the sub-segment of α connecting x to y by $[x, y]_\alpha$. That is, if $\alpha(s) = x$ and $\alpha(t) = y$ for $s \leq t$, then $[x, y]_\alpha = \alpha[s, t]$.

We often need to concatenate quasi-geodesics. Let $\alpha: [s_1, s_2] \rightarrow X$ and $\beta: [t_1, t_2] \rightarrow X$ be two quasi-geodesics such that $\alpha(s_2) = \beta(t_1)$. We denote the concatenation of α and β by $\alpha \cup \beta$ by which we mean the following quasi-geodesic:

$$\alpha \cup \beta: [s_1, t_2 - t_1 + s_2] \rightarrow X, \quad \alpha \cup \beta(t) = \begin{cases} \alpha(t) & \text{for } t \in [s_1, s_2] \\ \beta(t + t_1 - s_2) & \text{for } t \in [s_2, t_2 - t_1 + s_2] \end{cases}.$$

For $r > 0$, let $B_r^\circ \subset X$ be the open ball of radius r centered at \mathfrak{o} , let B_r be the closed ball centered at \mathfrak{o} and let $B_r^c = X - B_r^\circ$. For a \mathfrak{q} -ray α and $r > 0$, we let $t_r \geq 0$ denote the first time α intersects B_r^c and $T_r \geq t_r$ be the last time α intersects B_r . We denote $\alpha(t_r)$ by $\alpha_r \in X$. Also, let

$$\alpha|_r = \alpha[0, t_r] \quad \text{and} \quad \alpha|_{\geq r} = \alpha[T_r, \infty)$$

be the restrictions α to the intervals $[0, t_r]$ and $[T_r, \infty)$ respectively. That is, $\alpha|_r$ is the subsegment of α connecting \mathfrak{o} to α_r and $\alpha|_{\geq r}$ is the portion of α that starts at radius r but never returns to B_r . Lastly, if p is a point on a \mathfrak{q} -ray α , we also use $\alpha|_{[p, \infty)}$ to denote the tail of α starting from the point p .

We use some short hands for pairs of constants. For example, for a given pairs \mathfrak{q} and \mathfrak{q}' we write $\mathfrak{q} \leq \mathfrak{q}'$ if

$$\mathfrak{q} = (q, Q), \quad \mathfrak{q}' = (q', Q'), \quad q \leq q' \quad \text{and} \quad Q \leq Q'.$$

Also, $\mathfrak{q} = \max(\mathfrak{q}_1, \mathfrak{q}_2)$ means

$$\mathfrak{q}_1 = (q_1, Q_1), \quad \mathfrak{q}_2 = (q_2, Q_2), \quad q = \max(q_1, q_2) \quad \text{and} \quad Q = \max(Q_1, Q_2).$$

We also use $d(\cdot, \cdot)$ instead of $d_X(\cdot, \cdot)$ when the metric space X is fixed. For $x \in X$, $\|x\|$ denotes $d(\mathfrak{o}, x)$. Let $A \subset X$ be a set and $D > 0$, then

$$N_D(A) := \{x \in X \mid \exists a \in A \text{ where } d(x, a) \leq D\}$$

Preliminary assumption. General metric spaces could be very wild and difficult to work with. Hence, we make some assumptions about the space X which we show they holds for our main examples (see Section 8 and Section 11). Some of the statements in this paper can be stated in a more general setting, but these assumptions simplify the exposition and exclude certain exotic examples.

Assumption 0. (No dead ends) The metric space X is always assumed to be a proper, geodesic metric space. Furthermore, there exist a pair of constants \mathfrak{q}_0 such that every point $x \in X$ lies on an infinite \mathfrak{q}_0 -ray.

Recall that every proper quasi-geodesic metric space is quasi-isometric to a proper geodesic metric space (see for example [Löh18, Proposition 5.3.9]) via a process similar to Lemma 2.3. So the first condition in the Assumption 0 is not a strong assumption. The second condition in Assumption 0 holds for the spaces we are most concerned about, namely, any space quasi-isometric to a finitely generated group. Note that, many groups such that lamplighter groups, have dead ends in the sense that not every point lies on an infinite geodesic ray. In fact it is shown that there exists wreath products with unbounded dead-end length with respect to certain standard generating sets[CT05]. However we prove now that all spaces quasi-isometric to a finitely generated group satisfies Assumption 0. This result is proven independently in [GHW16, Theorem 3.3].

Lemma 2.5. *Let X be a proper geodesic metric space with a cocompact action by a finitely generated group G . Then X satisfies Assumption 0.*

Proof. If $\Phi: X \rightarrow Y$ is a quasi-isometry between two proper geodesic metric spaces, then Assumption 0 holds for X if and only if it holds for Y . This is because quasi-geodesics in X are mapped to quasi-isometric embeddings of intervals to Y which can be tamed using Lemma 2.3. Hence, it is enough to prove the lemma for the case when X is the Cayley graph of G with respect to a finite generating set.

We first argue that X contains a bi-infinite geodesic ray γ . Pick a sequence of point x_n such that $\|x_n\| \rightarrow \infty$, let γ_n be a geodesic in X connecting \mathfrak{o} to x_n and let y_n be a point on γ_n such that both $\|y_n\| \rightarrow \infty$ and $d(y_n, x_n) \rightarrow \infty$. Note that we can think of y_n as an element of G . Since X is proper, the sequence of geodesic segments $y_n^{-1}(\gamma_n)$ converges, up to taking a subsequence, to a bi-infinite geodesic γ passing through \mathfrak{o} .

Now let x be a point in X and let $\gamma_x = x \cdot \gamma$ be a bi-infinite geodesic passing through x . Let z be a point on γ_x that is closest to \mathfrak{o} . Then z divides γ_x into two half-infinite geodesics γ_x^+ and γ_x^- starting at z . Let γ_x^+ be the half-infinite geodesic that contains x . By Part (II) of Lemma 2.6, the concatenations $[\mathfrak{o}, z] \cup \gamma_x^+$ is a $(3, 0)$ -quasi-geodesic ray emanating from \mathfrak{o} passing through x . \square

2.1. Surgeries between quasi-geodesics. In this section we present several methods to produce a quasi-geodesic as a concatenation of other geodesics or quasi-geodesics. The statements are intuitively clear and the proofs are elementary. So, this subsection could be skipped on the first reading of the paper. First, we recall a few surgery lemmas from [QRT22] and [QRT23].

Lemma 2.6. *Let X be a metric space satisfying Assumption 0. The following statements are [QRT22, Lemma 2.5, Lemma 4.3] and [QRT23, Lemma 3.7] respectively.*

(I) *(Nearest-point projection) Consider a point $x \in X$ and a (q, Q) -quasi-geodesic segment β connecting a point $z \in X$ to a point $w \in X$. Let y be a closest point in β to x . Then*

$$\gamma = [x, y] \cup [y, z]_\beta$$

is a $(3q, Q)$ -quasi-geodesic.

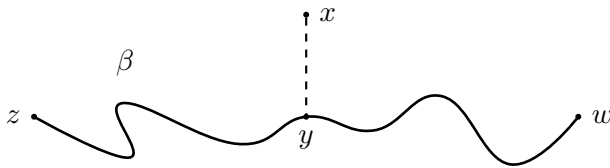


FIGURE 1. The concatenation of the geodesic segment $[x, y]$ and the quasi-geodesic segment $[y, z]_\beta$ is a quasi-geodesic.

(II) *(Quasi-geodesic ray to geodesic ray) Let β be a geodesic ray and γ be a (q, Q) -ray. For $r > 0$, assume that $d_X(\beta_r, \gamma) \leq r/2$. Then, there exists a $(9q, Q)$ -quasi-geodesic γ' where $\gamma'(t) = \beta(t)$ for large values of t and*

$$\gamma|_{r/2} = \gamma'|_{r/2}.$$

Lemma 2.7. (*Fellow travelling*) *Let X be a metric space satisfying Assumption 0. Let q -rays α, β , $t_0 > 0$ and $C > 0$ be such that, for all $t \leq t_0$, we have*

$$d(\alpha(t), \beta(t)) \leq C.$$

Then there exists a $(q, Q + C)$ -quasi-geodesic ray β' such that

$$\beta'|_{t_0} = \beta|_{t_0} \quad \text{and} \quad \beta'|_{(t_0+1, \infty)} = \alpha|_{(t_0, \infty)}.$$

Proof. Let $\gamma: [0, C] \rightarrow X$ be a geodesic segment connecting $\beta(t_0)$ and $\alpha(t_0)$. Define β' as:

$$\beta'(t) = \begin{cases} \beta(t) & \text{for } t \in [0, t_0] \\ \gamma(t - t_0) & \text{for } t \in [t_0, t_0 + C] \\ \alpha(t - C) & \text{for } t \geq t_0 + C \end{cases}.$$

We claim that β' is a $(q, Q + C)$ -quasi-geodesic ray. Given two points $\beta'(t_1)$ and $\beta'(t_2)$. First we consider the case when $t_1 < t_0$ and $t_2 \geq t_0 + C$. By assumption $\beta'(t_1) = \beta(t_1)$ and

$$d(\beta(t_1), \alpha(t_1)) \leq C.$$

Thus

$$d(\beta'(t_1), \beta'(t_2)) \leq d(\alpha(t_1), \alpha(t_2 - C)) + C \leq q(t_2 - C - t_1) + Q + C \leq q(t_2 - t_1) + Q.$$

On the other hand we have

$$\begin{aligned} d(\beta'(t_1), \beta'(t_2)) &\geq d(\alpha(t_1), \alpha(t_2 - C)) - C \\ &\geq \frac{1}{q}(t_2 - C - t_1) - Q - C. \end{aligned}$$

Therefore, β' is a $(q, Q + C)$ -quasi-geodesic redirecting β to α .

Another case to consider is when $t_1 \leq t_0$ and $t_0 \leq t_2 \leq t_0 + C$. In this case

$$d(\beta(t_0), \beta(t_2)) < C.$$

Thus we have

$$\begin{aligned} d(\beta(t_1), \beta(t_2)) &\leq d(\beta(t_1), \beta(t_0)) + C \\ &\leq q|t_0 - t_1| + Q + C \\ &\leq q|t_2 - C - t_1| + Q + C \leq q|t_2 - t_1| + Q. \end{aligned}$$

On the other side we have

$$\begin{aligned} d(\beta(t_1), \beta(t_2)) &\geq d(\beta(t_1), \beta(t_0)) - C \\ &\geq \frac{1}{q}(t_2 - C - t_1) - Q - C. \end{aligned}$$

The case where $t_0 \leq t_1 \leq t_0 + C$ and $t_2 \geq t_0 + C$ is analogous. Other cases are trivial. \square

Lemma 2.8 (*Pass through a nearby point*). *Let X be a metric space satisfying Assumption 0. Let α be a (q, Q) -ray, $x \in X$ and let $t_0 > 0$ be such that*

$$\ell := d(x, \alpha(t_0)) \leq 1.$$

Then there exists a $(q, Q + 3)$ -quasi-geodesic ray α' such that $x = \alpha'(t_0 + 1)$ and

$$\alpha'|_{t_0} = \alpha|_{t_0} \quad \text{and} \quad \alpha'|_{(t_0+2, \infty)} = \alpha|_{(t_0, \infty)}.$$

Proof. Let $\gamma: [0, 1] \rightarrow X$ be a geodesic segment connecting $\alpha(t)$ and x parametrized with constant speed. Define α' as:

$$\alpha'(t) = \begin{cases} \alpha(t) & \text{for } t \in [0, t_0], \\ \gamma(t - t_0) & \text{for } t \in [t_0, t_0 + 1], \\ \gamma(t - t_0 - 1) & \text{for } t \in [t_0 + 1, t_0 + 2], \\ \alpha(t - 2\ell) & \text{for } t \geq t_0 + 2. \end{cases}$$

We claim that α' is a $(q, Q + 3)$ -quasi-geodesic ray. Given two points $\alpha'(t_1)$ and $\alpha'(t_2)$. First we consider the case when $t_1 < t_0$ and $t_2 \geq t_0 + \ell$. By assumption $\alpha'(t_1) = \alpha(t_1)$ and $\alpha'(t_2) = \alpha(t_2 - 2)$. Thus

$$d(\alpha'(t_1), \alpha'(t_2)) = d(\alpha(t_1), \alpha(t_2 - 2)) \leq q(t_2 - 2 - t_1) + Q \leq q(t_2 - t_1) + Q.$$

On the other hand we have

$$\begin{aligned} d(\alpha'(t_1), \alpha'(t_2)) &\geq d(\alpha(t_1), \alpha(t_2 - 2)) \\ &\geq \frac{1}{q}(t_2 - 2 - t_1) - Q \\ &= \frac{1}{q}(t_2 - t_1) - Q - \frac{2}{q} \geq \frac{1}{q}(t_2 - t_1) - (Q + 2). \end{aligned}$$

Another case to consider is when $t_1 \leq t_0$ and $t_0 \leq t_2 \leq t_0 + 2$. In this case

$$\begin{aligned} d(\alpha'(t_1), \alpha'(t_2)) &\leq d(\alpha(t_1), \alpha(t_0)) + \ell \\ &\leq q|t_0 - t_1| + Q + \ell \leq q|t_2 - t_1| + (Q + 2). \end{aligned}$$

On the other side we have

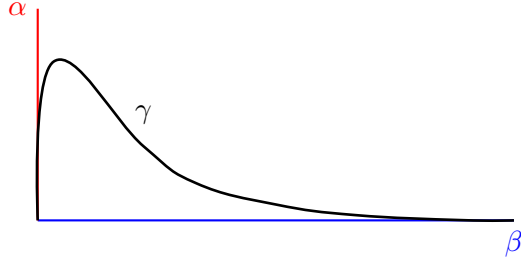
$$\begin{aligned} d(\alpha'(t_1), \alpha'(t_2)) &\geq d(\alpha(t_1), \alpha(t_0)) - \ell \\ &\geq \frac{1}{q}|t_0 - t_1| - Q - \ell \\ &\geq \frac{1}{q}(t_2 - t_1 - 2) - Q - \ell \geq \frac{1}{q}(t_2 - t_1) - (Q + 3). \end{aligned}$$

The case where $t_0 \leq t_1 \leq t_0 + 2\ell$ and $t_2 \geq t_0 + 2\ell$ is analogous. Other cases are trivial. \square

3. EQUIVALENCE CLASSES OF RAYS UP TO QUASI-REDIRECTION

As previously stated we assume throughout that Assumption 0 holds. In this section, we define a preorder \preceq on the set of quasi-geodesic rays. The set of equivalence classes associated to \preceq form a partially ordered set $P(X)$. Elements of $P(X)$ will later serve as points in our boundary ∂X (see Section 5). We will also show in Section 5 that equivalent classes associated to sublinearly Morse quasi-geodesics rays are the minimal elements with respect to this partial order.

Roughly speaking, for quasi-geodesic rays α and β , $\alpha \preceq \beta$ if α can be *quasi-redirected* to β , that is, if there is a family of quasi-geodesic rays with uniform constants that coincide with α in the beginning for an arbitrarily long time but eventually coincide with β .

FIGURE 2. The ray α can be quasi-redirected to β at radius r .

Definition 3.1. Let α, β and γ be quasi-geodesic rays. We say β *eventually coincide with* γ (and write $\gamma \stackrel{e}{=} \beta$) if there are times $t_\gamma > 0$ and t_β (which maybe negative) such that, for $t \geq t_\gamma$, we have

$$\gamma(t) = \beta(t + t_\beta).$$

For $r > 0$, we say γ *quasi-redirects* α to β at radius r if

$$\gamma|_r = \alpha|_r \quad \text{and} \quad \beta \stackrel{e}{=} \gamma.$$

If γ is a \mathbf{q} -ray, we say α *can be \mathbf{q} -redirected to β at radius r* . We refer to $(t_\beta + t_\gamma)$ as the *landing time*. We say $\alpha \preceq \beta$, if there is $\mathbf{q} \in [1, \infty) \times [0, \infty)$ such that, for every $r > 0$, α can be \mathbf{q} -redirected to β at radius r .

Lemma 3.2 (Quasi-redirection is transitive). *Let α, β, γ be quasi-geodesic rays. If α can be (q_1, Q_1) -redirected to β at every radius $r > 0$ and β can be (q_2, Q_2) -redirected to γ at every radius $r > 0$, then α can be (q_3, Q_3) -redirects to γ at every radius $r > 0$ where*

$$q_3 = \max \{q_2 + 1, q_1\}, \quad \text{and} \quad Q_3 = \max \{Q_1, Q_2\}.$$

Hence, the relation \preceq is transitive, that is, if $\alpha \preceq \beta$ and $\beta \preceq \gamma$ then $\alpha \preceq \gamma$.

Proof. Consider $r > 0$ and let t_1 be the first time such that the $\|\alpha(t_1)\| = r$. Let ζ_1 be a (q_1, Q_1) -quasi-geodesic ray quasi-redirecting α to β at radius r . Let $t_2 > 0$ and $s_1 \in \mathbb{R}$ be such that for all $t \in [t_2, \infty)$,

$$\zeta_1(t) = \beta(t + s_1).$$

Let $t_3 > 0$ be large enough such that

$$(1) \quad \left(\frac{1}{q_2} - \frac{1}{q_3} \right) t_3 \geq q_1 t_2 - Q_3 + Q_1 + Q_2 + \frac{|s_1|}{q_2},$$

and

$$(2) \quad (q_3 - q_2) t_3 \geq q_1 t_2 - Q_3 + Q_1 + Q_2 + q_2 |s_1|.$$

Let $r' := \|\beta(t_3)\|$ and let ζ_2 be a (q_2, Q_2) -quasi-geodesic ray redirecting β to γ at radius r' . Let $t_4 > 0$ and $s_2 \in \mathbb{R}$ be such that for all $t \in [t_4, \infty)$,

$$\zeta_2(t) = \gamma(t + s_2).$$

Now let ζ be a ray defined as follows:

$$\zeta: \mathbb{R}_+ \rightarrow X, \quad \zeta(t) = \begin{cases} \zeta_1(t) & \text{for } t \in [0, t_3], \\ \zeta_2(t + s_1) & \text{for } t \in [t_2, \infty). \end{cases}$$

Note that the two intervals $[0, t_3]$ and $[t_2, \infty)$ overlap. However, for $t \in [t_2, t_3]$, we have

$$\zeta(t) = \zeta_1(t) = \beta(t + s_1) = \zeta_2(t + s_1).$$

We claim that ζ is a (q_3, Q_3) -quasi-geodesic ray.

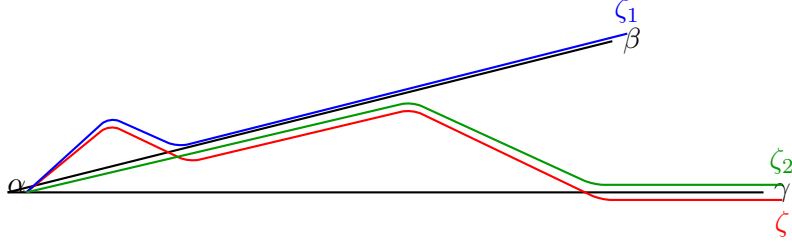


FIGURE 3. The ray ζ , which is constructed from ray ζ_1 and ζ_2 , quasi-redirects α to γ .

Let x, y be points along ζ where $x = \zeta(t_x)$ and $y = \zeta(t_y)$. There are several cases. If $t_x, t_y \leq t_3$, then $\zeta(t_x) = \zeta_1(t_x)$ and $\zeta(t_y) = \zeta_1(t_y)$, and hence

$$\frac{1}{q_1}(t_y - t_x) - Q_1 \leq d(x, y) = d(\zeta_1(t_x), \zeta_1(t_y)) \leq q_1(t_y - t_x) + Q_1.$$

But $q_1 \leq q_3$ and $Q_1 \leq Q_3$, hence the claim holds for these times.

Likewise if $t_x, t_y \geq t_2$, then $\zeta(t_x) = \zeta_2(t_x + s_1)$, $\zeta(t_y) = \zeta_2(t_y + s_1)$ and hence

$$\frac{1}{q_2}(t_y - t_x) - Q_2 \leq d(x, y) = d(\zeta_2(t_x + s_1), \zeta_2(t_y + s_1)) \leq q_2(t_y - t_x) + Q_2.$$

But $q_2 \leq q_3$ and $Q_2 \leq Q_3$, hence the claim also holds for these times.

It remains to consider the case where $t_x \in [0, t_2]$ and $t_y \in [t_3, \infty)$. We have

$$(3) \quad \frac{1}{q_1}t_x - Q_1 \leq \|x\| \leq q_1 t_x + Q_1 \quad \text{and} \quad \frac{1}{q_2}(t_y + s_1) - Q_2 \leq \|y\| \leq q_2(t_y + s_1) + Q_2.$$

Therefore,

$$\begin{aligned} \text{(triangle inequality)} \quad d(x, y) &\geq \|y\| - \|x\| \\ \text{(Equation (3))} \quad &\geq \frac{1}{q_2}(t_y + s_1) - Q_2 - q_1 t_x - Q_1 \\ \text{(Equation (1))} \quad &\geq \frac{1}{q_3}t_y - Q_3, \end{aligned}$$

and

$$\begin{aligned} \text{(triangle inequality)} \quad d(x, y) &\leq \|y\| + \|x\| \\ \text{(Equation (3))} \quad &\geq q_2(t_y + s_1) + Q_2 + q_1 t_x + Q_1 \\ \text{(Equation (2))} \quad &\geq q_3 t_y + Q_3. \end{aligned}$$

That is, ζ is a (q_3, Q_3) -quasi-geodesic ray. The argument holds for any $r > 0$. Hence α can be (q_3, Q_3) -redirected to γ at every radius $r > 0$. \square

Since we also have $\alpha \preceq \alpha$ for every quasi-geodesic ray, \preceq is a preorder on the set of quasi-geodesic rays.

Definition 3.3. Define $\alpha \simeq \beta$ if and only if $\alpha \preceq \beta$ and $\beta \preceq \alpha$. Then \simeq is an equivalence relation on the space of all quasi-geodesic rays in X . Let $P(X)$ denote the set of all equivalence classes of quasi-geodesic rays under \simeq . For a quasi-geodesic ray α , let $[\alpha] \in P(X)$ denote the equivalence class containing α . We extend \preceq to $P(X)$ by defining $[\alpha] \preceq [\beta]$ if $\alpha \preceq \beta$. Note that this does not depend on the representative chosen in the given class. The relation \preceq is a partial order on elements of $P(X)$.

We now check that the partially ordered set $P(X)$ is invariant under a quasi-isometry.

Proposition 3.4. *Let X, Y be proper geodesic metric spaces and let $\Phi: X \rightarrow Y$ be a (k, K) -quasi-isometry sending the base point $\mathfrak{o}_X \in X$ to the base point $\mathfrak{o}_Y \in Y$. Then there is a well-defined induced map*

$$\Phi^*: P(X) \rightarrow P(Y) \quad \text{where} \quad \Phi^*([\alpha]) = [\Phi \circ \alpha].$$

Furthermore, Φ^* preserves the partial order on $P(X)$ and $P(Y)$.

Proof. It suffices to argue that the relation \preceq is preserved by Φ . That is, for quasi-geodesic rays α and β in X where α can be quasi-redirected to β , we need to show that $\Phi \circ \alpha$ can be quasi-redirected to $\Phi \circ \beta$. Consider a pair of constants q and family of q -rays γ_r ($r > 0$) that respectively q -redirect α to β at radius r . Since $\alpha|_r = \gamma_r|_r$ we have

$$(\Phi \circ \alpha)|_{r'} = (\Phi \circ \gamma_r)|_{r'} \quad \text{for} \quad r' \geq \frac{r}{k} - K,$$

and since $\alpha \stackrel{c}{=} \gamma_r$ we have

$$\Phi \circ \beta \stackrel{c}{=} \Phi \circ \gamma_r.$$

Also, $r' \rightarrow \infty$ and $r \rightarrow \infty$ and, since γ_r are uniform quasi-geodesics, $\Phi \circ \gamma_r$ are uniform quasi-geodesics as well (note that we need to use Lemma 2.3 to tame $\Phi \circ \alpha$, $\Phi \circ \beta$ and $\Phi \circ \gamma_r$, but if portions of these quasi-geodesics coincide before taming, they will also coincide after taming). This finishes the proof. \square

It is desirable to have a geodesic representative in each quasi-redirecting class, however, this is not the case in general (see Example 3.7). But a weaker version holds which will be useful later.

Lemma 3.5. *Let X be a proper, geodesic, metric space and let α be a q -ray. Then there exists a geodesic ray $\alpha_0 \preceq \alpha$.*

Proof. Choose a sequence $r_i \rightarrow \infty$ and let x_i be starting point of the quasi-geodesic $\alpha|_{\geq r_i}$. Then x_i is also the closest point in $\alpha|_{\geq r_i}$ to \mathfrak{o} . Let

$$\alpha_i = [\mathfrak{o}, x_i] \cup \alpha|_{\geq r_i}.$$

By Part (I) of Lemma 2.6, α_i is a $(3q, Q)$ -quasi-geodesic ray. Up to taking a subsequence, the geodesic segments $[\mathfrak{o}, x_i]$ converge to a geodesic ray α_0 . That is, for every $r > 0$, assume i is large enough we have $[\mathfrak{o}, x_i]$ stays within distance 1 of α_0 up to radius r . By Lemma 2.7 (setting $C = 1$), α_0 can be $(3q, Q + 1)$ -redirected to α_i at radius r . But the tail of α_i is the same as the tail of α . Thus α_0 can be $(3q, Q + 1)$ -redirected to α at radius r for every $r > 0$. Thus, $\alpha_0 \preceq \alpha$ with quasi-redirecting constants $(3q, Q + 1)$. \square

Fundamental assumptions on redirecting. To continue, we need to make more assumptions about the metric space X . General metric spaces can be very wild with large holes in the middle. Later in the paper, we will show that for a large classes of groups, the Cayley graphs satisfy these assumptions. It would be interesting to know if this holds for all finitely generated group and whether these assumptions follow from a simpler, more geometric assumption on the metric space.

Assumption 1. (Quasi-geodesic representative) There is \mathfrak{q}_0 (by making it larger, we can assume it is the same at \mathfrak{q}_0 in Assumption 0) such that every equivalence class $\mathfrak{a} \in P(X)$ contains a \mathfrak{q}_0 -ray. We fix such a \mathfrak{q}_0 -ray, denote it by $\alpha_0 \in \mathfrak{a}$ and refer to it as the central element of the class \mathfrak{a} .

Assumption 2. (Uniform redirecting function) For every $\mathfrak{a} \in P(X)$, there is a function

$$f_{\mathfrak{a}}: [1, \infty) \times [0, \infty) \rightarrow [1, \infty) \times [0, \infty),$$

called the redirecting function of the class \mathfrak{a} , such that if $\mathfrak{b} \prec \mathfrak{a}$ then any \mathfrak{q} -ray $\beta \in \mathfrak{b}$ can be $f_{\mathfrak{a}}(\mathfrak{q})$ -redirected to α_0 .

Note that the function $f_{\mathfrak{a}}$ may depend on the choice of the central element. But such functions exist for every quasi-geodesic ray. That is:

Lemma 3.6. *Let X be a space where Assumptions 0, 1 and 2 hold. For every quasi-geodesic ray α , there is a function*

$$f_{\alpha}: [1, \infty) \times [0, \infty) \rightarrow [1, \infty) \times [0, \infty),$$

such that if $\beta \prec \alpha$ for a \mathfrak{q} -ray β , the β can be $f_{\alpha}(\mathfrak{q})$ -redirected to α .

Proof. Let α_0 be the central element in the class $[\alpha]$. Assume α_0 can be (q_1, Q_1) -redirected to α . By Assumption 2, β can be $f_{\mathfrak{a}}(\mathfrak{q})$ -redirected to α_0 . Now, Lemma 3.2 implies that for

$$f_{\alpha}(\mathfrak{q}) = \max(f_{\mathfrak{a}}(\mathfrak{q}), (q_1 + 1, Q_1))$$

the \mathfrak{q} -ray β can be $f_{\alpha}(\mathfrak{q})$ -redirected to α . □

We use the following example to show how Assumptions 1 may fail. To see how assumptions 2 may fail, see Example 7.2.

Example 3.7. The easiest way to generate examples is via connected, locally finite metric graphs since these are always proper geodesic metric spaces. For a simple construction of such an example, fix an integer $k > 0$. Attach two copies of \mathbb{R}_+ at a point \mathfrak{o} and denote them by α_0 and β . Then attach the point in α_0 that is distance n from \mathfrak{o} to the point in β that is distance n^2 along β with a segment of length n^2/k . Denote the resulting metric space by X_k .

Then α_0 is a geodesic in X_k but β is only a $(k, 0)$ -quasi-geodesic since the paths that go along α_0 a distance n and then switch to β give shortcuts for points in β . That is, every point in β lies on a quasi-geodesic ray but not on a geodesic rays.

Furthermore, $\alpha_0 \preceq \beta$ but $\beta \not\preceq \alpha_0$. Hence $P(X_k) = \{[\alpha_0], [\beta]\}$ and the relation is not symmetric. We also notice that $[\beta_0]$ does not contain a geodesic ray even though X_k is a geodesic metric space.

We can also use X_k to see how Assumption 1 can fail. Namely, consider \mathbb{R}^2 equipped with the Euclidean metric. For every $k > 0$ attach a copy of X_k to \mathbb{R}^2 along α_0 in a way that the resulting space is still proper. For example, attach α_0 in X_k to the line starting at $(1, k)$ with slope k . Then the geodesic β_k connecting $(0, 0)$ to $(1, k)$ and then traveling along β in X_k

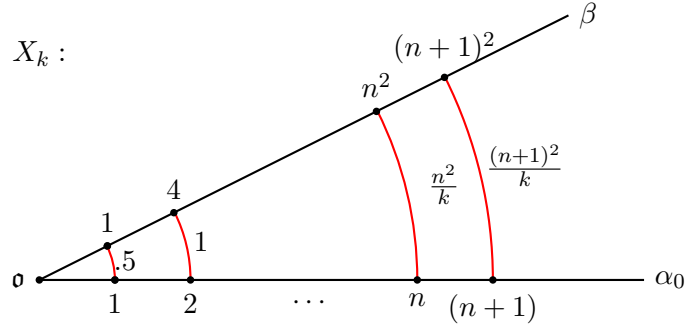


FIGURE 4. The space X_k is a proper geodesic metric space. However, $[\beta]$ does not have a geodesic representative.

is only a $(k, 0)$ -quasi-geodesic. In fact $[\beta_k]$ does not have a (q, Q) -representative for $q < k$. Hence, the Assumption 1 does not hold for any \mathfrak{q}_0 .

In Proposition 3.9, we establish some consequences of Assumption 1 and Assumption 2. First we need a lemma that follows only from Assumption 0.

Lemma 3.8 (Down sets in $P(X)$ are closed under point-wise convergence). *Let X be a metric space where Assumption 0 holds, let $\mathfrak{a} \in P(X)$ and let α_0 be the central element of \mathfrak{a} . Let $\alpha_n \in \mathfrak{a}$ be a sequence of \mathfrak{q} -rays such that $\alpha_n \rightarrow \beta$ point wise. That is, for all $t > 0$, $\alpha_n(t) \rightarrow \beta(t)$. Then β is a \mathfrak{q} -ray and β can be $(q, Q + 1)$ -redirected to α_0 .*

Proof. For $r > 0$, let t_r be the first time where $\|\alpha(t_r)\| = r$. Pick $n > 0$ large enough such that, for all $t \leq t_r$, we have

$$d(\alpha_n(t), \beta(t)) \leq 1$$

Let $\gamma: [0, 1] \rightarrow X$ be a geodesic segment connecting $\beta(t_r)$ and $\alpha_n(t_r)$. Define β_r as:

$$\beta_r(t) = \begin{cases} \beta(t) & \text{for } t \in [0, t_r], \\ \gamma(t - t_r) & \text{for } t \in [t_r, t_r + 1], \\ \alpha_n(t - 1) & \text{for } t \geq t_r + 1 \end{cases}$$

By Lemma 2.7, β_r is a $(q, Q + 1)$ -quasi-geodesic ray that redirects β to $\alpha_n \sim \alpha_0$. Thus by transitivity there exists \mathfrak{q}' that redirects β to α_0 at each radius r . Thus $\beta \preceq \alpha_0$. \square

Proposition 3.9. *Let X be a metric space where Assumption 0, 1 and 2 hold. For every $\mathfrak{a} \in P(X)$, and $\mathfrak{q}, \mathfrak{q}' \in [0, \infty) \rightarrow [1, \infty) \times [0, \infty)$ and $r > 0$, there are constants $\ell_{\mathfrak{a}}(\mathfrak{q}, r) > 0$ and $R_{\mathfrak{a}}(f_{\mathfrak{a}}(\mathfrak{q}) + (0, 1), \mathfrak{q}', r) > 0$ such that the followings hold:*

- (I) (Uniform landing function for each class) *If $\mathfrak{b} \prec \mathfrak{a}$, then every \mathfrak{q} -ray $\beta \in \mathfrak{b}$ can be $(f_{\mathfrak{a}}(\mathfrak{q}) + (0, 1))$ -redirected to α_0 with the landing time at most $\ell_{\mathfrak{a}}(\mathfrak{q}, r)$ (see Definition 3.1).*
- (II) (Redirecting at large distance implies uniform redirecting at small distant) *If β is a \mathfrak{q} -ray that \mathfrak{q}' -redirects to α_0 at radius $R \geq R_{\mathfrak{a}}(f_{\mathfrak{a}}(\mathfrak{q}) + (0, 1), \mathfrak{q}', r)$, then β can be $(f_{\mathfrak{a}}(\mathfrak{q}) + (0, 1))$ -redirected to α_0 at radius r .*
- (III) (Taming of the tail) *If $\beta \in \mathfrak{b} \prec \mathfrak{a}$ is a \mathfrak{q}' -ray where $\beta|_R$ is a \mathfrak{q} -quasi-geodesic segment for $R \geq R_{\mathfrak{a}}(f_{\mathfrak{a}}(\mathfrak{q}) + (0, 1), \mathfrak{q}', r)$ then there is a $(f_{\mathfrak{a}}(\mathfrak{q}) + (0, 1))$ -ray $\alpha \in \mathfrak{a}$ such that $\beta|_r = \alpha|_r$.*

Proof. We start with the proof of the first assertion. Assume, for contradiction, that there exists $r > 0$ and a sequence of \mathfrak{q} -rays $\alpha_n \prec \alpha_0$ and times $t_n \rightarrow \infty$ such that there does not exist a $f_a(\mathfrak{q}) + (0, 1)$ -quasi-redirection of α_n to α_0 at radius r that lands on α_0 before $\alpha_0(t_n)$. Since X is proper, up to taking a subsequence, we can assume that the sequence of rays α_n converges to some ray β . By Lemma 3.8, β is a \mathfrak{q} -ray with $\beta \prec \alpha_0$. By Assumption 2, there exists a $f_a(\mathfrak{q})$ -ray α that redirects β to α_0 at radius r ; let t_α be the landing time. Take n large enough such that α_n is within distance 1 of $\beta|_r = \alpha|_r$ and that $t_n > t_\alpha$. Let t_r be the time when $\alpha_r = \alpha(t_r)$ and apply Lemma 2.7 to α_n and α at the time t_r to construct a $(f_a(\mathfrak{q}) + (0, 1))$ -rays α'_n that quasi-redirects α_n to α_0 . Then α'_n lands on α_0 before t_n . This contradicts our assumptions and hence proves the first claim.

To see the second assertion we assume for contradiction that there exists $r > 0$, $\mathfrak{q}' \in [1, \infty) \times [0, \infty)$, a sequence of radii $R_n \rightarrow \infty$ and a sequence of \mathfrak{q} -rays β_n such that

- (S1) β_n can be \mathfrak{q}' -redirected to α_0 at a radius R_n , but
- (S2) β_n cannot be $(f_a(\mathfrak{q}) + (0, 1))$ -redirected to α_0 at radius r .

After taking a subsequence, we can assume that there exists a γ where $\beta_n \rightarrow \gamma$ point-wise. By Lemma 3.8, γ is a \mathfrak{q} -ray and it can be $(\mathfrak{q}', Q' + 1)$ -redirected to α_0 . Therefore, γ can be in fact be $f_a(\mathfrak{q})$ redirected to α_0 by Assumption 2. Let n_r be large enough so that β_{n_r} is within distance 1 of γ up to radius r . Applying Lemma 2.7 to β_{n_r} and γ at radius r , we produce $(f_a(\mathfrak{q}) + (0, 1))$ -rays γ_r redirecting β_{n_r} to α_0 at radius r . This contradicts (S2) and thus proves the second assertion.

The proof of the third assertion is nearly identical to above. We assume, for contradiction, that there exists $r > 0$, $\mathfrak{q}' \in [1, \infty) \times [0, \infty)$, a sequence of radii $R_n \rightarrow \infty$ and a sequence of \mathfrak{q}' -rays β_n such that

- (S3) $\beta_n|_{R_n}$ is a \mathfrak{q} -quasi-geodesic segment, but
- (S4) β_n cannot be $f_a(\mathfrak{q}) + (0, 1)$ -redirected to α_0 after radius r .

After taking a subsequence, we can assume that there exists a γ where $\beta_n \rightarrow \gamma$ point-wise. As before, γ is a \mathfrak{q} -ray that can be $(\mathfrak{q}', Q' + 1)$ -redirected to α_0 . Therefore, γ can in fact be $f_a(\mathfrak{q})$ redirected to α_0 by Assumption 2. We can argue identical to above to get a contradiction to (S4). This proves the third assertion. \square

4. MONO-DIRECTIONAL SPACES

The boundary we are defining is meant to generalize the Gromov boundary of a hyperbolic space and it captures the hyperbolicity in a metric space. Hence, when the space X has no hyperbolic directions, the space of direction $P(X)$ has only one point. In this section we concentrate on spaces without hyperbolic directions in this sense.

Definition 4.1. Let X be a metric space satisfying the Assumption 0, 1 and 2. We say X is *mono-directional* if $P(X)$ has only one point. That is, for every pair of quasi-geodesic rays α and β , we have $\alpha \prec \beta$.

The first classes of mono-directional spaces we consider are products of unbounded metric spaces.

Proposition 4.2. *Let $X = A \times B$, where A and B are proper geodesic metric spaces satisfying the Assumption 0, equipped with the L^∞ -metric. Then $P(X)$ is a point.*

Note that since $P(X)$ is invariant under quasi-isometries, the proposition also holds if we equip X with the L^p -metric, $p \geq 1$.

Proof. Consider a pair of \mathfrak{q} -rays ζ and ξ . For every $r > 0$ let $R = 4r$. Let $(a_1, b_1) = \zeta(t_r)$ be coordinates of the first time ζ hits the sphere of radius r in X and let $(a_2, b_2) = \xi(t_R)$ be the last time ξ hit the sphere of radius R in X . Either $\|a_1\| = r$ or $\|b_1\| = r$. We assume without loss of generality that $\|a_1\| = r$. Similarly, either $\|a_2\| = R$ or $\|b_2\| = R$. We assume $\|a_2\| = R$ which is the more complicated case.

Consider a \mathfrak{q}_0 -geodesic ray α_1 in A passing through a_1 and let a'_1 be a points along α_2 with $\|a'_1\| = 2r$. Similarly, consider a \mathfrak{q}_0 -geodesic ray α_2 in A passing through a_2 and let a'_2 be a points along α_2 with $\|a'_2\| = 3r$. Note that a'_2 lies between \mathfrak{o} and α_2 could just be a quasi-geodesic segment. Finally, let α be a \mathfrak{q}_0 -segment connecting a'_1 to a'_2 . Let b be any point in B with $\|b_2\| = 2r$. Let β_1 be a \mathfrak{q}_0 -geodesic segment connecting b_1 to b and β_2 be a \mathfrak{q}_0 -geodesic segment connecting b to b_2 . Let

$$\begin{aligned} \alpha_1: [t_{a_1}, t_{a_1} + s_{a_1}] &\rightarrow A && \text{be the parametrization of the segment} && [a_1, a'_1]_{\alpha_1}, \\ \alpha: [t_a, t_a + s_a] &\rightarrow A && \text{be the parametrization of the segment} && [a'_1, a'_2]_{\alpha}, \\ \alpha_2: [t_{a_2}, t_{a_2} + s_{a_2}] &\rightarrow A && \text{be the parametrization of the segment} && [a'_2, a_2]_{\alpha_2}, \\ \beta_1: [t_{b_1}, t_{b_1} + s_{b_1}] &\rightarrow B && \text{be the parametrization of the segment} && [b_1, b]_{\beta_1}, \end{aligned}$$

and

$$\beta_2: [t_{b_2}, t_{b_2} + s_{b_2}] \rightarrow B \quad \text{be the parametrization of the segment} \quad [b, b_2]_{\beta_2}.$$

Let

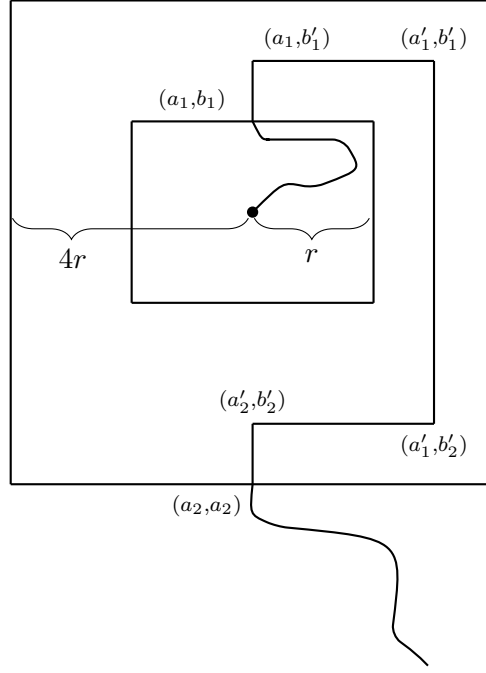
$$\begin{array}{lll} t_1 = t_r & t_2 = t_1 + s_{a_1} & t_3 = t_2 + s_{b_1} \\ t_4 = t_3 + s_a & t_5 = t_4 + s_{b_2} & t_6 = t_5 + s_{a_2} \end{array}$$

and define,

$$\gamma(t) = \begin{cases} \zeta(t) & \text{for } t \in [0, t_1], \\ (\alpha_1(t - t_1 + t_{a_1}), b_1) & \text{for } t \in [t_1, t_2], \\ (\alpha'_1, \beta_1(t - t_2 + t_{b_1})) & \text{for } t \in [t_2, t_3], \\ (\alpha(t - t_3 + t_a), b) & \text{for } t \in [t_3, t_4], \\ (a'_2, \beta_2(t - t_4 + t_{b_2})) & \text{for } t \in [t_4, t_5], \\ (\alpha_2(t - t_5 + t_{a_2}), b_2) & \text{for } t \in [t_5, t_6], \\ \xi(t) & \text{for } t \geq t_6. \end{cases}$$

Then γ is a quasi-geodesic where the constant depend only on \mathfrak{q} and \mathfrak{q}_0 . In fact, setting $t_0 = 0$ and $t_7 = \infty$, we have the restriction of γ to each interval $[t_{i-1}, t_{i+1}]$ ($i = 1, \dots, 6$) is a uniform quasi-geodesic by Part (II) of Lemma 2.6 since $\gamma(t_i)$ is a closest point from any point in $\gamma|_{[t_i, t_{i+1}]}$ to the segment $\gamma|_{[t_{i-1}, t_i]}$. And, for $|i - j| \geq 2$, any point in the segment $\gamma|_{[t_i, t_{i+1}]}$ and any point in the segment $\gamma|_{[t_j, t_{j+1}]}$ are at least r apart. Since the length of all these intervals are comparable to r and each one is a quasi-geodesic, γ is a also quasi-geodesic where the constants do not depend on r . Therefore, $\zeta \prec \xi$. The proof in the case $\|b_2\| = R$ is similar. \square

Example 4.3. The Baumslag-Solitar group. For a complete calculation of the redirecting boundary of the Baumslag-Solitar group, see [McM].



Morse and sublinearly Morse quasi-geodesic rays in a metric space X resemble geodesics in a Gromov hyperbolic space. In fact, we will later show in Proposition 6.5 that every equivalence class of κ -Morse quasi-geodesic rays, and hence every equivalence class of Morse geodesic rays (where the equivalence relations are those specified in the construction of sublinearly Morse boundary and Morse boundaries) are also equivalence classes in $P(X)$.

Question 4.4. *Assume X does not have any Morse geodesics. Does that imply that $P(X)$ is a single point?*

5. THE TOPOLOGY AND THE BOUNDARY

In this section, we build a topology on the set $X \cup P(X)$. We denote $P(X)$ equipped with the restriction of this topology to $P(X)$ by ∂X . This boundary is strongly analogous to the κ -Morse boundary of X (see [QRT23]) and it should be considered as an enlargement of the κ -Morse boundary. In fact, we will show in the Section 6 that ∂X contains every κ -Morse boundary as a topological subspace.

We define the topology on $X \cup P(X)$ by defining a system of neighbourhoods. Recall that points in $P(X)$ are equivalence classes of quasi-geodesic rays. To unify the treatment of point in X and $P(X)$, for every $x \in X$, we consider the set of quasi-geodesic rays that pass through x . Abusing the notation, we denote this set by \varkappa , that is

$$\varkappa = \left\{ \text{quasi-geodesics rays passing through } x \right\}.$$

We use the gothic letters $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ to denote elements of $P(X) \cup X$, that is, either a set of quasi-geodesic rays passing through a point $x \in X$ or an equivalence class of quasi-geodesic rays in $P(X)$. For $\mathfrak{a} \in P(X)$, define $F_{\mathfrak{a}}: [1, \infty) \times [0, \infty) \rightarrow [1, \infty) \times [0, \infty)$ by

$$(4) \quad F_{\mathfrak{a}}(\mathfrak{q}) = \max\{f_{\mathfrak{a}}(\mathfrak{q}) + (1, 1), (1, \mathfrak{q} + Q)\} \quad \text{for} \quad \mathfrak{q} \in [1, \infty) \times [0, \infty).$$

Definition 5.1. For $\mathbf{a} \in P(X)$ (with $\alpha_0 \in \mathbf{a}$ as a central element) and $r > 0$, define

$$\mathcal{U}(\mathbf{a}, r) := \left\{ \mathbf{b} \in P(X) \cup X \mid \text{every } \mathbf{q}\text{-ray in } \mathbf{b} \text{ can be } F_{\mathbf{a}}(\mathbf{q})\text{-redirected to } \alpha_0 \text{ at radius } r \right\}.$$

Remark 5.2. The sets $\mathcal{U}(\mathbf{a}, r)$ will define open neighborhoods around \mathbf{a} in our topology. This is meant to be a direct analogue of the cone topology for the Gromov boundary of hyperbolic spaces. Roughly speaking, we think a quasi-geodesic ray β is in a small neighborhood of a quasi-geodesic ray α if β can be redirected to α at a large radius. However, one might think it is more natural to require that α should be redirected to β at a large radius. This would suggest that there may exist an Out-topology which is different from our topology but equally valid. In Section 10.1, we will argue that this is not true and the Out-topology is the wrong definition. The main problem is that Lemma 5.7 below will not hold for the Out-topology.

In most arguments about a class of quasi-geodesic rays \mathbf{a} , it is enough to consider \mathbf{q} -rays where \mathbf{q} is not too big. We now make this precise.

Lemma 5.3. *For every $r > 0$ there is a pair of constants $\mathbf{q}_{\max}(r) \in [1, \infty) \times [0, \infty)$ such that if $\mathbf{q} \not\leq \mathbf{q}_{\max}$ then, for every $\mathbf{a} \in P(X)$, any \mathbf{q} -ray β can be $F_{\mathbf{a}}(\mathbf{q})$ -redirected to α_0 at radius r .*

Proof. For $\mathbf{q} = (q, Q)$ let $(q_1, Q_1) = F_{\mathbf{a}}(q, Q)$. From the definition of $F_{\mathbf{a}}$ we know that $Q_1 \geq \max(q, Q)$ and $q_1 \geq 2$. That is, if $q > 2r$ or $Q \geq 2r$, then $Q_1 \geq 2r$. Now, for a \mathbf{q} -ray β , and $\mathbf{a} \in P(X)$, let γ be the concatenation $\beta|_r$ followed by $\beta|_r$ traverses in reverse and then α_0 . Then γ is a $(2, 2r)$ -ray, $q_1 \geq 2$ and $Q_1 \geq 2r$. This finishes the proof. \square

We verify some basic properties of $\mathcal{U}(\mathbf{a}, r)$. Below $B(x, 1)$ is the ball of radius 1 around x .

Lemma 5.4. *Assume X satisfies Assumptions 0, 1 and 2. Then:*

- (I) *For $\mathbf{a}, \mathbf{b} \in P(X)$, if $\mathbf{b} \preceq \mathbf{a}$ then $\mathbf{b} \in \mathcal{U}(\mathbf{a}, r)$ for all $r > 0$. In particular, $\mathbf{a} \in \mathcal{U}(\mathbf{a}, r)$ for all $r > 0$.*
- (II) *For every $r_2 \geq r_1 > 0$, we have $\mathcal{U}(\mathbf{a}, r_2) \subseteq \mathcal{U}(\mathbf{a}, r_1)$.*
- (III) *For every $r > 0$, there is $r_{\mathbf{a}} > 0$ depending on \mathbf{a} and r such that*
 - (a) *For every $\mathbf{b} \in \mathcal{U}(\mathbf{a}, r_{\mathbf{a}})$, there is $r_{\mathbf{b}} > 0$ such that*

$$\mathcal{U}(\mathbf{b}, r_{\mathbf{b}}) \subset \mathcal{U}(\mathbf{a}, r).$$

- (b) *For every $\mathbf{x} \in \mathcal{U}(\mathbf{a}, r_{\mathbf{a}}) \cap X$,*

$$B(\mathbf{x}, 1) \subset \mathcal{U}(\mathbf{a}, r).$$

Proof. Parts (I) and (II) follow immediately from the definition of the neighborhood, Assumption 2 and the fact that $F_{\mathbf{a}}(\cdot) \geq f_{\mathbf{a}}(\cdot)$. We check (a) of (III).

Fix $r > 0$ and let $\mathbf{q}_{\max}(r)$ be as in Lemma 5.3. Assume for contradiction that there exist $\mathbf{q} \leq \mathbf{q}_{\max}(r)$, a sequence $r_{\mathbf{a}}^n \rightarrow \infty$ and $\mathbf{b}^n \in \mathcal{U}(\mathbf{a}, r_{\mathbf{a}}^n)$ such that for $r_{\mathbf{b}}^n = 2r_{\mathbf{a}}^n$, there exist $\mathbf{c}^n \in \mathcal{U}(\mathbf{b}^n, r_{\mathbf{b}}^n)$ and \mathbf{q} -rays $\gamma^n \in \mathbf{c}^n$ such that γ^n cannot be $F_{\mathbf{a}}(\mathbf{q})$ -redirected to α_0 . After taking a subsequence, we can assume the central elements $\beta_0^n \in \mathbf{b}^n$ and $\gamma^n \in \mathbf{c}^n$ point-wise converge. That is, for $t > 0$,

$$\lim_{n \rightarrow \infty} \beta_0^n(t) = \beta_0(t) \quad \text{and} \quad \lim_{n \rightarrow \infty} \gamma^n(t) = \gamma(t),$$

where β_0 is a \mathbf{q}_0 -ray and γ is a \mathbf{q} -ray.

By applying Lemma 2.7 we see that β_0 can be redirected to α_0 for all $R > 0$ hence $[\beta_0] \preceq \mathbf{a}$. Also γ can be redirected to β_0 for all $R > 0$ hence $\gamma \prec \beta_0$. Therefore, $[\gamma] \prec \mathbf{a}$ and,

by Assumption 2, γ can be $f_{\mathbf{a}}(\mathbf{q})$ redirected to α_0 for all $R > 0$. Applying Lemma 2.7 again, we see that γ^n can be $f_{\mathbf{a}}(\mathbf{q}) + (0, 1)$ redirected to α_0 at some radius that goes to infinity. But $f_{\mathbf{a}}(\mathbf{q}) + (0, 1) \leq F_{\mathbf{a}}(\mathbf{q})$ which contradicts the choice of γ^n . This proves part (a).

To see part (b), define

$$r_{\mathbf{a}} = \max_{\mathbf{q} \leq \mathbf{q}_{\max}(r)} R_{\mathbf{a}}(\mathbf{q}, F_{\mathbf{a}}(((q, Q + 3)), r) + 1.$$

Let z be a point in $B(\mathbf{z}, 1)$ and let ζ be any \mathbf{q} -ray in z (i.e. ζ passes through the point $z \in X$). By Lemma 2.8, there is a $(q, Q + 3)$ -ray ζ' that passes through x and $\zeta'|_{r_{\mathbf{a}}-1} = \zeta|_{r_{\mathbf{a}}-1}$. Since $x \in \mathcal{U}(\mathbf{a}, r_{\mathbf{a}})$, ζ' can be redirected to α_0 by a $F_{\mathbf{a}}(((q, Q + 3)))$ -ray. By the choice of $r_{\mathbf{a}}$ and Proposition 3.9, ζ can be $F_{\mathbf{a}}(\mathbf{q})$ -redirected to α_0 and thus $z \in \mathcal{U}(\mathbf{a}, r)$. \square

A system of neighbourhoods. For each $\mathbf{a} \in P(X)$, define

$$\mathcal{B}(\mathbf{a}) = \left\{ \mathcal{V} \subset X \cup P(X) \mid \mathcal{U}(\mathbf{a}, r) \subset \mathcal{V} \text{ for some } r > 0 \right\}$$

and for every $x \in X$, define

$$\mathcal{B}(x) = \left\{ \mathcal{V} \subset X \cup P(X) \mid B(x, r) \subset \mathcal{V} \text{ for some } r > 0 \right\}.$$

We claim that these sets form a fundamental system of neighbourhoods that can be used to define a topology on the set $X \cup P(X)$. We need to check they satisfy some basic properties.

Proposition 5.5. *For every $\mathbf{a} \in X \cup P(X)$, the set $\mathcal{B}(\mathbf{a})$ satisfies the following properties defining a pretopology on $X \cup \partial P(X)$:*

- (I) *Every subset of $X \cup P(X)$ which contains a set belonging to $\mathcal{B}(\mathbf{a})$ itself belongs to $\mathcal{B}(\mathbf{a})$.*
- (II) *Every finite intersection of sets of $\mathcal{B}(\mathbf{a})$ belongs to $\mathcal{B}(\mathbf{a})$.*
- (III) *The element \mathbf{a} is in every set of $\mathcal{B}(\mathbf{a})$.*
- (IV) *If $\mathcal{V} \in \mathcal{B}(\mathbf{a})$ then there is $\mathcal{W} \in \mathcal{B}(\mathbf{a})$ such that, for every $\mathbf{b} \in \mathcal{W}$, we have $\mathcal{V} \in \mathcal{B}(\mathbf{b})$.*

Proof. Property (I) is the definition of $\mathcal{B}(\mathbf{b})$. To see (II), consider sets $\mathcal{V}_1, \dots, \mathcal{V}_k \in \mathcal{B}(\mathbf{a})$. First assume $\mathbf{a} = \mathbf{a} \in P(X)$ and let r_i be such that $\mathcal{U}(\mathbf{a}, r_i) \subset \mathcal{V}_i$ and let $r = \max r_i$. Note that $\mathcal{U}(\mathbf{a}, r) \subseteq \mathcal{U}(\mathbf{b}, r_i)$ by part (II) of Lemma 5.4. Therefore,

$$\mathcal{U}(\mathbf{b}, r) \subset \bigcap_i \mathcal{V}_i$$

and hence the intersection is in $\mathcal{B}(\mathbf{b})$. If $\mathbf{a} = x \in X$, let r_i be such that $B(x, r_i) \subset \mathcal{V}_i$ and let $r = \min r_i$. Then

$$B(x, r) \subset \bigcap_i \mathcal{V}_i$$

and again the intersection is in $\mathcal{B}(\mathbf{b})$.

Property (III) is trivial when $\mathbf{a} \in X$ and it is part (I) of Lemma 5.4 when $\mathbf{a} \in P(X)$.

We now check that Property (IV). If $\mathbf{a} = \mathbf{z}$, we have $B(\mathbf{z}, r) \subset \mathcal{V}$ and we can take $\mathcal{W} = B(\mathbf{z}, r/2)$. Then for every $y \in \mathcal{W}$, we have $B(y, r/2) \subset \mathcal{V}$ and $\mathcal{V} \in \mathcal{B}(y)$.

Now assume $\mathbf{a} = \mathbf{a} \in P(X)$ and take $\mathcal{V} \in \mathcal{B}(\mathbf{a})$. Let $r > 0$ be such that $\mathcal{U}(\mathbf{a}, r) \subset \mathcal{V}$, let $r_{\mathbf{a}}$ be as in part (III) of Lemma 5.4 and let $\mathcal{W} = \mathcal{U}(\mathbf{a}, r_{\mathbf{a}})$. For every $\mathbf{b} \in \mathcal{W}$, part (III) of Lemma 5.4 implies that

$$\mathcal{U}(\mathbf{b}, r_{\mathbf{b}}) \subset \mathcal{U}(\mathbf{a}, r) \subset \mathcal{V}.$$

Thus $\mathcal{V} \subset \mathcal{B}(\mathbf{b})$ and we are done. \square

Our system of neighborhoods in fact defines a topology on $X \cup \partial X$:

Proposition 5.6 ([Bou98] Proposition 2). *If to each elements $\mathfrak{b} \in \partial X$ there corresponds a set $\mathcal{B}(\mathfrak{b})$ of subsets of ∂X such that properties (I) to (IV) above are satisfied, then there is a unique topological structure on ∂X such that for each $\mathfrak{b} \in \partial X$, $\mathcal{B}(\mathfrak{b})$ is the set of neighborhoods of \mathfrak{b} in this topology.*

We can argue, similar to above, to see that $\mathcal{U}(\mathfrak{a}, r) \cap \partial X$ defines a system of neighborhoods for ∂X and the resulting topology is the subspace topology induces by the topology defined on $X \cup \partial X$. That is, X , ∂X and $X \cup \partial X$ are all topological spaces.

5.1. Invariance under quasi-isometries. Definition 5.1 is written in such a way that neighborhoods $\mathcal{U}(\mathfrak{b}, r)$ are mapped to neighbourhoods under quasi-isometries, except, the functions $F_{\mathfrak{a}}(\cdot)$ may have to be replaced with larger functions. We now check that this does not impact the definition of the topology.

Lemma 5.7. *Let $F'_a: [1, \infty) \times [0, \infty) \rightarrow \mathbb{R}^2$ be a family of functions such that, for every $\mathfrak{a} \in P(X)$, $F'_a(\mathfrak{q}) \geq F_a(\mathfrak{q})$ for all \mathfrak{q} . Define*

$$\mathcal{U}(\mathfrak{a}, r, F'_a) := \left\{ \mathfrak{b} \in P(X) \cup X \mid \text{every } \mathfrak{q}\text{-ray in } \mathfrak{b} \text{ can be } F'_a(\mathfrak{q})\text{-redirected to } \alpha_0 \text{ at radius } r \right\}$$

Then, for every $r > 0$, there exists $R > 0$ such that

$$\mathcal{U}(\mathfrak{a}, R, F'_a) \subset \mathcal{U}(\mathfrak{a}, r).$$

Proof. For a given r let $q_{\max}(r)$ be as in Lemma 5.3. Let

$$R = \max_{\mathfrak{q} \leq q_{\max}} R_a(\mathfrak{q}, F'_a(\mathfrak{q}), r).$$

Let \mathfrak{b} be a point in $\mathcal{U}(\mathfrak{a}, R, F'_a)$ and let $\beta \in \mathfrak{a}$ be a \mathfrak{q} -ray with $\mathfrak{q} \leq q_{\max}$. By definition of $\mathcal{U}(\mathfrak{a}, R, F'_a)$, there is a $F'_a(\mathfrak{q})$ -ray $\alpha \in \mathfrak{a}$ such that $\beta|_R = \alpha|_R$. Now part (III) of Proposition 3.9 implies that there is $F_a(\mathfrak{q})$ -ray $\alpha' \in \mathfrak{a}$ such that $\alpha'|_r = \alpha|_r = \beta|_r$. Therefore, $\mathfrak{b} \in \mathcal{U}(\mathfrak{a}, r)$. \square

Remark 5.8. Note that, since $F'_a \geq F_a$, we have $\mathcal{U}(\mathfrak{a}, r) \subseteq \mathcal{U}(\mathfrak{a}, r, F'_a)$. Hence the above Lemma implies that the topology defined by the neighborhoods $\mathcal{U}(\mathfrak{b}, r, F')$ is the same as the topology defined by $\mathcal{U}(\mathfrak{b}, r)$. That is, the definition of topology on ∂X is robust and does not depend on the family of function F_a .

Theorem 5.9. *Let X be a metric space satisfying Assumptions 0, 1 and 2 and let*

$$\Psi: X \rightarrow Y$$

be a quasi-isometry. Then Y also satisfies Assumptions 0, 1 and 2 and hence the quasi-redirecting boundary exists for both X and Y . Furthermore, the induced map from

$$\Psi^*: \partial X \rightarrow \partial Y$$

is a homeomorphism.

Proof. Assumptions 0, 1 and 2 are written in a way that, the fact that they hold for X immediately implies they also hold for Y . We have shown in Proposition 3.4 that the map Ψ^* is a bijection. We only need to show that it is continuous. Let \mathfrak{a} be a point in ∂X and let $F_{\Psi(\mathfrak{a})}$ be the redirecting function defining neighborhoods for the class $\Psi(\mathfrak{a})$ in Y . For $r > 0$, consider a neighborhood $\mathcal{U}(\Psi(\mathfrak{a}), r) \cap \partial Y$ around $\Psi(\mathfrak{a})$ in ∂Y . It is immediate from the

definition of neighborhoods that, for some $r' > 0$ and some function $F': [1, \infty) \times [0, \infty) \rightarrow \mathbb{R}$, we have

$$\mathcal{U}(\mathbf{a}, r', F') \subset \Phi^{-1}(\mathcal{U}(\Psi(\mathbf{a}), r) \cap \partial Y).$$

Now define $F'' = \max(F_{\mathbf{a}}, F')$. Now, by Lemma 5.7, there is $R > 0$ such that

$$\mathcal{U}(\mathbf{a}, R, F'') \subset \mathcal{U}(\Psi(\mathbf{a}), r', F').$$

Also, since $F_{\mathbf{a}} \leq F''$, $\mathcal{U}(\mathbf{a}, R) \subset \mathcal{U}(\mathbf{a}, R, F'')$. Hence

$$\mathcal{U}(\mathbf{a}, R) \cap \partial X \subset \mathcal{U}(\Psi(\mathbf{a}), r', F') \cap \partial X \subset \Phi^{-1}(\mathcal{U}(\Psi(\mathbf{a}), r) \cap \partial Y) \cap \partial X.$$

This proves continuity of Ψ^* at \mathbf{a} . The proof of continuity of $(\Psi^*)^{-1}$ is similar. \square

6. THE SUBLINEARLY-MORSE BOUNDARY AND THE QUASI-REDIRECTING BOUNDARY

In this section we prove the sublinearly Morse boundary, defined in [QRT22] and [QRT23], is naturally a topological subspace of ∂X . Points in the sublinearly Morse boundary are defined via sublinear fellow traveling of quasi-geodesic rays. We first show that each sublinearly Morse equivalence class is also a redirecting equivalence class. We then show that the topology on the two spaces are compatible. In addition, we show that the sublinearly Morse classes are minimal elements in the partially set $P(X)$. First we recall the construction of the sublinearly Morse boundary.

6.1. Background on κ -Morse boundaries. Let (X, d_X) be a metric space satisfying the Assumption 0. We follow the notation of [QRT23] and refer the reader to [QRT23] for more details. By a *sublinear function* we mean a concave continuous function $\kappa: [0, \infty) \rightarrow [1, \infty)$ such that

$$\lim_{t \rightarrow \infty} \frac{\kappa(t)}{t} = 0.$$

For $x \in X$, define

$$\|x\| := d_X(\mathfrak{o}, x).$$

We often need to refer to $\kappa(\|x\|)$ and, for simplicity, we will write $\kappa(x)$ instead of $\kappa(\|x\|)$. Given a quasi-geodesic ray α (here we think of α as a subset of X , that is, we abuse the notation and let α represents both the map $\alpha: [0, \infty) \rightarrow X$ and the image of this map) and a constant $m > 1$, we define:

$$\mathcal{N}_\kappa(\alpha, m) := \left\{ x \in X : d_X(x, \alpha) \leq m \cdot \kappa(x) \right\}.$$

We recall two equivalent definitions of κ -Morse sets. Their equivalence is established in [QRT23, Proposition 3.10], hence we use whichever is needed as the definition.

Definition 6.1. (κ -Morse I) We say a quasi-geodesic ray α is κ -Morse I if there exists a proper function $m_\alpha: \mathbb{R}^2 \rightarrow \mathbb{R}$ such that for any $(\mathfrak{q}, \mathfrak{Q})$ -quasi-geodesic $\gamma: [s, t] \rightarrow X$ with endpoints on α , we have

$$\gamma([s, t]) \subset \mathcal{N}_\kappa(\alpha, m_\alpha(\mathfrak{q}, \mathfrak{Q})).$$

The function m_α will be called a κ -weakly Morse gauge of α .

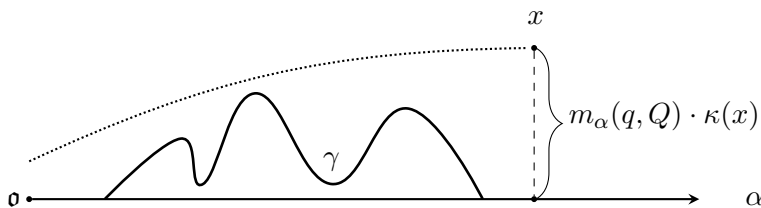


FIGURE 5. Weakly Morse: The (κ, n) -neighbourhood of the geodesic ray b .

Definition 6.2. (κ -Morse II) We say a quasi-geodesic ray α is κ -Morse II if there exists a proper function $m_\alpha : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that for any sublinear function κ' and for any $r > 0$, there exists R such that for any \mathfrak{q} -ray β with $m_Z(\mathfrak{q}) \leq \frac{r}{2\kappa(r)}$, if

$$d_X(\beta_R, \alpha) \leq \kappa'(R) \quad \text{then} \quad \beta|_r \subset \mathcal{N}_\kappa(\alpha, m_\alpha(\mathfrak{q})).$$

The function m_α will be called a κ -strongly Morse gauge of α .

Sublinear fellow traveling defines an equivalence relation between quasi-geodesic rays. We write,

$$\alpha \sim_s \beta \quad \iff \quad \lim_{r \rightarrow \infty} \frac{d(\alpha_r, \beta_r)}{r} = 0.$$

The sublinear fellow traveling equivalence class of α is denoted $[\alpha]_s$. We also recall the following basic facts about these classes from [QRT23, Lemma 3.4]:

Proposition 6.3. *Let α be a κ -Morse \mathfrak{q} -ray. Then, for any other \mathfrak{q} -ray β , if $\alpha \sim_s \beta$ then:*

- β is κ -Morse.
- there exists $m(\mathfrak{q})$ and $m'(\mathfrak{q})$ that depends only on $[\alpha]_s$ and \mathfrak{q} such that

$$\alpha \in \mathcal{N}_\kappa(\beta, m(\mathfrak{q}))$$

and

$$\beta \in \mathcal{N}_\kappa(\alpha, m'(\mathfrak{q})).$$

The boundary $\partial_\kappa X$, as a set, is the set of sublinear fellow traveling classes of κ -Morse geodesics. We now show that $\partial_\kappa X \subset P(X)$.

Lemma 6.4. *Let α be a geodesic ray that is κ -Morse and let $\beta \in [\alpha]_s$ be a \mathfrak{q} -ray, then $\beta \preceq \alpha$ and $\alpha \preceq \beta$. That is, $[\alpha]_s \subset [\alpha]$, where $[\alpha]$ is the equivalence class of α under quasi-redirecting.*

Proof. Since $\beta \in [\alpha]_s$, there is a sublinear function κ' such that, for every $R > 0$,

$$d(\beta_R, \alpha) \leq \kappa'(R).$$

Definition 6.2 implies that

$$\beta \in \mathcal{N}_\kappa(\alpha, m_\alpha(\mathfrak{q})).$$

Let r be large enough such that

$$d(\beta_r, \alpha) \leq m_\alpha(\mathfrak{q})\kappa(r) < \frac{r}{2}.$$

Surgery Lemma II implies that β can be redirected to α at radius r by a $(9q, Q)$ -quasi-geodesic ray where $\mathfrak{q} = (q, Q)$. Since this holds for all large values of r , we have that $\beta \preceq \alpha$.

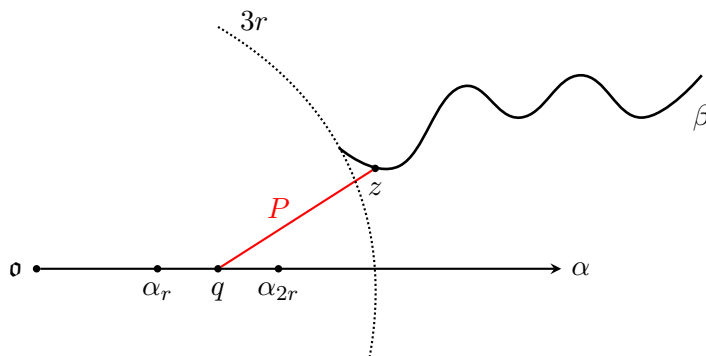


FIGURE 6. The path $[o, q] \cup P \cup \beta_{[z, \infty)}$ is a quasi-geodesic ray that redirects α to β .

We now show $\alpha \preceq \beta$. Let $y \in X$ be the last point on β that is in the ball of radius $3r$. We denote the portion of β from y onward by $\beta_{[y, \infty)}$.

Let P be the geodesic segment that realizes the set distance between $\alpha|_{2r}$ and $\beta_{[y, \infty)}$. Such a segment exists since X is proper. Suppose P has an end point q on $\alpha|_{2r}$ and an end point z on $\beta_{[y, \infty)}$. We denote the portion of β from z onward by $\beta_{[z, \infty)}$. By Surgery Lemma 2.6 I, $P \cup \beta_{[z, \infty)}$ is a $(3q, Q)$ -quasi-geodesic. Also, the closets point projection of any point in $[o, q]$ to $P \cup \beta_{[z, \infty)}$ is q , otherwise, P would not be the shortest path from $\beta_{[y, \infty)}$ to $\alpha|_{2r}$. Thus, again by Surgery Lemma 2.6 I, the concatenation

$$\ell := [o, q] \cup P \cup \beta_{[z, \infty)}$$

is a $(9q, Q)$ -ray.

It remains to prove that, for r large enough, we have $\|q\| \geq r$. First notice that

$$\|z\| \leq d(z, q) + d(q, o) \leq d(y, q) + 2r \leq d(y, o) + d(o, q) + 2r \leq 7r.$$

Choose r large enough such that

$$d(z, \alpha) \leq m_\alpha(q)\kappa(7r) < \frac{r}{2}.$$

Let z_α be a closest point in α from z . Then $\|z_\alpha\| \geq 3r - r/2 > 2r$, that is, α_{2r} lies in between q and z_α . We have

$$d(q, z_\alpha) - d(z_\alpha, z) \leq d(q, z) \leq d(\alpha_{2r}, z) \leq d(\alpha_{2r}, z_\alpha) + d(z_\alpha, z).$$

Therefore,

$$d(q, z_\alpha) - d(\alpha_{2r}, z_\alpha) \leq 2d(z_\alpha, z) \leq r.$$

This implies that $d(q, \alpha_{2r}) \leq r$ and hence $\|q\| \geq r$. We have shown that the geodesic α can be $(9q, Q)$ -redirected to β at radius r for all r large enough. \square

Proposition 6.5. *Let X be a proper geodesic metric space satisfying Assumption 0, let $\mathfrak{a} \in P(X)$ be a quasi-redirecting class and let $\alpha \in \mathfrak{a}$ be κ -Morse quasi-geodesic ray. Then $\mathfrak{a} = [\alpha]_s$ and, for an appropriate choice of a central element in \mathfrak{a} , we can take $f_{\mathfrak{a}}(q, Q) = (9q, Q)$. Moreover, $\mathfrak{a} = [\alpha]_s$ is minimal in the partial order of $P(X)$. That is to say, if \mathfrak{a} is κ Morse, then*

$$\mathfrak{b} \preceq \mathfrak{a} \implies \mathfrak{b} \sim \mathfrak{a}.$$

Proof. By [QRT23, Lemma 4.2], the class $[\alpha]_s$ always contains a geodesic representative α_0 which is also κ -Morse. Then $[\alpha]_s = [\alpha_0]_s$ and by Lemma 6.4, $[\alpha_0]_s \subseteq [\alpha_0]$. In particular, $\alpha \in [\alpha_0]$ and $[\alpha] = [\alpha_0]$. It remains to show that $[\alpha_0] \subseteq [\alpha_0]_s$.

Pick a \mathfrak{q} -ray $\beta \in [\alpha_0]$. Then there is \mathfrak{q}' and, for $r > 0$, there is a \mathfrak{q}' -ray, denoted γ_r , quasi-redirecting β to α_0 at radius r . Since α_0 is κ -weakly Morse (Definition 6.1), we have

$$\gamma_r \subset \mathcal{N}_\kappa(\alpha, m_\alpha(f_{\alpha_0}(\mathfrak{q}))).$$

But this holds for every r and $\beta \subset \cup_r \gamma_r$. Hence β sublinearly fellow travels α_0 and $\beta \in [\alpha_0]_s$.

Note that we just showed $\beta \prec \alpha_0$ implies $\beta \in [\alpha_0]_s = \mathfrak{a} = [\alpha]$. That is, any quasi-geodesic ray smaller than α is in \mathfrak{a} . This means \mathfrak{a} is minimal in $P(X)$. The assertion that

$$f_{\mathfrak{a}}(\mathfrak{q}) = (9q, Q)$$

follows from the proof of Lemma 6.4. □

6.2. Topology of $\partial_\kappa X$. Similar to $X \cup \partial X$, the topology in $X \cup \partial_\kappa X$ is defined using a neighborhood basis (see [QRT23] for more details). Namely, for $\mathfrak{a} \in \partial_\kappa X$, let m_{α_0} be the Morse gauge for α_0 (the central element in \mathfrak{a}). For $r > 0$, we define the set $\mathcal{U}_\kappa(\mathfrak{a}, r) \subseteq X \cup \partial_\kappa X$ as follows:

- An equivalence class $\mathfrak{b} \in \partial_\kappa X$ belongs to $\mathcal{U}(\mathfrak{a}, r)$ if, for any \mathfrak{q} -ray $\beta \in \mathfrak{b}$, where $m_{\alpha_0}(\mathfrak{q}) \leq \frac{r}{2\kappa(r)}$, we have

$$\beta|_r \subseteq \mathcal{N}_\kappa(\alpha_0, m_{\alpha_0}(\mathfrak{q})).$$

- A point $p \in X$ belongs to $\mathcal{U}_\kappa(\mathfrak{a}, r)$ if $d_X(\mathfrak{o}, p) \geq r$ and, for every \mathfrak{q} -quasi-geodesic segment β connecting \mathfrak{o} to p , where $m_{\alpha_0}(\mathfrak{q}) \leq \frac{r}{2\kappa(r)}$, we have

$$\beta|_r \subseteq \mathcal{N}_\kappa(\alpha_0, m_{\alpha_0}(\mathfrak{q})).$$

Theorem 6.6. *The bordification $X \cup \partial_\kappa X$ is a topological subspace of $X \cup \partial X$.*

Proof. The restriction of both topologies to X is the topology defined by the metric on X . We will prove the theorem with the following two claims:

Claim 6.7. *For every $\mathfrak{a} \in \partial_\kappa X$ and for all $r > 0$, there exists R such that*

$$(\mathcal{U}(\mathfrak{a}, R) \cap \partial_\kappa X) \subseteq \mathcal{U}_\kappa(\mathfrak{a}, r).$$

Proof of Claim. For $r > 0$, let

$$\kappa'(t) = \sup_{m_{\alpha_0}(\mathfrak{q}) \leq \frac{r}{2\kappa(r)}} m_{\alpha_0}(F_{\mathfrak{a}}(\mathfrak{q})) \cdot \kappa(t)$$

and let R be as in Definition 6.2 where α is α_0 . Let $\mathfrak{b} \in \mathcal{U}(\mathfrak{a}, R) \cap \partial_\kappa X$. Then every \mathfrak{q} -ray $\beta \in \mathfrak{b}$ can be $F_{\mathfrak{a}}(\mathfrak{q})$ -redirected to α_0 at radius R . That is, $\beta|_R$ is a subsegment of a $F_{\mathfrak{a}}(\mathfrak{q})$ -quasi-geodesic segment with end points on α_0 . But α_0 is κ -Morse and, by Definition 6.1,

$$\beta|_R \subset \mathcal{N}_\kappa(m_{\mathfrak{a}}(F_{\mathfrak{a}}(\mathfrak{q})), \alpha_0).$$

From definition of $\mathcal{U}_\kappa(\mathfrak{a}, r)$, we need to check that if $m_{\alpha_0}(\mathfrak{q}) \leq \frac{r}{2\kappa(r)}$ then

$$\beta|_r \subset \mathcal{N}_\kappa(m_{\mathfrak{a}}(\mathfrak{q}), \alpha_0).$$

But that is exactly what Definition 6.2 implies for R chosen as above. The proof for a point $x \in \mathcal{U}(\mathfrak{a}, R) \cap X$ is similar. ■

Claim 6.8. *For every $\mathfrak{a} \in \partial_\kappa X$ and $r > 0$ there exists $R > 0$ such that*

$$\mathcal{U}_\kappa(\mathfrak{a}, R) \subseteq \mathcal{U}(\mathfrak{a}, r).$$

Proof of Claim. We can use the same argument as in the proof of Lemma 6.4. In that proof, we needed to know that z with $\|z\| \leq 7r$ is still in the sublinear neighborhood of the central geodesic. Hence, for r large enough and $R \geq 7r$. The claim holds.

Again, the proof for a point $x \in \mathcal{U}(\mathfrak{a}, R) \cap X$ is similar is hence it is omitted. \blacksquare

Since the sets $\mathcal{U}(\mathfrak{a}, R)$ and $\mathcal{U}_\kappa(\mathfrak{a}, R)$ for neighborhood basis for $X \cup \partial_\kappa X$ and $X \cup \partial X$ respectively, the above claims prove the Proposition. \square

Corollary 6.9. *Let X be a proper Gromov hyperbolic space. Then $X \cup \partial X$ is homeomorphic to the Gromov compactification of X .*

Proof. Since all geodesics in X are Morse, the Gromov boundary is homeomorphic to the κ -boundary for $\kappa = 1$. Also, again because all geodesics are κ -Morse (where $\kappa=1$), the inclusion $\partial_\kappa \subset \partial X$ is in fact a bijection. Therefore, in view of Theorem 6.6, the Gromov boundary is also homeomorphic to ∂X . \square

7. SYMMETRY OF REDIRECTING

The partial order we defined on quasi-geodesics is not in general symmetric (see Section 10 for an example). However, symmetry is also a natural assumption to consider and it holds for our main class of examples that is discussed in Section 8. Here, we briefly investigate the consequences of the symmetry assumption in general terms (not specific examples). Notably, we will show that under this assumption, the boundary ∂X , when defined, is Hausdorff.

Assumption 3. (Symmetry of redirecting) For every pair of quasi-geodesic rays α and β ,

$$\alpha \preceq \beta \implies \beta \preceq \alpha.$$

First we note that Assumption 3 can be used instead of Assumption 1.

Lemma 7.1. *Assumption 0 and Assumption 3 imply Assumption 1.*

Proof. Consider a class $\mathfrak{a} = [\alpha] \in \partial X$ where α is a \mathfrak{q} -ray. By Assumption 0, there is geodesic segment α_i connecting \mathfrak{o} to $\alpha(i)$. Taking a subsequence, we can assume that segments α_i point-wise converge to a geodesic ray α_0 and we have as usual $\alpha_0 \preceq \alpha$. Hence by Assumption 3 $\alpha \preceq \alpha_0$ and thus $\alpha_0 \in \mathfrak{a}$. That is, the class \mathfrak{a} has a representative that is a geodesic ray. Therefore, Assumption 1 holds for $\mathfrak{q}_0 = (1, 0)$. \square

However, Assumption 2 is still necessary. We demonstrate that in the following example:

Example 7.2. Here we construct a geodesic metric space that satisfies Assumption 0, 1 and 3 but not Assumption 2. The space X consists of the following:

- All points in the xy -plane where $y \geq 0, x \geq 0$ and $y \leq \frac{1}{x}$ (shaded area).
- For every positive integer n , points in the xy -plane on the ray $y = nx, y \geq 0$ (we denote this ray by a_n).
- For every pair of positive integers n and k , a segment $\omega_{k,n}$ of length kn^2 connecting $a_n(kn) = (kn, kn^2)$ to $a_0(kn) = (kn, 0)$. This segment is not embedded in the xy -plane.

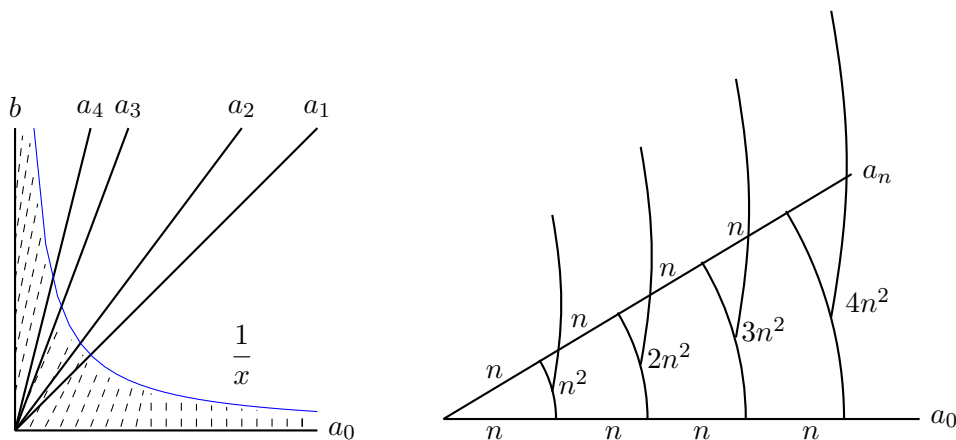


FIGURE 7. A copy of the graph on the right is attached to the pairs (a_0, a_n) for $n = 1, 2, 3, \text{etc.}$

- At the mid-point of each segment $\omega_{k,n}$ from $a_0(kn)$ to $a_n(kn)$, attach a *hair*, that is, an infinite geodesic ray.

The rays a_n is isometrically embedded in X . Also, the rays

$$a_n[0, kn] \cup \omega_{k,n} \cup a_0[kn, \infty) \quad \text{and} \quad a_0[0, kn] \cup \omega_{k,n} \cup a_n[kn, \infty)$$

are $((n+1), 0)$ -quasi-geodesic rays. That is, a_0 and a_n can be quasi-redirected to each other for $n = 0, 1, 2, \dots$ are hence all a_n are in the same class. Choosing a_0 as the central element, the redirecting constants from a_n to a_0 is $((n+1), 0)$. Thus Assumption 2 is not satisfied.

To see that X satisfies Assumption 0, we note that by construction X is a geodesic space. The region under the graph $y = 1/x$ ensures that the space X is proper. The attached hairs ensures every point outside of the shaded region lies on a geodesic ray. In fact, each hair represents its own equivalence class and each such class contain exactly two geodesic representatives. Furthermore, for q_0 large enough, we can ensure that every point in the shaded region lies on a q_0 -quasi-geodesic ray. Thus Assumption 0 is satisfied.

Note that a_0 can be quasi-redirected to every a_n and vice versa. Hence, all geodesic rays a_n belong to the same class \mathbf{a} . The geodesic ray b and rays associated to each hair represent incomparable classes, which is also not comparable to \mathbf{a} . Therefore, every class has a geodesic representative and hence, Assumption 1 holds. Also, since all the classes are incomparable, Assumption 3 holds. But as we saw in the preceding paragraph, Assumption 2 does not hold.

Lastly, we prove the main theorem of this section.

Theorem 7.3. *Let X be a proper geodesic metric space where Assumption 0, 2 and 3 hold. Then ∂X is a Hausdorff.*

Proof. Consider distinct points $\mathbf{a}, \mathbf{b} \in \partial X$. We will show that $\mathcal{U}(\mathbf{a}, r) \cap \mathcal{U}(\mathbf{b}, r) = \emptyset$ for some $r > 0$. Otherwise, there is a sequence $r_i \rightarrow \infty$ and a sequence of points

$$\mathbf{c}_i \in \mathcal{U}(\mathbf{a}, r_i) \cap \mathcal{U}(\mathbf{b}, r_i).$$

Let γ_i be a q_0 -ray in \mathbf{c}_i . Up to taking a subsequence, we can assume that γ_i point-wise converges to a q_0 -ray γ representing a class \mathbf{c} . Lemma 3.8 implies that $\gamma \prec \alpha_0$ and $\gamma \prec \beta_0$.

By symmetry α_0 and β_0 can also be quasi-redirected to γ and, by transitivity (Lemma 3.2), α_0 and β_0 can be quasi-redirected to each other. Therefore, $\mathbf{a} = \mathbf{b}$. That is, \mathbf{a} and \mathbf{b} could not have been distinct. \square

8. ASYMPTOTICALLY TREE-GRADED SPACES RELATIVE TO MONO-DIRECTIONAL SUBSETS

In this section we study a class of spaces called *asymptotically tree-graded spaces*. These spaces are first defined in [DS05] and are studied in [DS08, Sis12, Sis13b, OS05], among others. In this section we adopt the language and notation of [DS05]. These are metric space analogues of relatively hyperbolic groups. We focus on a sub-class where the peripheral subsets are mono-directional (see Section 4). We refer to these spaces as *asymptotically tree-graded spaces relative to mono-directional subsets*, or ATM spaces, for short. We show that these spaces satisfy Assumptions 0, 1 and 2 and thus have well-defined quasi-redirecting boundaries. We also show that if X is an ATM space then ∂X is metrizable and compact. Finally, we see that, when X is the Cayley graph of a group, ∂X is an alternate description of the Bowditch boundary.

8.1. Background on asymptotically tree-graded spaces. We start by recalling the necessary definitions and properties as laid out in [DS05]. We refer the readers to [DS05] for a more complete treatment. Asymptotically tree-graded spaces are metric spaces whose asymptotic cones are tree-graded. However, one of the main results of [DS05, Theorem 4.1] states that asymptotically tree-graded spaces can be characterized without referring to asymptotic cones. We use this characterization.

Definition 8.1. We say that a subset of X is a geodesic k -gon if it is a union of k geodesic segments p_1, \dots, p_k with pairs of endpoints $((p_i)_-, (p_i)_+)$ such that, $(p_i)_+ = (p_{i+1})_-$ for every $i = 1, \dots, (k-1)$ and $(p_k)_+ = (p_1)_-$. Also for each of $i = 1, \dots, k$, we denote the polygonal curve $P \setminus (p_{i-1} \cup p_i)$ by \mathcal{O}_{x_i} , where $x_i = (p_{i-1})_+ = (p_i)_-$.

Let $\vartheta > 0$ and $v \geq 8$ be constants. We say a k -gon P is $(\vartheta, 2, v)$ -fat if the following properties hold:

(F1) (Large comparison angles and large inscribed radii in interior points) For every edge p with endpoints $\{x, y\}$ we have

$$d(p \setminus \mathcal{N}_{2\vartheta}(x, y), P \setminus p) \geq \vartheta;$$

(F2) (Large edges and large inscribed radii in vertices) For every vertex x we have

$$d(x, \mathcal{O}_x) \geq v\vartheta.$$

Let (X, d) be a geodesic metric space and let \mathcal{A} be a collection of proper geodesic subsets of X . The metric space X is asymptotically tree-graded with respect to \mathcal{A} if and only if the following properties are satisfied:

- (AT1) (Isolated subsets) For every $\delta > 0$ there $D > 0$ such that, for distinct sets $A, A' \in \mathcal{A}$, the diameter of the intersection $\mathcal{N}_\delta(A) \cap \mathcal{N}_\delta(A')$ is bounded by D .
- (AT2) (Hyperbolicity outside of the special subsets) For every $\mathbf{q} = (q, Q)$ and every $\theta \in [0, 0.5)$ there exists a number $M_{\mathbf{q}} > 0$ such that for every \mathbf{q} -quasi-geodesic segment β defined on $[0, \ell]$ and every $A \in \mathcal{A}$ with $\beta(0), \beta(\ell) \in \mathcal{N}_{\theta\ell/q}(A)$ we have $\beta([0, \ell]) \cap \mathcal{N}_{M_{\mathbf{q}}}(A) \neq \emptyset$.

When $\mathbf{q} = (1, 0)$ we also write M_0 for when $\mathbf{q} = (1, 0)$. We note that even without assumption of cocompactness, the choice of $M_{\mathbf{q}}$ is uniform and does not depend on specific choices of A .

- (AT3) (Fat polygons are entirely in A .) For every $k \geq 2$ there exist $\vartheta > 0, v \geq 8$ and $\chi > 0$ such that every k -gon P with geodesic edges which is $(\vartheta, 2, v)$ -fat satisfies $P \subset \mathcal{N}_{\chi}(A)$ for some $A \in \mathcal{A}$.

Definition 8.2 (ATM). We say a space X is *ATM* if it satisfies Assumption 0 and it is asymptotically tree-graded with respect to a collection of mono-directional subspaces \mathcal{A} that each satisfies Assumption 2. We denote the redirecting function with respect to the central element of A by f_A . Note that the function f_A depends on the choice of base point, however, we do not include this in the notation.

Transition points and transient rays. We now recall some basic properties of ATM spaces. The following definitions and results are developed in [Sis12, Hru10].

Definition 8.3. [Sis12, Definition 3.9] Let α be a path in X . For $M, c > 0$, define the $\text{deep}_{M,c}(\alpha)$ to be the set of points $x \in \alpha$ such that there exists a subpath of α containing x with endpoints x_1, x_2 and $A \in \mathcal{A}$ where

$$x_1, x_2 \in N_M(A) \quad \text{and} \quad d(x, x_i) \geq c \quad \text{for } i = 1, 2.$$

Thinking of α as a subset of X , define

$$\text{trans}_{M,c}(\alpha) = \alpha - \text{deep}_{M,c}(\alpha)$$

to be the set of (M, c) -transition points of α .

Proposition 8.4. [Sis12, DS05] *Let X be a proper, geodesic, asymptotically tree-graded metric space. For every \mathbf{q} there exist constant $M = M(\mathbf{q}), c = c(\mathbf{q}), D = D(\mathbf{q})$ and $\rho(\mathbf{q})$ such that the followings hold. Let $\alpha: [a, b] \rightarrow X$ be a \mathbf{q} -quasi-geodesic segment.*

- (I) *The set $\text{deep}_{M,c}(\alpha)$ is a disjoint union of subpaths each contained in $N_{\rho M}(A)$ for distinct sets $A \in \mathcal{A}$.*
- (II) *For any pair of \mathbf{q} -quasi-geodesic segments α, β with the same endpoints, we have*

$$d_{\text{Haus}}(\text{trans}_{M,c}(\alpha), \text{trans}_{M,c}(\beta)) \leq D.$$

- (III) *Moreover, for every $A \in \mathcal{A}$ there are times $t, s \in [a, b]$ such that during the interval $[a, s]$ α approaches A at a linear speed, during the interval $[t, b]$ α moves away from A at a linear speed and $\alpha[s, t] \subset N_{\rho M}(A)$.*

The same also holds for quasi-geodesic rays.

The statements of (1) and (2) is contained [Sis12, Proposition 5.7]. The statement (2) follows from [DS05, Lemma 4.17].

Definition 8.5. Let α be a \mathbf{q} -quasi-geodesic segment or \mathbf{q} -ray in X . We say a point $\alpha(t)$ is a \mathbf{q} -transition point of α if

$$\alpha(t) \in \text{trans}_{M(\mathbf{q}),c(\mathbf{q})}(\alpha),$$

where $M(\mathbf{q}), c(\mathbf{q})$ are as Proposition 8.4.

Definition 8.6. Let α be a \mathbf{q} -ray. We say α is a \mathbf{q} -transient ray if, there is a sequence of times $t_i \rightarrow \infty$ such that $\alpha(t_i)$ is a \mathbf{q} -transition point of α .

Note that if $\mathfrak{q}' \geq \mathfrak{q}$ and α is a \mathfrak{q} -ray, then α is also a \mathfrak{q}' -ray. But, the set of \mathfrak{q} -transition points is not necessary a subset or a superset of the set of \mathfrak{q}' -transition points because, to ensure

$$\text{deep}_{M_1, c_1}(\alpha) \subset \text{deep}_{M_2, c_2}(\alpha)$$

we need $c_1 \geq c_2$ and $M_1 \leq M_2$. However, as we shall see, the quality of being a transient ray is independent of choice of \mathfrak{q} .

Lemma 8.7. *Let α be a \mathfrak{q} -ray and let M, c and ρ be as in Proposition 8.4. Then either α is a \mathfrak{q} -transient ray or α is eventually contained in $N_{\rho M}(A)$ for some $A \in \mathcal{A}$. Furthermore, if α is a \mathfrak{q} -transient ray and $\mathfrak{q}' \geq \mathfrak{q}$, then α is also a \mathfrak{q}' -transient ray.*

Proof. By definition, if α is not \mathfrak{q} -transient then α is eventually contained in $\text{deep}_{M, c}(\alpha)$. This means that there is $A \in \mathcal{A}$ and t_A such that $\alpha(t_A) \in N_M(A)$ and, for $t \geq t_A$, there is $s \geq t$ such that $\alpha(s) \in N_M(A)$. Hence, $\alpha[t_A, \infty) \subset N_{\rho M}(A)$.

If α is \mathfrak{q} -transient, then for every $A \in \mathcal{A}$, α eventually leaves every neighborhood of A (see part (III) of Proposition 8.4). In particular, α leaves the $(\rho(\mathfrak{q}') \cdot M(\mathfrak{q}'))$ -neighborhood of every $A \in \mathcal{A}$. The above argument shows that α is also \mathfrak{q}' -transient. \square

Part (II) of Proposition 8.4 can be restated as follows:

Corollary 8.8 (Relative fellow traveling property). *Suppose α, β are \mathfrak{q} -quasi-geodesic segments that start and end at the same point and let $x \in \alpha$ be a \mathfrak{q} -transition point of α . Then there exists a point $y \in \beta$ that is a \mathfrak{q} -transition point of β and*

$$d(p, q) \leq D(\mathfrak{q}).$$

Finally we recall the notion of *saturation* of a quasi-geodesic (see [DS05, Definition 4.20]).

Definition 8.9. Let α be a \mathfrak{q} -ray or \mathfrak{q} -segment. The *saturation* of α , denoted by $\text{Sat}(\alpha)$, is the union of α and all $A \in \mathcal{A}$ with $N_{M(\mathfrak{q})}(A) \cap \alpha \neq \emptyset$.

The saturation is quasi-convex (see [DS05, Lemma 4.25]).

Lemma 8.10 (Uniform quasi-convexity of saturations). *For every \mathfrak{q}' , there exists $\tau(\mathfrak{q}') > 0$ such that for every $L \geq 1$ and every \mathfrak{q} -ray or \mathfrak{q} -segment α , $\text{Sat}(\alpha)$ has the property that, for every \mathfrak{q}' -segment γ with endpoints $N_L(\text{Sat}(\alpha))$, we have*

$$\gamma \subset N_{\tau(\mathfrak{q}') \cdot L}(\text{Sat}(\alpha)).$$

Redirecting in ATM spaces. In this section we show that \preceq is a symmetric relation. Elements of $P(X)$ are divided to transient class with infinitely many transition points and non-transient class that are eventually contained in a bounded neighborhood of some $A \in \mathcal{A}$.

Lemma 8.11. *If α is a transient quasi-geodesic ray and $\beta \preceq \alpha$, then β is also a transient ray.*

Proof. We choose \mathfrak{q} large enough so that both α and β are \mathfrak{q} -rays and β can be \mathfrak{q} -redirected to α and let M, c and D be as in Proposition 8.4. Assume for contradiction that β is not transient. Then there a radius r_0 such that

$$\beta|_{\geq r_0} \subset \text{deep}_{M, c}(\beta).$$

In particular, for every $r > 0$, there are points $z_1, z_2 \in \beta$ such that $\|z_1\| \leq r_0 - c$, $\|z_2\| \geq r + c$ and $d(z_i, A) \leq M$ for $i = 1, 2$ and some $A \in \mathcal{A}$.

Since α is transient, there exists $x \in \alpha$ with $\|x\| \geq r_0 + D$ such that x is a \mathfrak{q} -transition point. Let $r = \|x\| + D$ and let z_1 and z_2 be as above. Let $R \geq \|z_2\|$ and let γ be a \mathfrak{q} -ray redirecting β to α at radius R .

By Corollary 8.8, there is a point $y \in \gamma$ with $d(x, y) \leq D$ such that y is a \mathfrak{q} -transition point of γ . Note that $r_0 \leq \|y\| \leq r$ which implies, $d(y, z_i) \geq c$ for $i = 1, 2$. Also,

$$B_R(\mathfrak{o}) \cap \gamma = B_R(\mathfrak{o}) \cap \beta,$$

thus, the segment $[z_1, z_2]_\beta$ is also a subsegment of γ . Hence $y \in \text{deep}_{M,c}(\gamma)$. This is a contradiction. \square

As a consequence, being transient is a property of an equivalence class. We say $\mathfrak{a} \in P(X)$ is transient if some quasi-geodesic ray in \mathfrak{a} is transient which implies all rays in \mathfrak{a} are transient.

Proposition 8.12. *Let $\mathfrak{a} \in P(X)$ be a transient class. Then \mathfrak{a} contains a geodesic ray α_0 . If we choose α_0 as the central element of \mathfrak{a} , we have $f_{\mathfrak{a}}(q, Q) = (9q, Q)$. Furthermore, if $\beta \preceq \alpha_0$ then $\beta \in \mathfrak{a}$.*

Proof. Let $\mathfrak{a} \in P(X)$ be a transient class and $\alpha \in \mathfrak{a}$ be a transient \mathfrak{q} -ray. By Lemma 3.5, there exists a geodesic ray $\alpha_0 \preceq \alpha$. Since α is transient, Lemma 8.11 implies that α_0 is also transient.

In fact, assume α_0 can be \mathfrak{q}' -redirected to α . Then there is a sequence of times $t_i \rightarrow \infty$ such that $\alpha(t_i)$ is a \mathfrak{q}' -transition point. Lemma 8.8 implies that, for every t_i , there is s_i such that $\alpha_0(s_i)$ is a \mathfrak{q}' -transition point and $d(\alpha_0(s_i), \alpha(t_i)) \leq D(\mathfrak{q}')$. For i large enough, we have

$$d(\alpha_0(s_i), \alpha(t_i)) \leq \frac{1}{2} \|\alpha_0(s_i)\|.$$

Thus by Part (II) of Surgery Lemma 2.6, α can be $(9q, Q)$ -redirected to α_0 . In particular, $\alpha \preceq \alpha_0$. Every other \mathfrak{q}' -ray in \mathfrak{a} can similarly be $f_{\mathfrak{a}}(\mathfrak{q}')$ -redirected to α using Part (II) of Lemma 2.6 for $f_{\mathfrak{a}}$ as in the statement of the proposition.

The proof of the last assertion is the same as above. If $\beta \preceq \alpha_0$ then, by Lemma 8.11, β is also transient and β and α_0 are near each other at all the transition points. Part (II) of Surgery Lemma 2.6 implies that α_0 can be quasi-redirected to β . \square

Lemma 8.13. *If α is a non-transient quasi-geodesic ray and $\beta \preceq \alpha$, then β is also a non-transient ray. In fact, there is $A \in \mathcal{A}$ such that both α and β are eventually contained in a bounded neighborhood of A .*

Proof. Let \mathfrak{q} be large enough such that α and β are \mathfrak{q} -rays and there is a family of \mathfrak{q} -rays γ_i redirecting β to α at radius r_i , with $r_i \rightarrow \infty$. Let M, c, D and ρ be as in Proposition 8.4.

The set $\text{deep}_{M,c}(\alpha)$ is a disjoint union of subpaths and there is a \mathfrak{q} -transition point between any two adjacent ones. Since α is non-transient, the tail of α after the last transition point is all in a deep segment which, by part (I) of Proposition 8.4, stays in a bounded neighborhood of some $A \in \mathcal{A}$.

If β contains \mathfrak{q} -transition points at arbitrarily large radii, then γ_i contain transition points at arbitrary radii for large enough i . But γ_i are eventually equal to α , hence (by Proposition 8.8) α contains \mathfrak{q} -transition point at arbitrarily large radius. This is a contradiction since α is not transient.

Therefore, β is non-transient and hence, for some $B \in \mathcal{A}$, β is eventually contained in a bounded neighborhood of B . If $B \neq A$, then γ_i stay near B for a long time and then near A .

This means γ_i has a \mathfrak{q} -transition at arbitrarily larger radii for large values of i . Hence α has infinitely many \mathfrak{q} -transition points which is again a contradiction. The contradiction implies $B = A$. This finishes the proof. \square

As a consequence, being non-transient is also a property of an equivalence class. We say $\mathfrak{a} \in P(X)$ is a non-transient if some quasi-geodesic ray in \mathfrak{a} is non-transient which implies all rays in \mathfrak{a} are non-transient and in fact, they all eventually stay in a bounded neighborhood of some $A \in \mathcal{A}$.

Proposition 8.14. *Let \mathfrak{a} be a non-transient class, where all the rays in \mathfrak{a} eventually stay in a bounded neighborhood of some $A \in \mathcal{A}$. Then every other quasi-geodesic ray that eventually stays in a bounded neighborhood of A is in \mathfrak{a} . Furthermore, \mathfrak{a} contains a geodesic ray and we can choose a function $f_{\mathfrak{a}}$ that depends only on f_A .*

Proof. Let α be a quasi-geodesic ray in \mathfrak{a} . By Lemma 3.5, there is a geodesic ray $\alpha_0 \preceq \alpha$. Lemma 8.13 implies that α_0 is non-transient and eventually stays in a bounded neighborhood of A . Let β be any \mathfrak{q} -ray that eventually stays in a bounded neighborhood of A . We will show that β can be quasi-redirected to α_0 where the redirecting constant depends uniformly on $f_A(\mathfrak{q})$. Note that this in particular proves (setting $\beta = \alpha$) that $\alpha_0 \in \mathfrak{a}$ and hence finishes the proof.

Let $\rho_0 = \rho(1,0)$, $M_0 = M(1,0)$, $\rho = \rho(\mathfrak{q})$ and $M = M(\mathfrak{q})$ be as in Proposition 8.4. Let t_0 be the first time α_0 enters the M_0 -neighborhood of A and let $\mathfrak{o}_A \in A$ be a closest point in A to $\alpha_0(t_0)$. Note that α_0 stays in the $\rho_0 M_0$ -neighborhood of A after t_0 . Let α_0^A be a quasi-geodesic ray in A starting at \mathfrak{o}_A that follow travels α_0 , namely, we can compose $\alpha_0|_{[t_0, \infty)}$ with the closest point projection to A and use Lemma 2.3 to tame the resulting quasi-geodesic.

Similarly, let t_β be the first time β enters the M -neighborhood of A . Consider $\text{Sat}(\alpha_0)$ (See Definition 8.9) and recall the uniform quasi-convexity of saturations [DS05, Lemma 4.25]. Since β is contained in a bounded neighborhood of the saturation of α_0 , β has to enter the M -neighborhood of A in a bounded distance from \mathfrak{o}_A where the bounds depend only on \mathfrak{q} . Let β^A be a quasi-geodesic ray in A starting at \mathfrak{o}_A that follow travels β constructed as above.

Now α_0^A and β^A are \mathfrak{q}' -rays in A where \mathfrak{q}' depends uniformly on \mathfrak{q} . Since A is mono-directional, β^A can be $f_A(\mathfrak{q}')$ redirected to α_0^A . Part (II) of Lemma 2.6 implies that β can be quasi-redirected to β^A , and α_0^A can be quasi-redirected to α_0 . Now, arguing as in Lemma 3.2, we see that β can be quasi-redirected to α_0 with quasi-redirection constant uniformly depending $f_A(\mathfrak{q}')$. This finishes the proof. \square

Corollary 8.15. *The relation \preceq is symmetric. In particular, different classes in $P(X)$ are not comparable.*

Proof. This follows immediately from Proposition 8.12 and Proposition 8.14. \square

Topological properties of ∂X . In this subsection, we show that ∂X is compact and metrizable. We note that X satisfies Assumption 0, 1 and 2; Assumption 0 holds by the definition of an ATM space and Assumptions 1 and 2 follow from Proposition 8.12 and Proposition 8.14.

Proposition 8.16. *Let X be an ATM space. Then ∂X is second countable.*

Proof. Take a sequence $r_i \rightarrow \infty$ and let

$$\mathcal{S}_i := \{x \in X \mid \|x\| = r_i\},$$

be the sphere for radius r_i in X . Since X is proper, there are finitely many points that are 1-dense on any given sphere. By Assumption 0, every point $x \in X$ lies on a \mathfrak{q}_0 -ray. Thus there exists a finite set \mathcal{E}_i of \mathfrak{q}_0 -rays such that the set

$$\{\alpha_{r_i} \mid \alpha \in \mathcal{E}_i\} \quad \text{is 1-dense in } \mathcal{S}_i.$$

In addition, let \mathcal{Z} be the set of non-transient classes in $P(X)$. Note that \mathcal{Z} is countable. Now, define

$$\mathcal{Y} := \left\{ \mathcal{U}(\mathfrak{a}, R) \mid \mathfrak{a} \in \mathcal{Z} \quad \text{and} \quad R \in \mathbb{N} \right\} \cup \left\{ \mathcal{U}([\alpha], R) \mid \alpha \in \bigcup_i \mathcal{E}_i \quad \text{and} \quad R \in \mathbb{N} \right\}.$$

This is a countable set of neighborhoods. To prove the proposition, it is sufficient to show that, for every $\mathfrak{b} \in \partial X$ and every $r > 0$, there exists $\mathcal{U}(\mathfrak{a}, R) \in \mathcal{Y}$ such that

$$(5) \quad \mathfrak{b} \in \mathcal{U}(\mathfrak{a}, R) \subset \mathcal{U}(\beta_0, r).$$

If \mathfrak{b} is a non-transient class, then $\mathfrak{b} \in \mathcal{Z}$ and we can take $\mathfrak{a} = \mathfrak{b}$ and choose any $R \geq r$. Thus, we assume \mathfrak{b} is a transient class.

Let the geodesic ray β_0 be the central element of \mathfrak{b} . For any $r > 0$, let $(q_r, Q_r) = \mathfrak{q}_{\max}(r)$ be as in Lemma 5.3, let $\mathfrak{q}_r = (9q_r, Q_r)$ and let $D_r = D(\mathfrak{q}_r)$ be as in Proposition 8.4. Let

$$r' \geq 2r + 6D_r$$

be such that $(\beta_0)_{r'}$ is a $(1, 0)$ -transition point for β_0 . Choose $R \gg r'$, let $(q_R, Q_R) = \mathfrak{q}_{\max}(R)$ be as in Lemma 5.3, $\mathfrak{q}_R = (9q_R, Q_R)$ and let $D_R = D(\mathfrak{q}_R)$ be as in Proposition 8.4. Again, let

$$R' \geq 2R + 6D_R$$

be such that $(\beta_0)_{R'}$ is a $(1, 0)$ -transition point for β_0 . Finally, choose $r_i \gg 2R'$ and let α be a \mathfrak{q}_0 -ray in \mathcal{E}_i such that $d((\beta_0)_{r_i}, \alpha_{r_i}) \leq 1$. Let $\mathfrak{a} = [\alpha]$. We check that (5) holds.

We start by showing that $\mathfrak{b} \in \mathcal{U}(\mathfrak{a}, R)$. Part (II) of Lemma 2.6 implies that α can be $(9q_0, Q_0)$ -redirected to β_0 at radius $r_i/2 \gg R'$. Since $(\beta_0)_{R'}$ is a transition point for β_0 , arguing as in the proof of Lemma 8.11, we see that there is a transition point $x \in \alpha$ with

$$d(x, (\beta_0)_{R'}) \leq D_R$$

that is a $(9q_0, Q_0)$ -transition point for α .

Let the geodesic ray $\alpha_0 \in \mathfrak{a}$ be the central element of \mathfrak{a} . Then there is a point $x_0 \in \alpha_0$ near x that is a transition point for α_0 , that is

$$d(x_0, (\beta_0)_{R'}) \leq d(x_0, x) + d(x, (\beta_0)_{R'}) \leq 2D_R.$$

Let $\beta \in \mathfrak{b}$ be a $\mathfrak{q} = (q, Q)$ -ray with $\mathfrak{q} \leq \mathfrak{q}_R$. Since $(\beta_0)_{R'}$ is a transition point for β_0 , there is a transition point $y \in \beta$ with $d(y, (\beta_0)_{R'}) \leq D_R$. This means

$$d(x_0, y) \leq d(x_0, (\beta_0)_{R'}) + d((\beta_0)_{R'}, y) \leq 3D_R \leq R'/2.$$

Part (II) of Lemma 2.6 implies that β can be $(9q, Q)$ -quasi-redirected to α_0 at radius $R'/2 \geq R$. Therefore $\mathfrak{b} \in \mathcal{U}(\mathfrak{a}, R)$.

We now show $\mathcal{U}(\mathfrak{a}, R) \subset \mathcal{U}(\beta_0, r)$. The proof is similar to above and uses the transition points. For the sake brevity, we omit some of the intermediate constants in our proof. Let \mathfrak{c} be a point in $\mathcal{U}(\mathfrak{a}, R)$ and let $\gamma \in \mathfrak{c}$ be a \mathfrak{q} -ray with $\mathfrak{q} \leq \mathfrak{q}_{\max}(r)$. We need to show that γ can be $(9q, Q)$ -redirected to β_0 at radius r . Note that, γ be the redirected to α_0 at radius R . Since $(\beta_0)_{r'}$ is a $(1, 0)$ -transition points, α has to have a nearby transition point, and thus α_0

has a nearby transition point which implies γ has a nearby transition point. That is, there is a point x in γ such that

$$d(x, (\beta_0)_{r'}) \leq 3D_r \leq r'/2.$$

Part (II) of Lemma 2.6 implies that γ can be $(9q, Q)$ -redirected to β_0 at radius $r'/2 \geq r$.

Since \mathcal{Y} is a countable basis for the topology we have that ∂X is a second countable topological space. \square

We use the notation

$$A \gg B$$

for quantities A and B to mean that there exists a sufficiently large constant $C > 0$ not depending on \mathfrak{q} such that when $A - B \geq C$ the argument holds. This simplifies the exposition when there are many additive errors accumulating but the errors do not get larger when the quasi-geodesic constant get larger.

We first establish an a criterion for an element \mathfrak{c} to be contained in a neighborhood $\mathcal{U}(\mathfrak{a}, r)$.

Lemma 8.17. *Let $\mathfrak{a} \in \partial X$ and $r > 0$ be given and let the geodesic ray α_0 be the central element of \mathfrak{a} . We can choose the redirecting function $f_{\mathfrak{a}}$ large enough such that the following holds. Then there exists $r' > r$ such that, if a geodesic γ can be $f_{\mathfrak{a}}(1, 0)$ -redirected to α_0 at radius r' and $\gamma \in \mathfrak{b}$ (not necessarily the central element of \mathfrak{b}) then $\mathfrak{b} \in \mathcal{U}(\mathfrak{a}, r)$.*

Proof. Suppose that \mathfrak{a} is a transient class. Let x_0 be a $(1, 0)$ -transition point along α_0 with

$$\|x_0\| \gg r \quad \text{and let} \quad r' \gg \|x_0\|.$$

Since γ can be redirected to α_0 at radius r' , Lemma 8.8 implies that γ has a $(1, 0)$ -transition point near x_0 . Similarly, any \mathfrak{q} -ray $\beta \in \mathfrak{b}$ ($\mathfrak{q} \leq \mathfrak{q}_{\max}(r)$) has a \mathfrak{q} -transition point y near x_0 . Choosing the $\|x_0\|$ large enough, we can ensure that

$$d(x_0, y) \leq \frac{\|x_0\|}{2}.$$

Then Part (II) of Lemma 2.6 implies that β can be $(9q, Q)$ -redirected to α_0 at radius $\|x_0\|/2 > r$. Since

$$(9q, Q) = f_{\mathfrak{a}}(q, Q) < F_{\mathfrak{a}}(q, Q),$$

and this holds for every quasi-geodesic ray $\beta \in \mathfrak{b}$, we have $\mathfrak{b} \in \mathcal{U}(\mathfrak{a}, r)$. Suppose otherwise that \mathfrak{a} is a non-transient class. Let

$$\begin{aligned} M_0 &= M(1, 0), c_0 = c(1, 0), \rho_0 = \rho(1, 0) \\ M &= M(\mathfrak{q}_{\max}(r)), c = c(\mathfrak{q}_{\max}(r)), \rho = \rho(\mathfrak{q}_{\max}(r)) \end{aligned}$$

be as in Proposition 8.4. For the rest of the proof, by a bounded constant, we mean a constant that depends uniformly on these constants. Since \mathfrak{a} is non-transient, there is $A \in \mathcal{A}$ and t_0 such that $\alpha_0|_{[t_0, \infty)}$ is in a $(\rho_0 M_0)$ -neighborhood of A . Let \mathfrak{o}_A be a closest point in A to $\alpha_0(t_0)$. Then any \mathfrak{q} -ray that enter an $M(\mathfrak{q})$ -neighborhood of A , does so a bounded distance away from \mathfrak{o}_A (see Lemma 8.10). Let α_0^A be a quasi-geodesic ray in A that follow travels α_0 (meaning they stay a bounded distance from each other in a parametrized manner).

We would like to argue that, every quasi-geodesic that stays near A for a long time can be redirected to α_0 at a radius r . We formulate this in terms of geodesic segments in A .

Claim 8.18. *For every $r_A > 0$ there is $r'_A \geq r_A$ such that every \mathfrak{q} -segment γ_A in A connecting \mathfrak{o}_A to x with $d(\mathfrak{o}_A, x) \geq r'_A$ can be $f_A(\mathfrak{q})$ -redirected to α_0^A at a radius r_A . That is, there is a $f_A(\mathfrak{q})$ -ray γ'_A in A that $\gamma_A|_{r_A} = \gamma'_A|_{r_A}$ and γ'_A is eventually equal to α_0^A .*

Proof of Claim. Let r_A be given and assume for contradiction there is a sequence of radii $r_i^A \rightarrow \infty$ and \mathfrak{q} -segments γ_i^A ($\mathfrak{q} \leq \mathfrak{q}_{\max}(r_A)$), that cannot be $f_A(\mathfrak{q})$ -redirected to α_0^A . Taking a subsequence, we find a \mathfrak{q} -ray γ_A that cannot be $f_A(\mathfrak{q})$ -redirected to α_0^A . This contradicts the fact that A is mono-directional. \blacksquare

Let $r_A = r - d(\mathfrak{o}, \mathfrak{o}_A) + 2\rho \cdot M$ and choose

$$r' \gg d(\mathfrak{o}, \mathfrak{o}_A) + r'_A + 2\rho \cdot M$$

where r'_A is as in the claim.

Let γ be a geodesic ray that redirects to α_0 at radius r' . Let \mathfrak{e}_γ be the points where γ exits the M -neighborhood of A . Let γ' be the $f_a(1, 0)$ -ray where $\gamma'|_{r'} = \gamma|_{r'}$ and γ is eventually in a bounded neighborhood of A . Then γ' is entirely contained in a ρM -neighborhood of A . Hence $\|\mathfrak{e}_\gamma\| - r'$ is bounded.

Let β be a \mathfrak{q} -ray in \mathfrak{b} with $\mathfrak{q} \leq \mathfrak{q}_{\max}(r)$. Then, by quasi-convexity of the saturation (Lemma 8.10), β has to also enter the M -neighborhood of A at a point x at most a bounded distance away from \mathfrak{o}_A and exit the neighborhood near at a point y that is at most a bounded distance away from \mathfrak{e}_γ . Lemma 8.10 implies that, $[x, y]_\beta$ stays in a bounded neighborhood of A . Let β_A be the \mathfrak{q}' -quasi-geodesic segment in A starting from \mathfrak{o}_A that fellow travels $[x, y]_\beta$ where \mathfrak{q}' depends only on \mathfrak{q} . By the claim, β_A can be quasi-redirected to α_0^A at radius r_A via a $f_A(\mathfrak{q}')$ -ray β' in A . Now $\beta|_r$ fellow travels $[\mathfrak{o}, \mathfrak{o}_A] \cup \beta'_A|_{r_A}$. By Lemma 2.7, β can be quasi-redirected to $[\mathfrak{o}, \mathfrak{o}_A] \cup \beta'_A|_{r_A}$. Also, $[\mathfrak{o}, \mathfrak{o}_A] \cup \beta'_A$ is eventually equal to α_0^A and hence fellow travels α_0 . Again, by Lemma 2.7, it can be quasi-redirected to α_0 at any radius. By transitivity, we see that β can be quasi-redirected to α_0 at radius r with uniform constant that depends only on $f_A(\mathfrak{q})$. Hence, $\mathfrak{b} \in \mathcal{U}(\mathfrak{a}, r)$ for an appropriate function F_a . \square

Recall that a topological space is regular if points can be separated from closed sets.

Proposition 8.19. *Let X be an ATM space. Then ∂X is regular.*

Proof. Let $\mathfrak{a} \in \partial X$ be a point and $B \subset \partial X$ be a closed set. Since, B is closed, there is $r > 0$ such that

$$(6) \quad B \cap \mathcal{U}(\mathfrak{a}, r) = \emptyset.$$

For a given $r > 0$, let $r' > 0$ be as in Lemma 8.17. We will show that, for every $\mathfrak{b} \in B$, there is $r_{\mathfrak{b}}$ such that

$$\mathcal{U}(\mathfrak{a}, r') \cap \mathcal{U}(\mathfrak{b}, r_{\mathfrak{b}}) = \emptyset.$$

Then, $\bigcup_{\mathfrak{b} \in B} \mathcal{U}(\mathfrak{b}, r_{\mathfrak{b}})$ contains an open neighborhood of B disjoint from $\mathcal{U}(\mathfrak{a}, r')$.

Pick $\mathfrak{b} \in B$ and let β_0 be the central element of \mathfrak{b} . Assume for contradiction that

$$\mathcal{U}(\mathfrak{a}, r') \cap \mathcal{U}(\mathfrak{b}, R) \neq \emptyset$$

for every R . Then, there is a sequence

$$\mathfrak{c}_i \in \mathcal{U}(\mathfrak{a}, r') \cap \mathcal{U}(\mathfrak{b}, R_i) \quad \text{where } R_i \rightarrow \infty.$$

Let the geodesic ray γ_i be the central element of \mathbf{c}_i . Up to taking a subsequence, γ_i point-wise converge to a geodesic ray γ . Since γ_i can be redirected to β_0 at radius R_i , γ can be quasi-redirected to β_0 and hence $\gamma \in \mathbf{b}$. This is a contradiction which shows $\mathcal{U}(\mathbf{a}, r') \cap \mathcal{U}(\mathbf{b}, r_{\mathbf{b}}) \neq \emptyset$ for some $r_{\mathbf{b}}$. \square

Theorem 8.20. *When X is an ATM space, ∂X is a compact, second countable and metrizable topological space.*

Proof. As we saw in the beginning of the subsection, X satisfies Assumptions 0, 1 and 2 and hence has a well-defined ∂X . We showed in Proposition 8.16 that ∂X is second countable and in Proposition 8.19 we showed that ∂X regular. Also, \preceq is symmetric by Corollary 8.15 and hence Hausdorff by Lemma 7.3. Urysohn Metrization Theorem implies that ∂X is metrizable.

A metrizable topological space is compact if and only if it is sequentially compact. Let $\{\mathbf{a}_i\}$ be a sequence of equivalence classes in ∂X and consider the associated geodesic representatives $\{\alpha_0^i\}$. Since X is proper, there exists a subsequence which we again index as $\{\alpha_0^i\}$ that converges point-wise to a geodesic β_0 . We will show that $\mathbf{a}_i \rightarrow \mathbf{b}$, that is,

For a given $r > 0$, let $r' > 0$ be as in Lemma 8.17. For i large enough, Lemma 2.7 implies that α_0^i can be $f_{\mathbf{b}}(1, 0)$ -redirected to β_0 . Again by Equation 4, $f_{\mathbf{b}}(1, 0) < F_{\mathbf{a}}(1, 0)$ and together with Lemma 8.17, we have that $\mathbf{a}_i \in \mathcal{U}(\mathbf{b}, r)$. Since this holds for every $r > 0$, we have $\mathbf{a}_i \rightarrow \mathbf{b}$. Therefore ∂X is compact. \square

9. A GEOMETRIC CHARACTERIZATION OF THE BOWDITCH BOUNDARY

In this section we examine the case when X is a Cayley graph of a relatively hyperbolic group pair (G, \mathcal{P}) where G is a group and \mathcal{P} is a collection of mono-directional subgroups. We show that, in this case, the quasi-redirecting boundary is naturally homeomorphic to the Bowditch boundary of (G, \mathcal{P}) . That is, for this class of groups, the Bowditch boundary can be constructed purely from the geometry of X without referencing the group structure or the dynamics of the group action. We begin by reviewing relative hyperbolic groups and the Bowditch boundary following [Bow12].

Definition 9.1. Fix a finite generating set S once and for all and let $\text{Cay}(G)$ denote the Cayley graph of G with respect to this generating set. We refer to the groups $P \in \mathcal{P}$ as *peripheral* subgroups. Let \mathcal{A} be the set of subgraphs of $\text{Cay}(G)$ associates to cosets of groups in \mathcal{P} . Namely, for $P \in \mathcal{P}$ and $g \in G$, $A_{P,g}$ is the induced subgraph of $\text{Cay}(G)$ with vertex set gP . We form the *coned-off* Cayley graph, denoted $K(G)$ or simply K , by adding a vertex $*p_A$ for each $A \in \mathcal{A}$, and adding edges of length $\frac{1}{2}$ from $*p_A$ to each vertex of A . Since $\text{Cay}(G)$ is a subgraph of K , for any two vertices $v, w \in \text{Cay}(G)$, we have

$$(7) \quad d_K(v, w) \leq d_{\text{Cay}(G)}.$$

Definition 9.2. A graph is *fine* if for each integer n , every edge belongs to only finitely many simple cycles of length n . If the coned-off Cayley graph is hyperbolic and is fine, then G is *relatively hyperbolic* relative to \mathcal{P} .

A key property of relative hyperbolic group is the *Bounded Coset Penetration* [Far98] which we state now. An oriented path $\ell \in K$ is said to *penetrate* $A \in \mathcal{A}$ if it passes through the cone point $*p_A$ of A ; its *entering* and *exiting* vertices are the vertices immediately before and after $*p_A$ on ℓ . The path is *without backtracking* if once it penetrates $A \in \mathcal{A}$, it does not

penetrate A again. If for each $q \geq 1$ there is a constant $a = a(q)$ such that if ζ and ζ' are $(q, 0)$ -quasi-geodesics without backtracking in K and with the same endpoints, then

- (1) if ζ penetrates some $A \in \mathcal{A}$, but ζ' does not, then the distance between the entering and exiting vertices of ζ in A is at most $a(q)$; and
- (2) if ζ and ζ' both penetrate $A \in \mathcal{A}$, then the distance between the entering vertices of ζ and ζ' in A is at most $a(q)$, and similarly for the exiting vertices.

Now we define the Bowditch boundary for relatively hyperbolic groups. Let ∂K denote the Gromov boundary of K . Let $V(K)$ denote the vertex set of K , let $V_\infty K = \{ *p_A, A \in \mathcal{A} \}$ and let $\Delta K = V_\infty(K) \cup \partial K$.

Definition 9.3. For $v, w \in (V(K) \cup \partial K)$, let $[v, w]_K$ denote a geodesic segment (or a geodesic ray) in K connecting v to w . Given any $v \in (V(K) \cup \partial K)$ and a finite set $W \subseteq V(K)$, we write

$$m(v, W) = \left\{ w \in \Delta K \mid W \cap [v, w]_K \subseteq \{v\} \text{ for every geodesic } [v, w]_K \right\}.$$

The Bowditch boundary $\partial_B G$ of the relative hyperbolic group G is the set ΔK equipped with a topology that is generated by the neighborhoods of the form $m(v, W)$.

Every geodesic in K can be associated to some geodesic in $\text{Cay}(G)$. Let ℓ be a path in K , a *lift* of ℓ , denoted $\bar{\ell}$, is a path formed from ℓ by replacing edges incident to vertices in $V_\infty(K)$ with a geodesic in $\text{Cay}(G)$.

Lemma 9.4. *There exists a uniform bound δ_0 such that, given any geodesic line or segment $\ell \subset K$ where $|\ell| \geq 3$, there exists a geodesic line or segment $\bar{\ell}_0$ in $\text{Cay}(G)$ such that, when considered as a subset of K , $\bar{\ell}_0$ is contained in a δ_0 -neighborhood of ℓ in K . If an endpoint of ℓ is $*p_A$, then $\bar{\ell}_0$ can be chosen to start at any vertex in $N_{M_0}(A)$. Furthermore, every $(1, 0)$ -transition point of $\bar{\ell}_0$ is δ_0 -close in $\text{Cay}(G)$ to some vertex of ℓ .*

Proof. Since vertices in $V_\infty(K)$ are not adjacent to each other in K , $|\ell| \geq 3$ implies ℓ contains at least one vertex of $\text{Cay}(G)$. By [Sis13a, Proposition 1.14], there exists bounded constants q_1 such that every ℓ has a lift $\bar{\ell}$ that is a q_1 -quasi-geodesic in $\text{Cay}(G)$. If $\bar{\ell}$ is a finite quasi-geodesic segment then by Assumption 0, there exists a geodesic connecting its end vertices and by [Hru10, Lemma 8.13] $\bar{\ell}$ is in a bounded neighborhood of ℓ in K . Otherwise, $\bar{\ell}$ is an infinite quasi-geodesic line or ray, then there exists an infinite set of longer and longer geodesic segments $\{\bar{\ell}_i\}$ in $\text{Cay}(G)$ with endpoints on $\bar{\ell}$ such that the end points converges to both ends of $\bar{\ell}$. Since $|\ell| \geq 3$, ℓ contains at least one vertex v not in $*p_A$ and thus by Proposition 8.4 (2) there exists a transition point $v_i \in \bar{\ell}_i$ such that for all i

$$d_{\text{Cay}(G)}(v, v_i) \leq L.$$

Thus by Arzelà–Ascoli Theorem, up to a subsequence, the set $\{\bar{\ell}_i\}$ converges to geodesic ray or line which we denote $\bar{\ell}_0$. By construction the projection of $\bar{\ell}_0$ to K is in a bounded neighborhood of ℓ . The last claim follows from [Hru10, Lemma 8.13] all transition points of $\bar{\ell}_0$ is boundedly close to points of ℓ . \square

Lastly we recall the *relative thin triangle* property [Sis12, Definition 3.11] and by [Sis12, Theorem 1.1], the condition holds for geodesic triangles in $\text{Cay}(G)$.

Proposition 9.5. *There exists a constant δ_1 such that the following holds. For point $x, y, z \in \text{Cay}(G)$ consider a geodesic triangle (x, y, z) and let w be a $(1, 0)$ -transition point along $[x, y]$. Then there exists $w' \in [x, z] \cup [z, y]$ such that $d_{\text{Cay}(G)}(w, w') \leq \delta_1$.*

By Osin-Sapir [DS05, Theorem 9.1], $\text{Cay}(G)$ is an asymptotically tree-graded space with respect to \mathcal{A} . Assume for the remainder of this section that each $A \in \mathcal{A}$ is mono-directional and we have that $\text{Cay}(G)$ is an ATM space. Thus the quasi-redirecting boundary ∂G exists. Recall from Proposition 8.12 and Proposition 8.14, each equivalence class in ∂G contains a central element that is a geodesic ray.

Definition 9.6. Define a map

$$\xi : \partial G \rightarrow \partial_B G$$

as follows. Let $\mathfrak{a} \in \partial G$ and $\alpha_0 \in \mathfrak{a}$ be the central element of \mathfrak{a} . If α_0 is not transient, then by Lemma 8.13 there exists a set $A \in \mathcal{A}$ such that a tail of α_0 is in a bounded neighborhood of A . In this case we define

$$\xi(\mathfrak{a}) := *p_A.$$

Otherwise, α_0 is transient. By the construction and hyperbolicity of K , α_0 is an unbounded unparameterized quasi-geodesic in K and hence converges to a point $\hat{\alpha}_0$ in ∂K . We define

$$\xi(\mathfrak{a}) := \hat{\alpha}_0.$$

Lemma 9.7. *The map $\xi : \partial G \rightarrow \partial_B G$ is a bijection.*

Proof. Suppose $\xi(\mathfrak{a}) = \xi(\mathfrak{b})$ for $\mathfrak{a}, \mathfrak{b} \in \partial G$ and let α_0 and β_0 be the central elements. If $\xi(\mathfrak{a}) = \xi(\mathfrak{b})$ is a vertex in $V_\infty(K)$, then α_0 and β_0 are eventually contained in a bounded neighborhood of A for some $A \in \mathcal{A}$. Lemma 8.14 shows $\alpha_0 \sim \beta_0$ in ∂G and hence $\mathfrak{a} = \mathfrak{b}$.

Otherwise, $\xi(\mathfrak{a}) = \xi(\mathfrak{b})$ is a point in ∂K . Then α_0 and β_0 are in a bounded Hausdorff distance from each other in K . But, by bounded coset penetration, α_0 and β_0 have to enter the neighborhood of each $A \in \mathcal{A}$ near (in $\text{Cay}(G)$ -metric) the point \mathfrak{o}_A . Therefore, α_0 and β_0 come boundedly close to each other (in $\text{Cay}(G)$ -metric) infinitely often. Part (II) of Lemma 2.6 implies that $\alpha_0 \sim \beta_0$ and hence $\mathfrak{a} = \mathfrak{b}$.

To see that the map is surjective, we also have two cases. Let $v \in V_\infty(K)$ be a point in the Bowditch boundary and let A be the associated set in \mathcal{A} . Let α be a quasi-geodesic ray that connects $[\mathfrak{o}, \mathfrak{o}_A]$ with a geodesic ray starting at \mathfrak{o}_A and lie entirely in A . By [DS05, Lemma 4.19] α is a bounded constant quasi-geodesic ray in the class of ∂A . Then it follows that $\xi([\alpha]) = v$. Otherwise, let v be a point in ∂K . Since K is hyperbolic, there exists an equivalence class of quasi-geodesic rays associated with v and in fact there exists a geodesic representative in this class (for instance by Arzelà-Ascoli Theorem), which we refer to as α . Since α is a geodesic ray in K , by [Sis13a, Proposition 1.14], there exists a bounded constant quasi-geodesic ray α' in $\text{Cay}(G)$ that is a lift of α . We claim that, for $\mathfrak{a} = [\alpha']$, we have

$$\xi(\mathfrak{a}) = v.$$

Indeed, the central element α_0 of \mathfrak{a} is a geodesic in $\text{Cay}(G)$, and an unparameterized quasi-geodesic in K . Thus it stays in a bounded neighborhood of α and hence converges to v . This finishes the proof. \square

We now show that ξ and ξ^{-1} are both continuous. First we show that for every $v \in \Delta(K)$ and every finite subset $W \subset V(K)$, $m(v, W)$ is open in ∂G . It suffices to verify this for when W has one element as a finite intersection of open sets is open.

Lemma 9.8. *For every $\mathbf{b} \in \partial G$ and $p \in V(K)$ there exists $r > 0$ such that*

$$\xi(\mathcal{U}(\mathbf{b}, r)) \subset m(\xi(\mathbf{b}), p).$$

Therefore, ξ is continuous.

Proof. Let the geodesic ray β_0 be the central element of \mathbf{b} .

Case I. Assume that \mathbf{b} is transient. Consider β_0 as a subset of K and let $\pi_{\xi(\mathbf{b})}(p)$ be the closest point projection of p to β_0 in K (see Figure 8). Since K is hyperbolic, $\pi_{\xi(\mathbf{b})}(p)$ has a bounded diameter in K . Since \mathbf{b} is transient, β_0 has transition point that are arbitrarily far from \mathfrak{o} . Choose $r > 0$ such that, $(\beta_0)_r$ is a $(1, 0)$ -transition point of β_0 and

$$(8) \quad d_K(\mathfrak{o}, (\beta_0)_r) \gg d_K(\mathfrak{o}, \pi_{\xi(\mathbf{b})}(p)) + D(9, 0) + 2\delta,$$

where δ is the hyperbolicity constant of K , $D(9, 0)$ is as in Corollary 8.8 and $d_K(\mathfrak{o}, \pi_{\xi(\mathbf{b})}(p))$ is the maximum distance in K between any point in $\pi_{\xi(\mathbf{b})}(p)$ to \mathfrak{o} .

Let $\mathbf{a} \in \mathcal{U}(\mathbf{b}, r)$ and let α_0 be the central element in \mathbf{a} . Since $(\beta_0)_r$ is a transition point, there exists points $q \in \alpha_0$ such that

$$d(q, (\beta_0)_r) < D((9, 0)),$$

Thus $\|q\| \geq r - D((9, 0))$. Since K is hyperbolic, there exists either a geodesic ℓ in K connecting $\xi(\mathbf{a})$ to $\xi(\mathbf{b})$. The line ℓ is an edge in the ideal quadrilateral $((\beta_0)_r, \xi(\mathbf{b}), \xi(\mathbf{a}), q)$ (see Figure 8) hence it stays in a bounded neighborhood of

$$\beta_0|_{\geq r} \cup \alpha_0|_{\geq r} \cup [(\beta_0)_r, q].$$

Hence, ℓ is far from p in K and hence does not pass through p . Therefore, $\xi(\mathbf{a}) \in m(\xi(\mathbf{b}), p)$.

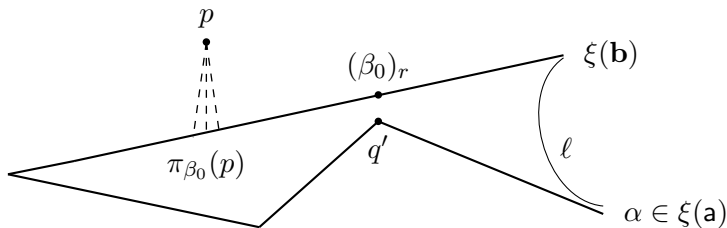


FIGURE 8. A transition point $(\beta_0)_r$ separates the point p and any geodesic line that connects $\xi(\mathbf{b})$ and $\xi(\mathbf{a})$.

Case II. Suppose otherwise that \mathbf{b} is not transient. By Lemma 8.11 there exists a unique set $A \in \mathcal{A}$ such that $\xi(\mathbf{b}) = *p_A$. Let β_0 be the central element of \mathbf{b} . Let

$$r \gg 2(\|\mathfrak{o}_A\| + \|p\|).$$

Let $\mathbf{a} \in \mathcal{U}(\mathbf{b}, r)$ and let α_0 be the central element of \mathbf{a} . Then α_0 can be $f_{\mathbf{b}}(1, 0)$ -redirect to β_0 at radius r . Let $\mathfrak{e} \in A$ be the point near where α_0 leaves the M_0 -neighborhood of A .

Consider any geodesic segment or ray ℓ in K connecting $\xi(\mathbf{a})$ to $*p_A$. By [Hru10, Proposition 8.13], ℓ enters $N_{r(f_{\mathbf{b}}(\mathfrak{q}))}(A)$ at a point that is boundedly close to \mathfrak{e} . Since $*p_A$ is the final point in ℓ , $*p_A$ does not appear in interior of ℓ and hence, for any other vertex x in ℓ , we have $\|x\| \geq \|\mathfrak{e}\| - D(1, 0)$. This implies $\|x\| \gg \|p\|$ and hence ℓ does not pass through p . Therefore,

$$\mathbf{a} \in m(\xi(\mathbf{b}), p)$$

and hence $\mathcal{U}(\mathbf{b}, r) \subset m(\xi(\mathbf{b}), p)$. \square

Lemma 9.9. *For any $\mathbf{b} \in \partial G$ and $r > 0$, there exists a finite set of vertices $W \subset V(K)$ such that for every \mathbf{a}*

$$\xi(\mathbf{a}) \in m(\xi(\mathbf{b}), W) \implies \mathbf{a} \in \mathcal{U}(\mathbf{b}, r).$$

Therefore, ξ^{-1} is continuous.

Proof. Let the geodesic ray β_0 be the central element of \mathbf{b} .

Case I. Suppose that β_0 is a transient geodesic ray. Given any r , let r' be as in Lemma 8.17 and let $(\beta_0)_{r_1}$ be a transition point of β_0 where $r_1 \geq 2r'$. Let δ_0 be the constant from Lemma 9.4 and let δ_1 be the constants of Proposition 9.5 accordingly. Let

$$(9) \quad r_2 > \delta_0 + \delta_1.$$

Let

$$W = B((\beta_0)_{r_1}, r_2) \subset \text{Cay}(G) \subset K$$

be the ball of radius r_2 in $\text{Cay}(G)$ centered at the vertex β_{r_1} . However, we consider W as a subset of K . Note that Since $\text{Cay}(G)$ is proper, $|B((\beta_0)_{r_1}, r_2)| < \infty$.

Consider any element $\mathbf{a} \in \partial_B G$ such that $\xi(\mathbf{a}) \in m(\xi(\mathbf{b}), W)$. That is, there exists a bi-infinite geodesic line ℓ in K from $\xi(\mathbf{a})$ to $\xi(\mathbf{b})$ avoiding W . Since \mathbf{b} is transient $|\ell| \geq 3$, by Lemma 9.4 there exists $\bar{\ell}_0$ whose projection to K is boundedly close to ℓ . If α_0 is transient, then the geodesics α_0 (the central element of \mathbf{a}) β_0 and $\bar{\ell}_0$ form an ideal geodesic triangle in $\text{Cay}(G)$. If α_0 is not transient, then there exists $A_{\mathbf{a}}$ such that α_0 eventually stays in $A_{\mathbf{a}}$. Then by Lemma 9.4 we can chose the starting point of $\bar{\ell}_0$ to be any vertex in $\alpha_0 \cap N_{M_0}(A_{\mathbf{a}})$, and we have that α_0 , β_0 and $\bar{\ell}_0$ also form a semi-ideal geodesic triangle. Proposition 9.5 implies that $(\beta_0)_{r_1}$ is δ_1 -close to either a transition point of either α_0 or a transition point in $\bar{\ell}_0$.

If $(\beta_0)_{r_1}$ is δ_1 -close to any transition point of $\bar{\ell}_0$, then by Lemma 9.4, all transition points of $\bar{\ell}_0$ is δ_0 close to points of ℓ , we have that $(\beta_0)_{r_1}$ is $(\delta_1 + \delta_0)$ -close to points of ℓ in the metric of $\text{Cay}(G)$. This contradicts the choice of r_2 . Thus $(\beta_0)_{r_1}$ is only δ_1 -close to a transition point in α_0 . This implies that α_0 can be $(3, 0)$ -redirected to β_0 at radius $r_1/2 \geq r'$. Observe that $(3, 0) < f_{\mathbf{b}}((1, 0)) + (0, 1)$. Thus Lemma 8.17 implies that $\mathbf{a} \in \mathcal{U}(\mathbf{b}, r)$.

Case II. Suppose on the other hand \mathbf{b} is not transient. Let r_0 be such that $\beta|_{\geq r_0}$ is in a bounded neighbourhood of A and $\xi(\mathbf{b}) = *p_A$. Let

$$f_G(\mathbf{q}) := \max\{(9q, Q), f_A(\mathbf{q}) \forall A \in \mathcal{A}\}.$$

Such a maximum function exists because all A 's are translations of a finite set of subgroups. For a give $r > 0$ let r' be as in Lemma 8.17 and let

$$r'' := R_{\mathbf{b}}(f_{\mathbf{b}}(1, 0) + (0, 1), f_{\mathbf{b}}(\mathbf{q}_0), r').$$

$$r_1 = \max\{2r'', 2r_0\}.$$

Let

$$R \gg r_1 + \delta_0 + D(f_G(1, 0))$$

and let $W = B(\mathfrak{o}, R)$ be the ball of radius R in $\text{Cay}(G)$ centered at \mathfrak{o} . Again, W contains finite number of vertices. We now think of W as a subset of K .

Let $\xi(\mathbf{a}) \in m(\xi(\mathbf{b}), W)$ with α_0 as a central element. If $\xi(\mathbf{a}) = \xi(\mathbf{b})$, the case is trivial. Thus we assume that α_0 eventually leaves the mono-directional set A at a point x with a $(1, 0)$ -transition point for α_0 .

Since $\xi(\mathbf{a}) \in m(\xi(\mathbf{b}), W)$, there exists a geodesic ray or segment ℓ in K that connects $\xi(\mathbf{a})$ to $\xi(\mathbf{b})$ and is disjoint from W . Again since $\mathbf{b} \neq \mathbf{a}$, $|\ell| \geq 3$. Thus by Lemma 9.4 there exists geodesic line $\bar{\ell}_0$ whose projection to K is boundedly close to ℓ . In particular, we can choose for α_0 to start at a vertex $x' \neq x$ in A . Since α_0 and $\bar{\ell}_0$ are both geodesic lines connecting $\xi(\mathbf{b})$ to $\xi(\mathbf{a})$, by Lemma 8.4, there exists a $(1, 0)$ -transition point $p \in \bar{\ell}_0$ such that

$$d(p, x) \leq D(f_{\mathbf{a}}(1, 0)).$$

Furthermore, by Lemma 9.4,

$$d(p, \ell) \leq \delta_0.$$

Therefore,

$$d(x, \ell) \leq D(f_{\mathbf{a}}(1, 0)) + \delta_0.$$

Since ℓ avoid W , all points of x has Cay(G)-norm greater than R and thus

$$\|x\| \geq R - D(f_{\mathbf{a}}(1, 0)) + \delta_0 \geq R - D(f_G(1, 0)) + \delta_0 \geq r_1 \geq r''.$$

Since $x \in A$ there exists a \mathbf{q}_0 -ray in A which can be $f_{\mathbf{b}}(\mathbf{q}_0)$ redirected to β_0 after r'' . Thus α_0 can be $f_{\mathbf{b}}(\mathbf{q}_0)$ -redirected to β_0 at

$$r'' = R_{\mathbf{b}}(f_{\mathbf{b}}(1, 0) + (0, 1), f_{\mathbf{b}}(\mathbf{q}_0), r'),$$

thus α_0 can be $(f_{\mathbf{b}}(1, 0) + (0, 1))$ -redirected to β_0 at radius r' . Thus Lemma 8.17 implies that $\mathbf{a} \in \mathcal{U}(\mathbf{b}, r)$. And we are done. \square

The combination of all of the preceding results proves the homeomorphism:

Theorem 9.10. *Let G be a relatively hyperbolic group with respect to subgroups P_1, P_2, \dots, P_k . Assume that the Cayley graphs of the subgroups P_i 's are mono-directional sets, then the quasi-redirecting boundary ∂G is homeomorphic to $\partial_B G$.*

Proof. Since the map $\xi: \partial X \rightarrow \partial_B X$ is 1-1, onto, and both ξ and ξ^{-1} is continuous, we conclude that $\xi: \partial G \rightarrow \partial_B G$ is a homeomorphism. \square

10. THE HAIRY PARKING LOT

In this section we analyze the quasi-redirecting boundary of a metric space X which demonstrates how ∂X could be non-Hausdorff when the space is not an ATM space. The space X we construct here is a proper CAT(0) metric space and it is a modification of an example first analyzed by Cashen [Cas16]. We show that the partial relation in $P(X)$ is not symmetric which means ∂X is not Hausdorff. However, ∂X is still compact. We also use to this example to justify our definition of the topology for ∂X (see Subsection 10.1).

Let \mathbb{E}^2 be the Euclidean plane and $B(1)$ be the open ball of radius 1 around the origin. Let $Y = \mathbb{E}^2 - B(1)$. In the polar coordinate in \mathbb{E}^2 , we can write

$$Y = \{(\rho, \theta) \mid \rho \geq 1\}.$$

Let Z be the union of Y and an infinite number of rays, attached at a net of points in Y (say, the points in Y with integer Cartesian coordinates). Then Z resembles the plane with a hole in the center and infinitely many hairs attached. Let $X = \tilde{Z}$ be the universal cover of Z . We fix a lift of the point $(1, 0)$ in Z and denote it by \mathbf{o} . This defines a polar coordinate for $\tilde{Y} \subset X$ where $\theta \in (-\infty, \infty)$, $\rho \geq 1$ and $\mathbf{o} = (1, 0)$.

Note that X is CAT(0). We fix 3 geodesic rays in X (note that the first coordinate is the radius and the second coordinate is the angle):

$$\begin{aligned}\alpha_+ : \mathbb{R}_+ &\rightarrow X, & \alpha_+(t) &= (1, t) \\ \alpha_- : \mathbb{R}_+ &\rightarrow X, & \alpha_-(t) &= (1, -t) \\ \zeta : \mathbb{R}_+ &\rightarrow X, & \zeta(t) &= (t + 1, 0)\end{aligned}$$

Also, for every hair attached at the point (ρ, θ) (which we refer to as the (ρ, θ) -hair) we pick a quasi-geodesic $\gamma_{\rho, \theta}$ exiting this hair, namely,

$$\gamma_{\rho, \theta} : \mathbb{R}_+ \rightarrow X, \quad \gamma_{\rho, \theta}(t) = \begin{cases} (1, t) & t \in [0, \theta] \\ (t - \theta + 1, \theta) & t \in [\theta, \theta + \rho - 1] \\ \text{exits along the hair} & t \geq \theta + \rho - 1 \end{cases}.$$

This is a uniform quasi-geodesic ray and the geodesic ray exiting the (ρ, θ) -hair stays uniformly close to $\gamma_{\rho, \theta}$, however, $\gamma_{\rho, \theta}$ is easier to describe. Let $\mathbf{a}_\pm = [\alpha_\pm]$, $\mathbf{z} = [\zeta]$ and $\mathbf{c}_{\rho, \theta} = [\gamma_{\rho, \theta}]$.

We claim that

$$P(X) = \{\mathbf{a}_+, \mathbf{a}_-, \mathbf{z}\} \cup \{\mathbf{c}_{r, \theta}\}_{(r, \theta)\text{-hairs}}.$$

Also, $\mathbf{a}_\pm \preceq \mathbf{z}$ and otherwise, no other classes are comparable. To see this, we note that all quasi-geodesic rays that exit the (ρ, θ) -hair are in the same class (since the tails of these rays coincide) and are not comparable to other quasi-geodesics (since the hairs have only one point of contact with the rest of the space). We only need to show that every quasi-geodesic ray that stays in \tilde{Z} can be quasi-redirected to ζ . This can be done using a logarithmic spiral. We do this explicitly for the α_+ . For $T \geq 1$, define

$$(10) \quad \alpha_T : \mathbb{R}_+ \rightarrow X, \quad \alpha_T(t) = \begin{cases} \alpha_+(t) = (1, t) & t \in [0, T] \\ ((t - T + 1), T - \ln(t - T + 1)) & t \in [T, (e^T + T - 1)] \\ ((t - T + 1), 0) & t \in [(e^T + T - 1), \infty) \end{cases}.$$

To see that α_T is a quasi-geodesic, note that, for $t \in [T, (e^T + T - 1)]$, we have

$$(11) \quad \left\| \frac{d}{dt} \alpha_T(t) \right\| \leq \left| \frac{d\rho}{dt} \right| + \rho \left| \frac{d\theta}{dt} \right| = 1 + (t - T + 1) \cdot \frac{1}{(t - T + 1)} \leq 2.$$

For other values of t , $\|\dot{\alpha}_T(t)\| = 1$. To see the lower bound, we use the fact that

$$d((r, \theta), (r', \theta')) \geq \max(|r - r'|, |\theta - \theta'|) \geq \frac{1}{2}(|r - r'| + |\theta - \theta'|).$$

For example, we can estimate the distance between \mathfrak{o} and $\alpha_T(t)$, for $t \in [T, (e^T + T - 1)]$, by

$$(12) \quad d(\mathfrak{o}, \alpha_T(t)) \geq \frac{1}{2}((t - T) + (T - \ln(t - T + 1))) = \frac{1}{2}(t - \ln(t - T + 1)).$$

Recall that, for $A > 0$, we have (the right hand side is the equation of a tangent line to the graph of $y = \ln(x)$)

$$\ln(x) \leq \frac{x - A}{A} + \ln(A).$$

Therefore, setting $x = (t - T + 1)$ and $A = 2$, we have

$$(13) \quad t - \ln(t - T + 1) \geq t - \left(\frac{t - T - 1}{2} \right) - \ln(2) \geq \frac{t}{2} - \ln(2).$$

Combining (12) and (13) (and rounding up the additive error) we get

$$d(\mathbf{a}, \alpha_T(t)) \geq \frac{t}{4} - 1.$$

A similar calculation for other pairs of points shows that α_T is a $(4, 1)$ -ray.

Since the family α_T gives uniform quasi-redirections from α_+ to ζ at every radius, we can conclude $\mathbf{a}_+ \preceq \mathbf{z}$. The proof of $\mathbf{a}_- \preceq \mathbf{z}$ is similar. However, as it was noted in [Cas16], the geodesic α_+ is Morse. Hence, any quasi-geodesic with end points on α_+ stays in a bounded neighborhood of α_+ . Thus, ζ cannot be quasi-redirected to α_+ . The same also holds for α_- . Hence, \mathbf{z} , \mathbf{a}_+ and \mathbf{a}_- are distinct classes.

The same argument as above shows that every $\gamma_{\rho, \theta}$ can be $(4, 1)$ -redirected to ζ at radius comparable to $(\rho + \theta)$. That is, for a sequence $\mathbf{c}_{\rho_n, \theta_n}$, we have

$$\mathbf{c}_{\rho_n, \theta_n} \rightarrow \mathbf{z} \quad \iff \quad (\rho_n + \theta_n) \rightarrow \infty.$$

In fact, if $(\rho + \theta) \geq r$ then $\mathbf{c}_{\rho, \theta} \in \mathcal{U}(\zeta, r)$ and hence $P(X) - \mathcal{U}(\zeta, r)$ is finite. In particular, ∂X is compact.

10.1. Is there an Out-Topology? Our goal throughout this paper has been to define a simple natural analogue of Gromov boundary. The idea that $P(X)$ should be considered as the set of points at infinity seems quite natural to us since the directions that can be quasi-redirected to each other do not represent truly different directions. We then defined a notion of a cone topology on these classes essentially by saying that classes that are close to $\mathbf{a} \in P(X)$ are those that can still be redirected to \mathbf{a} at a large radius.

However, a priori, one could define the cone topology in the opposite way, namely, classes that are close to $\mathbf{a} \in P(X)$ are those that \mathbf{a} can be redirected to at a large radius. It seems like we have made an arbitrary choice here that needs justification. Why should a neighborhood be defined by quasi-redirection towards \mathbf{a} (the In-topology) as opposed to away from \mathbf{a} (the Out-topology)? Is there a well-defined notion of Out-topology?

To be more precise, let X be a metric space where the assumptions 0, 1 and 2 hold. We can attempt to define neighborhoods for the Out-topology as before: For $\mathbf{a} \in P(X)$, $r > 0$ and $F: [1, \infty) \times [0, \infty) \rightarrow [1, \infty) \times [0, \infty)$, define

$$\mathcal{U}_{\text{Out}}(\mathbf{a}, r, F) := \left\{ \mathbf{b} \in P(X) \cup X \mid \begin{array}{l} \text{every } \mathbf{q}\text{-ray } \alpha \in \mathbf{a} \text{ can be } F_{\mathbf{a}}(\mathbf{q})\text{-redirected} \\ \text{to central element } \beta_0 \in \mathbf{b} \text{ at radius } r. \end{array} \right\}$$

In Section 5, the choice of $F_{\mathbf{a}}$ was not important as long as $F_{\mathbf{a}}$ was large enough due to Lemma 5.7 which is why F is not included in the notation $\mathcal{U}(\mathbf{a}, r)$. However, the analogue of Lemma 5.7 does not hold for sets \mathcal{U}_{Out} .

To see this in the hairy parking lot example, we consider such neighborhoods around \mathbf{a}_+ for a given function F . It turns out, if we enlarge F slightly, say set $F' = F + (1, 0)$, then $\mathcal{U}_{\text{Out}}(\mathbf{a}_+, r, F)$ does not contain $\mathcal{U}_{\text{Out}}(\mathbf{a}_+, R, F')$ no matter how large R is.

In fact, let us analyze what subset of X can be reached via a (q, Q) -ray that matches α_+ up to a radius r . Of course, all point (ρ, θ) where $\theta \geq r$ can be reached. But to make θ smaller, we have to move away from the set $\{\rho = 1\}$ at a linear rate. We argue that the most

efficient way is along a log spiral as in (10). Consider, for $A > 0$ and $f: \mathbb{R}_+ \rightarrow \mathbb{R}_+$, the ray

$$(14) \quad \gamma: \mathbb{R}_+ \rightarrow X, \quad \gamma(t) = \begin{cases} \alpha_+(t) = (1, t) & t \in [0, r] \\ (A \cdot (t - r + 1), r - f(t - r)) & t \geq r \end{cases}.$$

This path follows α_+ up to a radius r and then moves away from the set $\{\rho = 1\}$ at a constant speed of A while reducing θ by a function f . Let us analyze how fast γ can unwind around α_+ (that is, how large f can be) and stay a quasi-geodesic. As in (11), for $t \geq r$, we have

$$\|\dot{\gamma}(t)\| \leq \left| \frac{d\rho}{dt} \right| + \rho \left| \frac{d\theta}{dt} \right| = A + (t - r + 1)f'(t - r).$$

If γ is a \mathfrak{q} -ray, $\|\dot{\gamma}(t)\|$ has to be bounded with a bound depending on \mathfrak{q} . That is, for a constants $B_{\mathfrak{q}}$ depending on \mathfrak{q} , we have

$$(f(0) = 0) \quad f'(t - r) \leq \frac{B_{\mathfrak{q}}}{t - r + 1} \implies f(t - r) \leq B_{\mathfrak{q}} \ln(t - r + 1).$$

Note that $B_{\mathfrak{q}}$ is increasing as a function of \mathfrak{q} .

If (θ, ρ) is a point on γ , for $\theta \in (-\infty, r]$, we have

$$\begin{aligned} \theta = r - f(t - r) \geq r - B_{\mathfrak{q}} \ln(t - r + 1) &\implies \ln(t - r + 1) \geq \frac{r - \theta}{B_{\mathfrak{q}}} \\ &\implies (t - r + 1) \geq e^{\frac{r - \theta}{B_{\mathfrak{q}}}}. \end{aligned}$$

and

$$(15) \quad \rho = A \cdot (t - r + 1) \geq A \cdot e^{\frac{r - \theta}{B_{\mathfrak{q}}}} \geq C_{r, \mathfrak{q}} \cdot e^{\frac{-\theta}{B_{\mathfrak{q}}}}.$$

For some function $C_{r, \mathfrak{q}}$ depending on r and \mathfrak{q} . Similarly, if γ' is a \mathfrak{q}' -ray that matches α_+ up to a radius R for any (ρ, θ) on such a geodesic we have

$$(16) \quad \rho \geq C_{R, \mathfrak{q}'} \cdot e^{\frac{-\theta}{B_{\mathfrak{q}'}}}.$$

If $\mathfrak{q} = (q, Q)$, $\mathfrak{q}' = (q', Q')$ and $q' > q$, we have $B_{\mathfrak{q}'} > B_{\mathfrak{q}}$. Therefore, no matter how large R is compared to r , as $\theta \rightarrow -\infty$,

$$C_{R, \mathfrak{q}'} \cdot e^{\frac{-\theta}{B_{\mathfrak{q}'}}} \leq C_{r, \mathfrak{q}} e^{\frac{-\theta}{B_{\mathfrak{q}}}}.$$

That is, the set of point that could be on γ' is not contained in the set of point that could be on γ and, for every $R > r$, we can find a point (ρ, θ) where (15) holds but (16) does not. If (ρ, θ) is any such point, then

$$\mathbf{c}_{\rho, \theta} \notin \mathcal{U}_{\text{Out}}(\mathbf{a}, r, F_{\mathbf{a}}) \quad \text{but} \quad \mathbf{c}_{\rho, \theta} \in \mathcal{U}_{\text{Out}}(\mathbf{a}, R, F').$$

This causes some complication with the definition of Out-topology. For the sets \mathcal{U}_{Out} to define a topology, in analogy with Lemma 5.4, we need to know that the following: There is a family of functions F_{\cdot} and, for every $r > 0$ and every $\mathbf{b} \in \mathcal{U}_{\text{Out}}(\mathbf{a}, r)$, there is $R > 0$ such that

$$\mathcal{U}_{\text{Out}}(\mathbf{b}, R, F_{\mathbf{b}}) \subset \mathcal{U}_{\text{Out}}(\mathbf{a}, r, F_{\mathbf{a}}).$$

The argument above can be modified to show that this does not hold in general for arbitrarily large functions F_{\cdot} . For example, we can set $\mathbf{a} = \mathbf{a}_+$, and choose $\gamma_{1, R}$ to be a hair attached to

α_+ at a distance R from \mathfrak{o} . Then $\gamma_{1,R} \in \mathcal{U}_{\text{Out}}(\mathfrak{a}_+, R)$, but if $F_{\mathfrak{c}_{1,R}}$ is even slightly larger than $F_{\mathfrak{a}}$, we have

$$\mathcal{U}_{\text{Out}}(\mathfrak{c}_{1,R}, R, F_{\mathfrak{c}_{1,R}}) \not\subset \mathcal{U}_{\text{Out}}(\mathfrak{a}, r, F_{\mathfrak{a}}).$$

That is, if we want to have a Out-Topology, we have to choose the functions F in a very delicate way and it is not clear if this is possible. And the Out-topology would change if you changed the functions $F_{\mathfrak{a}}$.

Another issue is that, even if it was possible to define an Out-topology, the restriction of the Out-topology to natural subspaces such as the sublinearly Morse boundary or even the contracting boundary (see [CM19]) does not match the previously defined topologies on these spaces. For the sake of brevity, we leave the details to reader except to mention that, similar calculations as above shows that, for every F , it is possible to find a sequence ρ_n, θ_n, R_n such that

$$R_n \rightarrow \infty, \quad \theta_n \rightarrow -\infty \quad \text{and} \quad \mathfrak{c}_{\rho_n, \theta_n} \in \mathcal{U}_{\text{Out}}(\alpha_+, R_n, F).$$

However, if $\theta_n \rightarrow -\infty$, the sequence $\mathfrak{c}_{\rho_n, \theta_n}$ does not converge to α_+ in the usual topology of the Morse-boundary (following either [CM19] or [QRT23]).

11. THE CROKE-KLEINER GROUP

In this section, we show that the hairy parking lot example (Section 10) is not an artificial example and indeed the same phenomena happens in a finitely presented group, in particular, in right angled Artin groups. As a special case, we closely examine the Croke-Kleiner Group acting on the universal cover of the Salvetti Complex (denoted, as usual, by X). Our main goal is to prove Theorem 11.1. The proof showcases the rich structure of space of quasi-geodesic rays in the Croke-Kleiner Group. We also give a classification of the set $P(X)$ by examining the growth rate of the *excursion* of geodesic rays (see Theorem 11.9). In the hairy parking lot example, every direction is either sublinearly Morse (in fact Morse) or non-Hausdorff point. However, it turns out that, in the Croke-Kleiner Group, there exists unexpected equivalence classes of directions that are rank-one and QI-invariant, but are not sublinearly Morse. In addition, these points are not the non-Hausdorff points of the space. This adds to the knowledge that rank-one is not a good predictor of QI-invariant, as a class of rank-one directions is exhibited to not be QI-invariant in [Qin16]. Lastly, this example shows that neither the Bass-Serre trees nor the hierarchically hyperbolic structures of such spaces is in general accurate in identifying QI-invariant directions at boundary at infinity.

In the interest of brevity, the exact calculation of the constants of quasi-geodesic rays constructed in this section are omitted since these calculations are standard and are similar to the arguments presented in the paper so far.

The Croke-Kleiner Group. For background on the Croke-Kleiner group, see [CK00, CK02] and [Qin16]. Here we follow the notation from [CK00]. The Croke-Kleiner group is a group G with the following presentation:

$$G = \langle a, b, c, d \mid [a, b], [b, c], [c, d] \rangle.$$

Consider a tori complex as follows.

Let \mathfrak{o} be a 0-cell. Let a, b, c, d be four oriented 1-cells, each an isometric copy of $[0, 1]$, whose boundary is \mathfrak{o} . Let S_1 be a torus that is an isometric copy of $[0, 1] \times [0, 1]$, where the opposite sides are glued together to the loops a and b . Likewise, let S_2 be a Euclidean square torus with 1-skeleton b, c and S_3 be a Euclidean square torus with 1-skeleton c, d . This

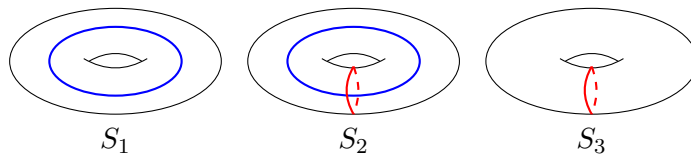


FIGURE 9. the tori are glued along the curves isometrically .

tori complex is the *Salvetti complex* associated with this group. The universal cover of the Salvetti complex is the main space we study in this section and we denote it by X . In this section, we show

Theorem 11.1. *The space X satisfies Assumptions 0-2 and hence ∂X is defined. The relation \preceq is not symmetric and $P(X)$ has a unique maximal element. Therefore, ∂X is not Hausdorff.*

Note that Assumption 0 holds by Lemma 2.5. To check Assumptions 1 and 2, we need to identify elements of $P(X)$, choose a geodesic representative α_0 in each class $\mathfrak{a} \in P(X)$ and show that every \mathfrak{q} -ray $\alpha \in \mathfrak{a}$ can be $f_{\mathfrak{a}}(\mathfrak{q})$ -redirected to α_0 for some function $f_{\mathfrak{a}}$ depending only on the class \mathfrak{a} .

Blocks and the Bass-Serre tree. A *block* in X is a convex infinite subset of X that is a lift of either $S_1 \cup S_2$, or $S_2 \cup S_3$. Thus a block is isometric to the universal cover of the Salvetti complex of either of the following groups

$$G_1 = \langle a, b, c \mid [a, b], [b, c] \rangle \quad \text{or} \quad G_2 = \langle b, c, d \mid [b, c], [c, d] \rangle.$$

In other words, each blocks comes with a co-compact action of a conjugate copy of either G_1 or G_2 . A *flat* in X is a lift of S_1, S_2 or S_3 . Given a pair of blocks, their intersection in X is either empty or a *flat* (in fact, always a lift of S_2). That is, a flat comes with a compact action of a conjugate of the group $\langle b, c \mid [b, c] \rangle = \mathbb{Z}^2$. We refer to these as *bc-flats*.

One can construct a graph where vertices are blocks and two vertices are connected by an edge if and only if two blocks intersect in a flat. The resulting graph, which we denote by T , is the Bass-Serre tree associated with amalgamated product decomposition of G :

$$G = G_1 *_{\langle b, c \mid [b, c] \rangle} G_2.$$

For $x \in X$, the *a-axis* passing through x is the bi-infinite geodesic ray passing through points $x \cdot a^n$, for $n \in \mathbb{Z}$. The *b-axis*, *c-axis* and *d-axis* are similarly defined.

Note that blocks have a product structure. A G_1 -block containing a point $x \in X$ is a Euclidean product of the tree generated by $\langle a, c \rangle$ and the *b-axis* centered at the point x . Similarly, a G_2 -block containing $x \in X$ is a Euclidean product of the tree generated by $\langle b, d \rangle$ and the *c-axis* centered at the point x . We can use this product structure to quasi-redirect quasi-geodesics to each other. For example, the arguments in Proposition 4.2 can be used to prove the following lemma.

Lemma 11.2. *For every $\mathfrak{q} \in [1, \infty] \times [0, \infty)$ and $\rho > 0$, there is a $\mathfrak{q}' \in [1, \infty] \times [0, \infty)$ such that the following holds. Let B be a block, $R \geq (1 + \rho) \cdot r > 0$ be a pair of radii and α and β be two \mathfrak{q} -rays. Assume $\alpha_r \in B$ and that $\beta|_{\geq R}$ starts at a point in B . Then, α can be \mathfrak{q}' -redirected to β at radius r .*

That is, we can transition from $\alpha|_r$ to $\beta|_{\geq R}$ as long as there is buffer between them that has a product structure and a thickness that is a linear function of r .

The unique maximal element. We now start analyzing the set $P(X)$. Let \mathfrak{o} also denote the point in X associated with the identity element in G . Let $\zeta: \mathbb{R}_+ \rightarrow X$ be a geodesic ray in X such that, for a positive integer n ,

$$\zeta(n) = \mathfrak{o} \cdot b^n.$$

That is, ζ follows the positive b -axis passing through \mathfrak{o} . We start by showing that $\mathbf{z} = [\zeta]$ is the unique maximal element in $P(X)$. The choice of b here is arbitrary. The geodesic ζ_c following the c -axis is in the same block at ζ and hence $\zeta_c \in \mathbf{z}$. Similarly, if ζ_a and ζ_d are geodesics following the a -axis and the d -axis respectively, we have $\zeta_a \in [\zeta]$ since ζ and ζ_a are in the same block and $\zeta_d \in [\zeta_c] = [\zeta]$ since ζ_c and ζ_d are in the same block. Recall that \mathfrak{q} -rays of X are assumed to be a continuous and emanating from \mathfrak{o} .

Proposition 11.3. *Let α be a \mathfrak{q} -ray in X . Then α can be \mathfrak{q}' -redirected to ζ where \mathfrak{q}' depends only on \mathfrak{q} . In particular, $\alpha \preceq \zeta$.*

Proof. After perturbing α by a bounded amount, we can assume that α lies in the 1-skeleton of the universal cover of the Salvetti complex. In fact, we assume, there is a sequence $s_n \in \{a, b, c, d\}$, for $n \in \mathbb{N}$, such that $\alpha(n) = s_1 \dots s_n$. Let $r > 0$ be given. We construct a quasi-geodesic ray redirecting α to ζ at radius r .

Using Lemma 11.2, we can find a quasi-geodesic ray α' , with constants depending only on \mathfrak{q} , where $\alpha_r = \alpha'_r$ and such that α' eventually follows some b -axis. That is, there is $T > 0$, such that for $t \geq T$, we have $\alpha'(t+1) = \alpha'(t) \cdot b$.

Again, we can assume that α' lies in the 1-skeleton of the universal cover of the Salvetti complex. We now construct a spiral quasi-geodesic redirecting α' to ζ . This is, in spirit, the same idea as Equation (14). Meaning, we try to backtrack along the segment $\alpha'[0, T]$, however, to stay a quasi-geodesic, we move exponentially far away from α' in each block. To be more precise, let

$$\alpha'(T) = v_1 w_1 \dots v_k w_k, \quad \text{where } v_i \notin G_2 \text{ and } w_i \notin G_1.$$

It is possible that we have to start with a word in G_2 or end with a word in G_1 , but in these cases the construction is similar. For $t \in [0, 2T]$, we define $\gamma(t) = \alpha'(t)$. That is,

$$\gamma(2T) = v_1 w_1 \dots v_k w_k \cdot b^T.$$

We start by moving along the c -axis to

$$\gamma(4T) = v_1 w_1 \dots v_k w_k \cdot b^T \cdot c^{2T} = v_1 w_1 \dots v_k \cdot c^{2T} \cdot w_k b^T.$$

This is still a quasi-geodesic. At the worst case moving along the c -axis cancels with w_k . But the presence of b^T in the end ensures that, any point along this segment is at least a distance T from $\alpha'[0, T]$. Now we can undo $w_k b^T$ to get:

$$\gamma(5T + |w_k|) = v_1 w_1 \dots v_k \cdot c^{2T}.$$

This is also a quasi-geodesic since the distance between this segment and $\gamma[0, 2T]$ is at least T . We can now proceed in this way: first we add a large power of b to get

$$\gamma(9T + |w_k|) = v_1 w_1 \dots w_{k-1} v_k \cdot c^{2T} \cdot b^{4T} = v_1 w_1 \dots w_{k-1} \cdot b^{4T} \cdot v_k c^{2T},$$

and then we undo $v_k c^{2T}$ to get

$$\gamma(11T + |w_k| + |v_k|) = v_1 w_1 \dots w_{k-1} \cdot b^{4T}.$$

As before, the large power of b ensures that this path is still a quasi-geodesic. We can let $T_1 = 11T + |w_k| + |v_k|$ and proceed inductively. To establish the pattern, we repeat the above procedure one more time. We will add c^{T_1} and undo $w_{k-1} b^{4T}$ to reach

$$\gamma(2T_1 + 4T + |w_{k-1}|) = v_1 w_1 \dots w_{k-2} v_{k-1} \cdot c^{T_1}.$$

We then add b^{2T_1} and undo $v_{k-1} \cdot c^{T_1}$ to reach

$$\gamma(5T_1 + 4T + |w_{k-1}| + |v_{k-1}|) = v_1 w_1 \dots w_{k-2} \cdot b^{2T_1}.$$

This is the same at multiplying $\gamma(T_1)$ on the right by

$$c^{T_1} \cdot b^{-4T} \cdot w_{k-1}^{-1} \cdot b^{2T_1} \cdot c^{-T_1} \cdot v_{k-1}^{-1}$$

one letter at a time. Note that the powers of b do not cancel since $w_k \notin G_1$. And then we set

$$T_2 = 5T_1 + 4T + |w_{k-1}| + |v_{k-1}|$$

and proceed as before. When all v_i and w_i disappear, we have $\gamma(T_k)$ is on ζ , and we can continue along ζ for $t \geq T_k$. \square

Recurrent quasi-geodesic rays. We define a map $\Psi_\alpha: \mathbb{R}_+ \rightarrow T$ (recall that T is the Bass-Serre tree) as follows: Let $A_0 \in T$ be the G_1 -blocks containing \mathfrak{o} . Let the $u_1 > 0$ be the supremum of times t such that $\alpha(t) \in A_0$ and let $A_1 \neq A_0$ be the block α enters immediately after it exits A_0 . We now define u_i and A_i inductively. Let $u_i > u_{i-1}$ be the supremum of times t such that $\alpha(t)$ is contained in the block A_{i-1} and let $A_i \neq A_{i-1}$ be the block α enters immediately afterwards. Now define

$$\Psi_\alpha(t) = A_i \quad \text{for} \quad t \in [u_i, u_{i+1}).$$

This is a quasi-Lipschitz map where constants depend on \mathfrak{q} .

It is possible that $u_i = \infty$ for some i . This means that α visits the block A_i infinitely many times. In this case, we say α is *recurrent*. Otherwise, we say α is *transient*.

Lemma 11.4. *If α is recurrent, then $\alpha \in \mathfrak{z}$.*

Proof. Let A_i be the terminal block for α . Let β be a quasi-geodesic that eventually stays in B following the b -axis. Lemma 11.2 implies that $\beta \preceq \alpha$. Also, we can use the argument of Proposition 11.3 to build a spiral from ζ to β . That is, $\zeta \preceq \beta$. Hence, $\zeta \preceq \alpha$. But we already know $\alpha \preceq \zeta$ by Proposition 11.3. This finishes the proof. \square

Geodesic representatives in each class. Now assume α is not recurrent. Note that the sequence A_i is an embedded path in T , that is, a sequence of vertices where A_i is adjacent to A_{i+1} without repeating. Since T is a tree, this path is a geodesic in T limiting to some end ξ of T . By [CK02, Corollary 5.26], there is at least one geodesic ray $\alpha_0 \in X$ whose itinerary is the sequence A_i . We show that α_0 can always be redirected to α . First, we recall the following lemma that follows from [Qin16, Proposition 3.10] and [CK02, Section 4.1].

Lemma 11.5. [Qin16, CK02] *Let x, y be two points in A_{i+2} , let $[\mathfrak{o}, x']$ and $[\mathfrak{o}, y']$ be subsegments of geodesic segments $[\mathfrak{o}, x]$ and $[\mathfrak{o}, y]$ such that $x', y' \in A_i$. Let $[\mathfrak{o}, z]$ be the intersection $[\mathfrak{o}, x'] \cap [\mathfrak{o}, y']$ (possibly $z = \mathfrak{o}$). Then, there is an embedded Euclidean triangle in X where the edges are the geodesic segments $[z, x']$, $[z, y']$ and $[x', y']$.*

Lemma 11.6. *For α and α_0 as above, we have $\alpha_0 \preceq \alpha$. In particular, all geodesics with itinerary A_i are in the same class.*

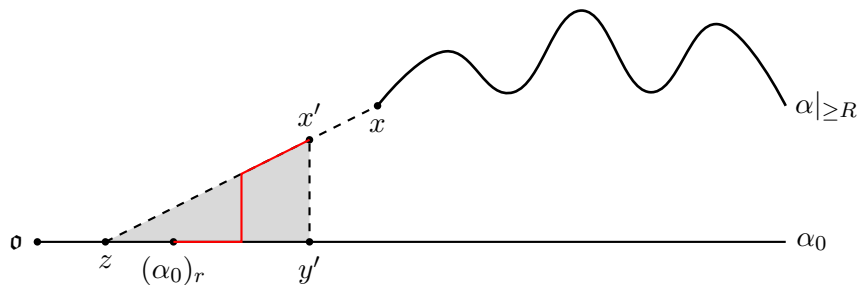


FIGURE 10. α_0 can be redirected to α at radius r .

Proof. Pick $r > 0$. Let $(\alpha_0)_r$ be in A_i . We choose $R > 2r$ and $j > i + 2$ such that $\alpha|_{\geq R}$ starts at a point $x \in A_j$ and there is a point y' on $\alpha_0 \cap A_{j-2}$ with $\|y'\| \geq 2r$. It follows from Part (I) of Lemma 2.6 that the concatenation $\gamma = [o, x] \cup \alpha|_{\geq R}$ is a $(3q, Q)$ -quasi-geodesic. Also, by Lemma 11.5, for a point $x' \in [o, x] \cap A_{j-2}$, there is a flat triangle containing points z, x' and y' . If $\|z\| \geq r$, then γ already gives a redirection from α_0 to α at radius r . Otherwise, the Euclidean triangle can be used, similar to Theorem 4.2, to redirect α_0 to γ at radius r . (See Figure 10). This finishes the proof. \square

Proposition 11.7. *For every q , there is q' such that the following holds. Assume α is a q -ray that is transient with itinerary A_i and let α_0 be a geodesic with the same itinerary. Then either $\alpha \in \mathbf{z}$ or α can be q' -redirected to α_0 .*

Proof. We use Part (II) of Lemma 2.6. If there is a sequence of radii $r_i \rightarrow \infty$ such that $d_X((\alpha_0)_{r_i}, \alpha) \leq r_i/2$, then γ can be $(9q, Q)$ -redirected to α_0 and we are done. Otherwise, there is r_0 such that, for every $r \geq r_0$,

$$(17) \quad d_X((\alpha_0)_r, \alpha) \geq r/2.$$

We show that, in this case, ζ can be redirected to α . Since α can always be redirected to ζ , this implies that $\alpha \in \mathbf{z}$.

By [CK02, Section 4.1], the itinerary of α_0 determines the combinatorial path. Using the language of [CK02], let x_k be the points on the bc -flat shared by A_k and A_{k+1} that determines the quarter-planes on the walls. Let $w_k \in G$ be the word such that $x_k \cdot w_k = x_{k+1}$. Then the concatenation of words w_k is a quasi-geodesic rays α_1 with uniform constants q_0 and with the same itinerary as α_0 . That is, we can write the q_0 -ray α_1 as an infinite sequence

$$w_1 w_2 w_3 \dots \quad \text{where} \quad w_{2i+1} \notin G_2 \quad \text{and} \quad w_{2i} \notin G_1,$$

or perhaps we start with G_2 instead of G_1 . Since we assume in this paper without loss of generality that quasi-geodesics are continuous, it follows from the template structure of X that the entry point of α into A_k is aligned with x_k either along the b -axis of the c -axis. That is, there are powers $p_k \in \mathbb{Z}$ such that

$$\alpha(u_k) = x_k \cdot b^{p_k} \quad \text{for even } k \quad \text{and} \quad \alpha(u_k) = x_k \cdot c^{p_k} \quad \text{for odd } k,$$

or, again, with opposite parity.

Let $\alpha_1^{A_i}$ be the point where α_1 enters A_i , note that if a \mathfrak{q} -ray satisfies Equation (17), p_k is comparable and differ by a uniform multiplicative constant to the $d(\alpha_1^{A_k}, \alpha(u_k))$. Thus we observe that p_k (powers of b or c) comparable and differ by a uniform multiplicative constant to u_k . Recall also that from $\alpha(u_k)$ to $\alpha(u_{k+1})$, the path α has to replace the powers of b with the powers of c (or vice versa) and also it has to travel along w_k . Therefore, there is a constant $\rho_0 > 0$ depending on \mathfrak{q} such that, for $k > 0$ large enough,

$$(18) \quad u_{k+1} - u_k \geq \rho_0 \cdot (u_k + |w_k|).$$

Let $r > 0$ be given. We attempt to build a spiral quasi-geodesic γ chasing after α following the outline of the proof of Proposition 11.3 . We build the quasi-geodesic γ inductively. For $t \in [0, r]$, define $\gamma(t) = \zeta(t) = b^t$. Then we proceed exactly as in the proof of Proposition 11.3 trying to get to the block A_i as fast possible while staying a quasi-geodesic. To accomplish this, the length of time γ spends in each A_i has to grow exponentially. However, we do not have to double the length each time; the powers of b and c in the proof of Proposition 11.3 can grow as a slow exponential function. And how slow these powers grow will determine the quasi-geodesic constants of γ .

That is, for every $\rho > 0$, there is \mathfrak{q}_ρ , such that the amount of time γ stays in each A_i grows by a factor of $(1 + \rho)$ only, and then we can proceed along the word w_i with a speed of $1/\rho$. Here $\mathfrak{q}_\rho \rightarrow \infty$ as $\rho \rightarrow 0$. However, for a fixed ρ , we can build a spiral that visits the block A_k at time t_k where

$$(19) \quad t_0 = r \quad \text{and} \quad t_{k+1} \leq (1 + \rho) \cdot t_k + \rho \cdot |w_k|.$$

If we choose $\rho < \rho_0$, Equations (18) and (19) imply the sequence t_k grows more slowly than the sequence u_k . That is, for k large enough

$$\|\alpha(u_{k+1})\| \geq 2\|\gamma(t_k)\|.$$

Now Lemma 11.2 implies that γ can be redirected to α . Since this is true for every r , ζ can be redirected to α and, in view of Proposition 11.3 , $\alpha \in \mathbf{z}$. \square

Corollary 11.8. *For every quasi-geodesic ray α , the class $[\alpha]$ contains a geodesic representative and α can be uniformly redirected to this geodesic.*

Proof. Every quasi-geodesic can be uniformly quasi-redirected to ζ . Hence, the Corollary holds if $\alpha \in \mathbf{z}$. Otherwise, by Lemma 11.6, $\alpha_0 \preceq \alpha$ and by Proposition 11.7 α can be uniformly quasi-redirected to α_0 . That is, $[\alpha_0] = [\alpha]$ and the corollary holds. \square

Excursion. For a transient quasi-geodesic ray α , whether or not ζ can be quasi-redirected to α depends the excursion function of α , namely, the amount of progress α makes in each block. We now make this precise. We work with the combinatorial path introduces in the previous section. That is, we assume α_1 is an infinite sequence

$$w_1 w_2 w_3 \dots \quad \text{where} \quad w_{2i+1} \notin G_2 \quad \text{and} \quad w_{2i} \notin G_2.$$

We say the excursion is *sublinear with respect to the distance in X* if

$$\lim_{i \rightarrow \infty} \frac{|w_k|}{\sum_{i=1}^{k-1} |w_i|} = 0.$$

In [QRT22, Theorem A.12], these class of rays were studied and it was proven than they are sublinearly Morse. As we saw in Section 6, every element of a sublinearly Morse class $[\alpha_1]$ has to fellow travel α_1 sublinearly. Therefore, for such quasi-geodesics, $[\alpha_1] \neq \mathbf{z}$.

Proof of Theorem 11.1. Assumption 0 follows from Lemma 2.5 and Assumption 1 and 2 follow from Corollary 11.8. By Proposition 11.3, \mathbf{z} is the unique maximal element in $P(X)$. And since the sublinearly Morse direction are different from \mathbf{z} , $P(X)$ has more than one point. That is \preceq is not symmetric and, by Theorem 7.3, ∂X is not Hausdorff. \square

However, it turns out that sublinear excursion with respect to the distance in X does not give a characterization of directions that are different from \mathbf{z} . Indeed, we say the excursion of α_1 is *sub-exponential with respect to the distance in T* if

$$\lim_{i \rightarrow \infty} \frac{\log |w_i|}{i} = 0.$$

Theorem 11.9. *For a transient quasi-geodesic ray α_1 , $[\alpha_1] \neq \mathbf{z}$ if and only if the excursion of α_1 is sub-exponential with respect to the distance in T .*

Proof. First we show that if the excursion of α_1 is sub-exponential with respect to the distance in T then $[\alpha_1] \neq \mathbf{z}$. We need to show that ζ cannot be quasi-redirected to α_1 . That is, for every \mathbf{q} , there is an $r > 0$ such that there is no \mathbf{q} -ray γ with $\gamma|_r = \zeta|_r$ that is eventually equal to α_1 .

Assume, for contradiction that such γ exists for every r . The proof is similar to Proposition 11.7 with γ in this proof playing the role of α in the previous one. As before, let x_k be the points on the bc -flat shared by A_k and A_{k+1} that determines the quarter-planes on the walls. In particular $x_{k+1} = x_k \cdot w_k$. Let t_k be the first time γ enters the bc -flat containing x_k and let $\ell_k = d(\gamma(t_k), x_k)$. Then $\gamma(t_k) \cdot x_k^{-1}$ is either a power b or a power of c depending on parity. That is, if the difference is a power of b for x_k then $\gamma(t_{k+1}) \cdot x_{k+1}^{-1}$ is a power of c and vice versa. Hence, during the interval $[t_k, t_{k+1}]$, the path γ has to add ℓ_{k+1} powers of c , undo ℓ_k powers of b and travel along w_k . Since γ is a \mathbf{q} -ray, there is $\rho_{\mathbf{q}} > 0$ depending on \mathbf{q} such that

$$(20) \quad t_0 = r \quad \text{and} \quad t_{k+1} - t_k \geq \rho_{\mathbf{q}} \cdot (\ell_k + \ell_{k+1} + |w_k|).$$

Another way to travel to $\gamma(t_k)$ is to go along the segments w_i to x_k and then a distance of ℓ_k in the bc -plane containing x_k . Again, since γ is a \mathbf{q} -ray, we have

$$(21) \quad \left(\ell_k + \sum_{i=1}^{k-1} |w_i| \right) \geq \rho_{\mathbf{q}} \cdot t_k.$$

Let

$$\rho_1 = \frac{\rho_{\mathbf{q}}}{2} \quad \text{and} \quad 0 < \rho_0 < \rho_1.$$

We will show by induction that, for r large enough,

$$(22) \quad t_k \geq r \cdot (1 + \rho_1)^k \quad \text{and} \quad \ell_k \geq r \cdot \frac{\rho_{\mathbf{q}}}{2} \cdot (1 + \rho_1)^k.$$

The base case follows immediately from (20) and (21). Assuming these inequalities for k , we have (from (20)) that

$$t_{k+1} \geq t_k + \rho_{\mathbf{q}} \cdot \ell_k \geq r \cdot (1 + \rho_1)^k + r \cdot \frac{\rho_{\mathbf{q}}^2}{2} \cdot (1 + \rho_1)^k = r \cdot (1 + \rho_1)^{k+1}.$$

On the other hand, since the excursion of α_1 is sub-exponential with respect to the distance in T , there is C depending on ρ_0 such that $|w_i| \leq C \cdot (1 + \rho_0)^i$. Hence,

$$\sum_{i=1}^{k-1} |w_i| \leq C \sum_{i=1}^{k-1} \frac{(1 + \rho_0)^k - 1}{(1 + \rho_0) - 1} \leq \frac{C}{\rho_0} \cdot (1 + \rho_0)^k.$$

Choose r large enough such that

$$\frac{r \cdot \rho_{\mathfrak{q}}}{2} \geq \frac{C}{\rho_0}.$$

Then, by (21), we have

$$\ell_{k+1} \geq \rho_{\mathfrak{q}} \cdot t_{k+1} - \sum_{i=1}^k |w_i| \geq \rho_{\mathfrak{q}} \cdot r \cdot (1 + \rho_1)^{k+1} - \frac{C}{\rho_0} \cdot (1 + \rho_0)^{k+1} \geq r \cdot \frac{\rho_{\mathfrak{q}}}{2} \cdot (1 + \rho_1)^{k+1}.$$

This finishes the proof of (22). Since $\rho_0 < \rho_1$, this implies that γ arrives in A_i long after α_1 has left A_i and the distance between γ and α_1 goes to infinity (instead of zero). Thus, there does not exist a \mathfrak{q} -ray redirecting ζ to α_1 . But this holds for every \mathfrak{q} . Hence, $\alpha_1 \notin \mathbf{z}$.

We now prove the other direction. Assume that the excursion of α_1 is not sub-exponential with respect to the distance in T . This implies that

$$(23) \quad \exists \rho > 0 \quad \forall r > 0 \quad \exists k > 0 \quad \text{such that} \quad |w_k| \geq r \cdot (1 + \rho)^k.$$

Otherwise,

$$\forall \rho > 0 \quad \exists r > 0 \quad \forall k > 0 \quad \text{we have} \quad |w_k| \leq r \cdot \rho \cdot (1 + \rho)^k,$$

which implies

$$\forall \rho > 0 \quad \lim_{k \rightarrow \infty} \frac{\log |w_k|}{k} \leq \rho \quad \implies \quad \lim_{k \rightarrow \infty} \frac{\log |w_k|}{k} = 0.$$

Let ρ be as in (23) and $r > 0$ be given. We construct a \mathfrak{q} -ray γ (where \mathfrak{q} depends on ρ) that quasi-redirect ζ to α_1 at radius r . Let k be the first index where the inequality in (23) holds. That is,

$$|w_k| \geq r \cdot (1 + \rho)^k \quad \text{and} \quad |w_i| < r \cdot (1 + \rho)^i, \quad \text{for} \quad 1 \leq i < k.$$

Set $t_0 = r$ and, for $i = 1, \dots, (k-1)$ set

$$t_i = r \cdot (1 + \rho)^i \quad \text{and} \quad \ell_i = r \cdot \rho \cdot (1 + \rho)^i$$

Then, assuming ρ is small enough such that

$$\rho + \rho(1 + \rho) + 1 \leq 2,$$

we have

$$\begin{aligned} t_i + \frac{\rho}{2}(\ell_i + \ell_{i+1} + |w_i|) &\leq r \cdot (1 + \rho)^i + r \cdot \frac{\rho}{2} \left(\rho \cdot (1 + \rho)^i + \rho \cdot (1 + \rho)^{i+1} + (1 + \rho)^i \right) \\ &\leq r \cdot (1 + \rho)^i + r \cdot \rho \cdot (1 + 4\rho)^i = t_{i+1}. \end{aligned}$$

Therefore, for an appropriate \mathfrak{q} depending on ρ , there is enough time for a \mathfrak{q} -ray γ that starts a distance ℓ_i from x_i to reach a point that is distance ℓ_{i+1} from x_{i+1} while maintaining a distance ℓ_i from the α_1 . Since $\ell_i \geq \rho t_i$, this path is indeed a quasi-geodesic. Note that $\rho/2$ is playing the role of $\rho_{\mathfrak{q}}$ in (20).

We have shown that there exists a quasi-geodesic ray γ where $\gamma|_r = \zeta|_r$ such that γ that reached A_k at time t_k and a distance from x_k is

$$\ell_k = r \cdot \rho \cdot (1 + \rho)^k < r \cdot (1 + \rho)^k = |w_k|.$$

Hence, by Lemma 11.2, we can connect γ to x_{k+1} and continue along α_1 while staying a quasi-geodesic. This gives a redirection from ζ to α_1 at radius r .

Since this holds for every r , we can conclude that if the excursion of α_1 is not sub-exponential with respect to the distance in T then $\alpha_1 \in \mathbf{z}$. \square

Description of $P(X)$. Let us review what we know about $P(X)$ so far. To every transient quasi-geodesic ray α , we can associated an itinerary A_i , which is an infinite embedded path in T exiting an end ξ . Given such ξ , there is a preferred quasi-geodesic α_1 exiting ξ that passes through the points x_k . We set $w_k = x_{k+1} \cdot x_k^{-1}$ and refer to $|w_k|$ as the excursion of α_1 in the block A_i . Then $[\alpha_1]$ is different from \mathbf{z} if and only if the excursion of α_1 is sub-exponential with respect to the distance in T . That is, every class in $P(X)$ is either \mathbf{z} or $[\alpha_1]$ for such α_1 . To finish the description of $P(X)$ we need to show all these classes are different.

Lemma 11.10. *Let α and β be transient quasi-geodesic rays with different itineraries. If $\beta \preceq \alpha$ then $\zeta \preceq \alpha$.*

Proof. Let A_i be the itinerary for α , B_i be the itinerary for β and let k be the largest index where $A_k = B_k$. Let $r > 0$ be given. Consider a quasi-geodesic ray γ constructed as before such that $\gamma|_r = \zeta|_r$ and γ follows the itinerary A_i . Let t_k be the first time $\gamma(t_k) \in A_k$ and $\ell_k = d(\gamma(t_k), x_k)$. Now choose $R \gg \ell_k$ and consider a quasi-geodesic β' quasi-redirecting β to α at radius R . Then β' arrives at and leaves A_k much later than γ . Hence, by Lemma 11.2, we can redirect γ to β' , that is, construct a quasi-geodesic ray γ' where $\gamma[0, t_k] = \gamma'[0, t_k]$ and γ' is eventually equal to β' . Since β' is eventually equal to α , γ' quasi-redirects ζ to α at radius r . This can be done for every r with uniform constants. Hence $\zeta \preceq \alpha$. \square

Corollary 11.11. *If α_1 and β_1 are transient quasi-geodesic rays whose excursion is sub-exponential with respect to distance in T , then $[\alpha_1] \neq [\beta_1]$.*

Proof. If $\alpha \preceq \beta_1$, then by Lemma 11.10, $\zeta \prec \alpha_1$ which implies $[\alpha_1] = \mathbf{z}$. But this contradicts Theorem 11.9. \square

Therefore, we can think of $P(X)$ as a quotient of ∂T where all the direction whose excursion is not sub-exponential with respect to distance in T are collapsed into one point ζ .

An enlargement of the sublinearly Morse boundary. Finally, we check that not all directions are sublinearly Morse. Choose a sequence $\rho_n > 0$ such that $\rho_n \rightarrow 0$ as $n \rightarrow \infty$. Fix a constant $C \geq 1$ and, for each n , choose a power k_n large enough such that

$$(24) \quad (1 + \rho_n)^{k_n} \geq C k_n + \sum_{m=1}^{n-1} (1 + \rho_m)^{k_m}.$$

Then construct a quasi-geodesic α_1 as before where

$$|w_i| = \begin{cases} (1 + \rho_n)^{k_n} & \text{if } i = k_n \text{ for some } n \\ C & \text{otherwise} \end{cases}.$$

Equation (24) implies that, for every n ,

$$\frac{|w_{k_n}|}{\sum_{i=1}^{k_n-1} |w_i|} \geq 1,$$

meaning, the excursion is not sublinear with respect to distance in X . Hence α_1 is not sublinearly Morse. However, since $\rho_n \rightarrow 0$, we have

$$\frac{\log |w_i|}{i} \leq \rho_n \quad \text{for } i \geq k_n \quad \implies \quad \lim_{i \rightarrow \infty} \frac{\log |w_i|}{i} = 0.$$

That is, the excursion is sub-exponential with respect to distance in T . These seem to be interesting new classes of quasi-geodesic rays that warrant more study.

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