SUBLINEAR BILIPSCHITZ EQUIVALENCE AND SUBLINEARLY MORSE BOUNDARY

GABRIEL PALLIER AND YULAN QING

Abstract. A sublinearly biLipschitz equivalence (SBE) between metric spaces is a map from one space to another that distort distances with bounded multiplicative constants and sublinear additive error. Sublinear Morse boundaries are defined for all proper metric spaces as a quasi-isometrically invariant and metrizable topological set of quasi-geodesic rays. In this paper, we show that κ-Morse quasi-geodesics are mapped to uniformly sublinear neighbourhoods of κ-Morse geodesic rays. Thus a κ boundary of a proper metric space is invariant under any SBE. As an application we show that with mild assumptions random walks on countable groups are (q, θ)-rays.

1. Introduction

Sublinear biLipschitz equivalence between metric spaces is a generalization of quasiisometry that is natural with respect to asymptotic cones; it appeared first without a name, and then more explicitely in the work of Cornulier on asymptotic cones of Lie groups [Cor08, Cor11], before being studied for its own sake [Cor19, Pal20]. Following Cornulier we will abbreviate this relation as SBE. Because Gromov hyperbolicity admits a characterization in terms of asymptotic cones, it is an SBE-invariant among compactly generated locally compact groups; this was noted by Cornulier [Cor08, Theorem 4.3].

In this project we extend this result to groups and spaces beyond the ones that are Gromov hyperbolic. Precisely, this generalization is made in the following sense. Given a proper geodesic space that is not strictly Gromov hyperbolic, one can study its large scale hyperbolicity-like structure by describing the *sublinearly Morse boundary* of the group [QRT20]. The latter is a topological space collecting a large set of geodesic rays behaving in a hyperbolic fashion; when the space is Gromov-hyperbolic, this is simply the Gromov boundary. In this paper we show that this set of directions is invariant under sublinear bilipschitz equivalence. We then go further to obtain a simultaneous generalization of two distinct previous results in the litterature :

(1) (See §1.1 below) Qing, Rafi and Tiozzo's theorem that the homeomorphism type of the sublinearly Morse boundary is a quasi-isometry invariant among proper geodesic metric spaces [QRT20, Theorem A (2)].
(2) (See §1.2 below) Cornulier's theorem that the homeomorphism type of the Gromov boundary is SBE invariant among Gromov-hyperbolic groups [Cor19].

We recall some context on these two theorems in the following.

1.1. Boundaries.

In his seminal article [Gro87], Gromov introduced the class of hyperbolic groups and attached to such groups an equivariant bordification, now called the Gromov boundary. The class of Gromov-hyperbolic group is closed under quasiisometry, and the quasiisometries extend equivariantly to the Gromov boundaries.

Date: August 15, 2022.

The class of Gromov hyperbolic groups and spaces is however not vast enough to include natural examples such as CAT(0) groups, or the mapping class groups of surfaces of finite type, despite the fact that these exhibit some of the symptoms of nonpositive or negative curvature; for instance, their asymptotic cones are CAT(0) spaces for some word metric. For these, the visual boundary (the set of all geodesic rays emanating from a fixed base-point $(X, \mathfrak{o})$, up to fellow travel) does not provide a good large-scale invariant (as indicated by the works from Croke-Kleiner [CK00] to Qing [Qin16]). In [QRT19], the second named author and Rafi consider the set of quasi-geodesic rays whose Morse property is weakened compared to that of geodesic rays in Gromov spaces. In particular, given a sublinear function κ, Qing-Rafi define a quasi-geodesic ray γ to be *sublinearly κ-Morse* if any other geodesic segment with endpoints on γ is uniformly κ-close to γ, i.e. their distances to γ is bounded above by $n(\mathsf{q}, \mathsf{Q})\kappa(\|x\|)$, where $n(\mathsf{q}, \mathsf{Q})$ is a constant depends only on the quasi-geodesic constants (q, Q) of the segment, and the distance of each point on the segment to the origin, $\|x\| = d(\mathfrak{o}, x)$. The collection of all such quasi-geodesic rays, together with a coarse cone topology, is refered to as the κ boundary of X, and denoted $\partial_\kappa X$. These boundaries are shown to be a quasi-isometrically invariant topological space for all proper geodesic spaces [QRT20]. Therefore one can denote a κ-boundary of a group with $\partial_\kappa G$. Furthermore, they are metrizable topological spaces ([QRT20]). Since its introduction, sublinearly Morse boundaries are studied and compared to Gromov boundaries in various ways, such as via visibility, divergence and contracting properties (See[MQZ21], [IZ] and [Zal21]).

One important application of the sublinear boundaries is that, for appropriately chosen κ, $\partial_\kappa G$ is a topological model for the Poisson boundaries of simple random walks on various groups, such as right-angled Artin groups [QRT19], mapping class groups and relative hyperbolic groups [QRT20], hierarchically hyperbolic groups [NQ22], CAT(0) groups [GQR22] and Teichmüller spaces [GQR22]. The sublinearly Morse directions are also shown to be generic in Patterson Sullivan measure under suitable conditions [GQR22]. Most recently, Choi [Cho] claim this result to hold for all groups with two independent isometries with contracting axes.

1.2. **Sublinear bilipschitz equivalence.** Sublinear bilipschitz equivalence (SBE) appeared in works of Cornulier, where it was motivated by the quasiisometry classification of connected Lie groups. Cornulier noted that while the quasiisometry classification of all such groups reduces to that of closed subgroups of real upper triangular matrices, the sublinear billipschitz classification reduces to that of a smaller class, and that it was completely treated by the literature in the nilpotent case [Cor11]. In [Cor19] Cornulier asked about the SBE classification for other classes of groups, especially the word-hyperbolic groups.

As mentionned above, the first SBE invariant is the asymptotic cone; it is also the most natural one, since SBE may be defined as the largest class of maps inducing bilipschitz homeomorphisms between asymptotic cones with fixed basepoints. A geodesic metric space X is Gromov-hyperbolic if and only if all its asymptotic cones are real trees, for every choice of sequence of base-points [Gro93, 2.A] [Dru02, Proposition 3.A.4]. It follows that Gromov-hyperbolicity is an SBE-invariant among compactly generated locally compact groups, and especially among finitely generated groups)[Cor19, Theorem 4.3]. Cornulier proved that SBEs between Gromov-hyperbolic groups induce biHölder homeomorphisms between their Gromov boundaries; this was slightly improved by the first named author showing that a sublinear conformal structure in the Gromov boundary is actually preserved [Cor19, Pal20].

Theorem A. *(Theorem 4.4) Let Φ be an SBE between two proper geodesic spaces X and Y. Then Φ induces a bijection $\Phi_\star \colon \partial_\kappa X \to \partial_\kappa Y$. Moreover $\Phi_\star$ is a homeomorphism.*

The proof of Theorem 4.4 makes use of (L,θ)-rays, introduced in [Pal20] (there called $(\lambda, O(v))$-rays). These are images of the half-line under a map that is *sublinear bi-Lipschitz*. As such, they are analogues of quasigeodesics, but with an additive constant of quasigedoesicity that grows sublinearly. The degree of growth is precised by the function θ, while L is the large-scale Lipschitz constant. (L,θ)-rays also play an important role in the statement of our second result.

1.3. **Random walks.** Similar to the development of sublinearly Morse boundaries, an important motivation behind this project stems from simple random walk on finitely generated groups. In [Tio15], Tiozzo show that given a surface S, in the Teichmüller space $\mathcal{T}(S)$, a generic random walk tracks a geodesic sublinearly. In Section 5 of this paper, we show that this is equivalent to say that a generic random walk is a (L,κ)-ray for some sublinear function κ.

Theorem B. *(Theorem 5.1) Let G be the mapping class group $Mod(S)$ of a finite type surface, or let G be a relatively hyperbolic group. Let (X,d) be a Cayley graph of G. Let μ be a probability measure on G with finite first moment with respect to the metric d, such that the semigroup generated by the support of μ is a non-amenable group. Then there exists a constant A such that almost every sample path (w_n) is such that $(w_n\mathfrak{o})$ is a (A,κ)-ray. Moreover, one can take $\kappa(r) = \log(2+r)$.*

In fact, this result hold in more generality: we combine the proof of Theorem 5.1 together Theorem 6 in [Tio15] to obtain Theorem 5.2.

While Theorem 5.1 is logically independent from Theorem 4.4, both theorems promote the use of (L,θ)-rays. Whereas a generic random walk is not a quasigeodesic, it is an (L,θ) ray, and many of the geometric techniques devised for quasigeodesics can be employed to treat (L,θ)-rays as if they were quasigeodesics.

1.4. **Organization of the paper.** Section 2 collects preliminary material; especially we recall the relevant definitions and facts concerning sublinear bilipschitz equivalence, rays, and the sublinearly Morse boundaries. Beware that we chose to adapt the notation from [QRT20] so that the notation for sublinear bilipschitz equivalence is not the usual one. Section 3 is a preparation for Section 4, which is itself devoted to the proof of Theorem 4.4. Section 5 is about Theorem B. It only builds on the preliminaries and can be read without Sections 3 and 4.

2. Preliminaries

2.1. **Notation and convention for the sublinear functions.** Throughout this paper, let X and Y denote pointed proper geodesic metric spaces. The basepoints in both spaces are denoted $\mathfrak{o}$. The distance to the basepoint is denoted $\|x\| = d(x,\mathfrak{o})$ for all $x \in X$ or $x \in Y$. Let κ be a concave nondecreasing and strictly sublinear function. The last condition means that $\kappa(r)/r$ goes to 0 as r tends to $+\infty$. We also assume $\kappa \geqslant 1$.

By $\kappa(x)$ for $x \in X$ or Y we mean $\kappa(\|x\|)$. Let $Z \subseteq X$ be a closed subspace of X and $\mathsf{D} > 0$, then $\mathcal{N}_\kappa(Z,\mathsf{D})$ will denote $\{x \in X : d(x,Z) \leqslant \mathsf{D}\kappa(x)\}$. We say that D is *small with respect to* r, and write $\mathsf{D} \ll r$, if $\mathsf{D} \leqslant r/(2\kappa(r))$.

2.2. **Sublinear estimates.** The following basic sublinear estimate that is needed:

Lemma 2.1 (Sublinear Estimation Lemma). *For any $\mathsf{D}_0 > 0$, there exists $\mathsf{D}_1, \mathsf{D}_2 > 0$ depending on D_0 and κ so that, for $x, y \in X$,*

$$d(x,y) \le \mathsf{D}_0 \cdot \kappa(x) \Longrightarrow \mathsf{D}_1\kappa(x) \le \kappa(y) \le \mathsf{D}_2\kappa(x).$$

Proof. Since κ is sublinear, there is $R > 0$ such that $\kappa(x) \leqslant \frac{1}{2\mathsf{D}_0}\|x\|$ as soon as $\|x\| \geqslant R$. And then $d(x,y) \leqslant \mathsf{D}_0 \kappa(x)$ implies that $\|y\| \leqslant 3\|x\|2$ by the triangle inequality, so that, in all cases,

$$\kappa(y) \leqslant \left[\sup_{r \geqslant R} \frac{\kappa(3r/2)}{\kappa(r)} + \kappa(3R/2) \right] \kappa(x)$$

where we used that $\kappa(x) \geqslant 1$. We may define $\mathsf{D}_2 = \sup_{r \geqslant R} \frac{\kappa(3r/2)}{\kappa(r)} + \kappa(3R/2)$. On the other hand, if $\|x\| \geqslant R$ then $\|y\| \geqslant \|x\|/2$ so that

$$\kappa(y) \geqslant \left[\inf_{r \geqslant R} \frac{\kappa(r/2)}{\kappa(r)} \right] \kappa(x)$$

Setting $\mathsf{D}_1 = \min(\inf_{r \geqslant R} \frac{\kappa(r/2)}{\kappa(r)}, 1/\kappa(R))$ finishes the proof. □

2.3. **Quasigeodesics and θ-rays.** Here, θ is a function with the same properties as κ, that were specified in §2.1.

Definition 2.2 (θ-ray)**.** Let X be a proper pointed geodesic metric space. Let $L \geqslant 1$ be a constant. Say that $\gamma : [0, +\infty) \to X$ with $\gamma(0) = \mathfrak{o}$ is a (L, θ)-ray, or a θ-ray for short, if for every $s, t \in [0, +\infty)$

$$\frac{1}{L}|s-t| - \theta(\max(s,t)) \leqslant d(\gamma(s), \gamma(t)) \leqslant L|s-t| + \theta(\max(s,t)). \tag{2.1}$$

Beware that in [Pal20] we did not ask $\gamma(0) = \mathfrak{o}$ in the definition of a ray but we do it here. The difference is not a serious one, as one may simply advance the function θ (i.e. replace θ with $\theta(\mathsf{D} + \cdot)$ to accomodate for the change.

Remark 2.3 (compare [Pal20, Lemma 3.2]). If γ is a (L, θ)-ray then for s large enough $\frac{1}{2L}|s| \leqslant \|\gamma(s)\| \leqslant 2L|s|$ (apply (2.1) with fixed t), hence there exists $\widehat{\theta} = O(\theta)$ such that for s and t large enough,

$$\frac{1}{L}|s-t| - \widehat{\theta}(\max(\|\gamma(s)\|, \|\gamma(t)\|)) \leqslant d(\gamma(s), \gamma(t)) \leqslant L|s-t| + \widehat{\theta}(\max(\|\gamma(s)\|, \|\gamma(t)\|)). \tag{2.2}$$

When θ is a constant, the Definition 2.2 is that of a quasigeodesic ray. For our purposes, it is however necessary to treat the latter specifically because they play a special role in the definition of the sublinearly Morse boundary. Hence, we will use q for L and Q for θ when we want to denote a quasigeodesic ray.

Lemma 2.4 (Connect-the-dots for θ-rays)**.** *Let γ be a (L, θ)-ray into a proper geodesic metric space X. Then there exists $n > 0$ and $\widehat{\gamma}$ which is a $(L, n \cdot \theta)$-ray into X with the property that*

- *$\gamma(t) = \widehat{\gamma}(t)$ for all nonnegative integer t.*
- *$\widehat{\gamma}$ is continuous.*

Moreover, there exists $n > 0$ such that

$$d(\gamma(t), \widehat{\gamma}(t)) \leqslant n \cdot \theta(t) \tag{2.3}$$

for all t.

We will refer to $\widehat{\gamma}$ as a continuous completion of γ.

Proof. For every $t \in \mathbf{N}$, choose a geodesic segment σ_t from $\gamma(t)$ to $\gamma(t+1)$ at unit speed and denote its length ℓ_t. Note that $\ell(t) \leqslant L + \theta(t+1)$ by the inequality on the right in (2.1). Now for all $t \in [0, +\infty)$ set $\widehat{\gamma}(t) = \sigma_t(\ell_t \cdot \{t\})$ where $\{t\}$ denotes the fractional part of t. In this way $\widehat{\gamma}$ is

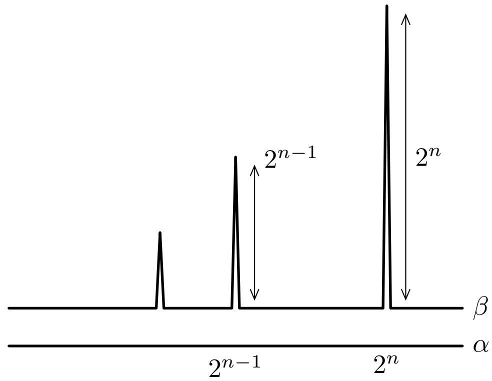

FIGURE 1. Counterexample to the conclusion of Lemma 2.6 when the left inequality of (2.1) is not respected by one of the rays (namely, β). Note that both $\|\alpha(t)\|$ and $\|\beta(t)\|$ here are bounded between linear functions of t.

continuous by construction. Let $t, s \in [0, +\infty)$ be such that $t \leqslant s$. Then, either $\lfloor s \rfloor = \lfloor t \rfloor$, in which case $d(\widehat{(}\gamma(t), \gamma(s)) \leqslant \ell_{\lfloor t \rfloor} \leqslant \theta(t+1) \leqslant n_0 \theta(t)$ for some n_0 by the properties of θ, or

$$\begin{aligned} d(\gamma(t), \gamma(s)) &\leqslant d(\gamma(t), \gamma(\lceil t \rceil)) + d(\gamma(\lceil t \rceil), \gamma(\lfloor s \rfloor)) + d(\gamma(\lfloor s \rfloor), \gamma(s)) \\ &\leqslant L(\lfloor s \rfloor - \lceil t \rceil)) + 2\theta(s+1) \\ &\leqslant L|s-t| + 2\theta(s+1). \end{aligned}$$

and

$$\begin{aligned} d(\gamma(t), \gamma(s)) &\geqslant d(\gamma(\lceil s \rceil), \gamma(\lfloor t \rfloor)) - d(\gamma(s), \gamma(\lceil s \rceil)) - d(\gamma(\lfloor t \rfloor), \gamma(t)) \\ &\geqslant L^{-1}(\lceil s \rceil - \lfloor t \rfloor)) - 2\theta(s+1) \\ &\geqslant L^{-1}|s-t| + 2\theta(s+1). \end{aligned}$$

Finally, $2\theta(s+1) \leqslant n_1 \theta(s)$ for some n_1, it remains to set $n = \max(n_0, n_1)$. ☐

Definition 2.5. Let α, β be quasi-geodesic rays or (L, θ)-rays for some L and θ into a proper geodesic space. We say $\alpha \sim \beta$ if either of the following holds:

(1) $\lim_{t \to \infty} \frac{d(\alpha(t), \beta)}{t} = 0$.
(2) $\lim_{t \to \infty} \frac{d(\beta(t), \alpha)}{t} = 0$.

Lemma 2.6. *(1) and (2) are equivalent.*

Beware that Lemma 2.6 is false if one replaces α and β with arbitrary maps, even proper maps from the half-line to X that respect the inequality on the right hand side of (2.1). For instance, if one considers the plane parametric curve β parametrized by arclength pogressing along the horizontal axis and making jumps of height 2^n at time 2^n for all $n \geqslant 0$, and the parametrization of the horizontal axis α, then this pair has (1) but not (2). See Figure 1.

Proof. Assume (1) holds and define $\eta(s) = d(\alpha(s), \beta)$ for $s \in [0, +\infty)$; this function η is sublinear. Let us introduce the set

$$\mathcal{T} = \{t \in [0, +\infty) : \exists s \in [0, +\infty), d(\alpha(s), \beta(t)) \leqslant 2d(\alpha(s), \beta)\}.$$

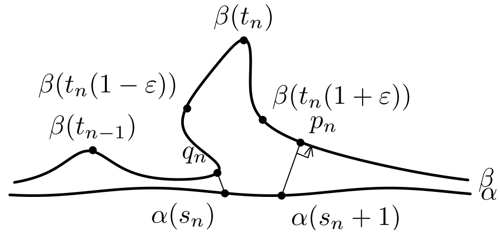

FIGURE 2. Proof of Lemma 2.6.

We claim that $\mathcal{T}$ cannot have large holes, more precisely there is no $\varepsilon > 0$ and sequence (t_n) with limit $+\infty$ such that

$$(t_n(1-\varepsilon); t_n(1+\varepsilon)) \cap \mathcal{T} = \emptyset \tag{2.4}$$

for all n. Indeed, assume the contrary and define

$$s_n = \sup\{s : \exists t < t_n, d(\alpha(s), \beta(t)) \leqslant 2d(\alpha(s), \beta)\}$$

It can be checked that s_n is well defined for all n and tends to $+\infty$, since α is proper while $\beta[0, t_n]$ is bounded and X is proper. Now consider a nearest point projection p_n of $\alpha(s_n+1)$ on β. Necessarily, $t > t_n(1+\varepsilon)$ for every t such that $p_n = \beta(t)$, in view of the definition of s_n and t_n. Let q_n be a nearest-point projection of $\alpha(s_n)$ on β; by the same argument, if $q_n = \beta(t)$ then $t < t_n(1-\varepsilon)$. Now, on the one hand by the triangle inequality,

$$\begin{aligned} d(p_n, q_n) &\leqslant d(p_n, \alpha(s_n+1) + d(\alpha(s_n+1), \alpha(s_n)) + d(\alpha(s_n), q_n)) \\ &\leqslant L + \theta(s_{n+1}) + \eta(s_n) + \eta(s_n+1). \end{aligned} \tag{2.5}$$

On the other hand, by the left-hand side of (2.1),

$$d(p_n, q_n) \geqslant 2L^{-1}\varepsilon t_n - \widehat{\theta}(\max(\|p_n\|, \|q_n\|)) \tag{2.6}$$

where $\widehat{\theta} = O(\theta)$. However, there is a constant $M > 0$ such that $M^{-1}t_n \leqslant s_n \leqslant Mt_n$ for n large enough; one can take $M = 2L^2(1+\varepsilon)$, by considering the first inequality in Remark 2.3.

Making $n \to +\infty$ and using that θ, $\widehat{\theta}$ and η are sublinear, (2.5) and (2.6) are in contradiction with one another. Thus (2.4) cannot be. It follows that there is a sublinear function μ such that for all $t \geqslant 0$, there is t' with $|t-t'| \leqslant \mu(t)$ and $t' \in \mathcal{T}$. And then $d(\beta(t), \alpha) \leqslant d(\beta(t), \beta(t')) + d(\beta(t'), \alpha)$, which is bounded above by a sublinear function of t involving η, μ and L in view of the definition of $\mathcal{T}$ and the linear control between s and t when $\beta(t')$ is a closest point projection of $\alpha(s)$ on β. We proved that (1) implies (2) for (L, θ) rays; the converse implication holds by symmetry, and quasigeodesics are θ-rays, hence the Lemma is proved.

□

2.4. **Definition of the sublinearly Morse boundary ∂X.** For more extensive references on sublinearly Morse boundaries, see [QRT19] and [QRT20]. A (q, Q)-quasigeodesic is a map $\gamma : [0, +\infty) \to X$ such that $\frac{1}{\mathsf{q}}d(x,y) - \mathsf{Q} \leqslant d(\gamma(x), \gamma(y)) \leqslant \mathsf{q}d(x,y) + \mathsf{Q}$ for all $x, y \in [0, +\infty)$.

Definition 2.7 (κ-Morse quasigeodesic [QRT20, Definition 3.2])**.** Let $Z \subseteq X$ be a closed subspace. Let $m_Z : \mathbb{R}^2 \to \mathbb{R}$ be a proper function. Say that Z is κ-Morse with Morse gauge m_Z if for every sublinear function κ', for every $r > 0$ such that $m_Z(\mathsf{q},\mathsf{Q}) \ll_\kappa r$, there exists $R > 0$ such that if β is a (q,Q)-quasigeodesic then

$$d_X(\beta_R, Z) \leqslant \kappa'(R) \Longrightarrow \beta|_r \subset \mathcal{N}_\kappa(Z, m_Z(\mathsf{q},\mathsf{Q})).$$

Proposition 2.8 ([QRT20, Lemma 3.4 and Corollary 3.5]). *Let α and β are quasi-gedodesic rays into X, such that β is (q,Q)-quasi-geodesic and α is κ-Morse. If $\alpha \sim \beta$ then β is κ-Morse with gauge $m_\alpha + 4m_\alpha(\mathsf{q},\mathsf{Q})$.*

We will reprove this in a greater generality in Proposition 3.1.

Define $\partial_\kappa X$ as the set of κ-Morse quasigeodesic up to $\sim$. For any κ-Morse β , define $\partial\mathcal{U}(\beta, r)$ as

$$\{\mathbf{a} : \alpha \in \mathbf{a} \text{ is a } (\mathsf{q},\mathsf{Q})\text{-quasigeodesic and } r \gg_\kappa m_\beta(\mathsf{q},\mathsf{Q}) \Longrightarrow \alpha_{|r} \subseteq \mathcal{N}_\kappa(\beta, m_\beta(\mathsf{q},\mathsf{Q}))\},$$

and then

$$\partial\mathcal{B}(\mathbf{b}) = \{\mathcal{V} \subseteq \partial_\kappa X : \exists \beta \in \mathbf{b}, \exists r > 0, \mathcal{V} \supseteq \partial\mathcal{U}(\beta, r)\}.$$

The topology on $\partial_\kappa X$ is defined as the unique one so that the $\partial\mathcal{B}(\mathbf{b})$ are the neighborhood systems at $\mathbf{b}$ [QRT20, Lemma 4.5]; it is metrizable [QRT20, Lemma 4.8]. Finally, the sublinearly Morse boundary is defined as $\partial X = \cup_\kappa \uparrow \partial_\kappa X$.

2.5. **Sublinear bilipschitz equivalence.** In this paragraph, θ is a sublinear function with the same properties as κ.

Definition 2.9. Let Z and Z' be two closed unbounded subsets in X. Say that Z and Z' linearly separate if $d(Z \cap S(\mathfrak{o}, r), Z') - r$ stays bounded as $r \to +\infty$.

Definition 2.10 (θ-SBE)**.** Let $(X, \mathfrak{o})$ and $(Y, \mathfrak{o})$ be proper geodesic pointed metric spaces. Let $L \geqslant 1$ be a constant, and let θ be a sublinear function as before. Say that $\Phi : X \to Y$ is a (L, θ)-sublinear bilipschitz equivalence (θ-SBE for short) if

$$\frac{1}{L} d(x_1, x_2) - \theta(\max(\|x_1\|, \|x_2\|)) \leqslant d(\Phi(x_1), \Phi(x_2))) \leqslant L d(x_1, x_2) + \theta(\max(\|x_1\|, \|x_2\|))$$

and $Y = \mathcal{N}_\theta(\Phi(X), D)$ for some $D \geqslant 0$.

Proposition 2.11 (Inverses)**.** *Let $\Phi : X \to Y$ be a θ-SBE. Then there exists $\overline{\Phi} : Y \to X$ a θ-SBE and $n > 0$ such that for all $x \in X$,*

$$d(x, \overline{\Phi}(\Phi(x))) \leqslant n \cdot \theta(x) \tag{2.7}$$

and for all $y \in Y$,

$$d(y, \Phi(\overline{\Phi}(y))) \leqslant n \cdot \theta(y). \tag{2.8}$$

Proof. This follows from [Cor19, Proposition 2.4], however, let us give here a self-contained proof. For simplicity, let us assume for now that $\Phi(X)$ is closed in Y; we will see in the end how to remove this assumption if necessary. For every $y \in Y$, define $\overline{\Phi}(y)$ as some $x \in X$ such that $\Phi(X)$ is a nearest-point projection of y on $\Phi(X)$. By assumption, Φ is a θ-SBE, hence applying the last property in Definition 2.10 one gets that

$$d(y, \Phi(\overline{\Phi}(y))) \leqslant D\theta(\Phi(x)), \tag{2.9}$$

for some $D \geqslant 0$. This is almost (2.8); let us rework this inequality slightly. Since we know that $d(y, \Phi(x)) \leqslant D\theta(\|y\|)$, when y is far enough from $\mathfrak{o}$, there is some constant K so that as soon as $\|y\| \geqslant K$, $\|y\|/2 \leqslant \|\Phi(x)\| \leqslant 2\|y\|$. It follows that for some constant K', for all y, $\|\Phi(x)\| \leqslant 2\|y\| + K'$. Hence

$$d(y, \Phi(\overline{\Phi}(y))) \leqslant D\theta(\Phi(x)) \leqslant D\theta(2\|y\| + K') = O(\theta(y)).$$

Thus we proved (2.8). Now for (2.7), note that x and $\overline{\Phi}(\Phi(x))$ have the same image, namely $\Phi(x)$, through Φ. So $\frac{1}{L}d(x, \overline{\Phi}(\Phi(x)) - \theta(\max(\|x\|, \|\overline{\Phi}(\Phi(x))\|) \leqslant d(\Phi(x), \Phi(x)) = 0$, whence

$$d(x, \overline{\Phi}(\Phi(x)) \leqslant L\theta(\max(\|x\|, \|\overline{\Phi}(\Phi(x))\|). \tag{2.10}$$

But also $\frac{1}{L}\|\overline{\Phi}(\Phi(x)\| - \theta(\overline{\Phi}(\Phi(x))) \leqslant \|\Phi(x)\| \leqslant L\|x\| + \theta(x)$. There exists K such that $\theta(r) \leqslant K + r/(2L))$, and then

$$\frac{1}{2L}\|\overline{\Phi}(\Phi(x)\| \leqslant L\|x\| + \frac{\|x\|}{2L} + 2K.$$

Plugging this into (2.10),

$$d(x, \overline{\Phi}(\Phi(x)) \leqslant L\theta(\max(\|x\|, (2L^2+1)\|x\| + 2LK) = O(\theta(\|x\|)).$$

Finally we need to prove that $\overline{\Phi}$ is a θ-SBE. Applying the inequality on the right in Definition 2.10 for Φ,

$$\begin{aligned} d(\overline{\Phi}(y), \overline{\Phi}(y')) &\leqslant Ld(\Phi\overline{\Phi}(y), \Phi\overline{\Phi}(y')) + \theta(\max(\|\overline{\Phi}(y)\|, \|\overline{\Phi}(y')\|)) \\ &\leqslant Ld(y, y') + 2\theta(\max(\|y\|, \|y'\|, \|\overline{\Phi}(y)\|, \|\overline{\Phi}(y')\|)) \end{aligned}$$

Note that

$$\frac{1}{L}\|\overline{\Phi}(y)\| - \theta(\|\overline{\Phi}(y)\|) \leqslant d(\Phi\overline{\Phi}(y), \Phi(\mathfrak{o})) \leqslant \|y\| + O(\theta(\|y\|)).$$

so that $\|\overline{\Phi}(y)\| \leqslant 2L\|y\| + M$ for some constant M. Thus

$$d(\overline{\Phi}(y), \overline{\Phi}(y')) \leqslant Ld(y, y') + 2\theta(\max(\|y\|, \|y'\|, \|\overline{\Phi}(y)\|, \|\overline{\Phi}(y')\|)) \leqslant Ld(y, y') + 2P\theta(\max(\|y\|, \|y'\|)$$

for some $P > 0$. This proves the inequality on the right in Definition 2.10 for $\overline{\Phi}$; in the exact same way, the left inequality on the left is obtained by using the inequality on the left for Φ.

It remains to check that $X = \mathcal{N}_\theta(\overline{\Phi}(Y), \overline{D})$ for some $\overline{D} \geqslant 0$. This follows from (2.7) exactly the same way that we deduced (2.8) from (2.9).

Finally, $\Phi(X)$ may not be closed in Y, but in the construction of $\overline{\Phi}(y)$, we can relax the condition defining $\overline{\Phi}(y)$ by replacing the nearest-point projection of y with some point at distance at most $2d(y, \Phi(X))$ from y. All the estimates afterwards go through with additional multiplicative constants. ☐

Remark 2.12. When $\theta = 1$, the proof is easier and it is one of the first exercises on quasiisometries in textbooks; see e.g. [DK18, Exercise 8.12].

Lemma 2.13. *Let α be an (L, θ)-ray. Let Φ and $\overline{\Phi}$ be as in Proposition 2.11. Let $\widehat{\alpha}$ and $\widehat{\overline{\Phi}\Phi\alpha}$ be continuous completions of α and $\overline{\Phi}\Phi\alpha$. Then there exists n depending on L and θ and n' depending on L, θ and Φ, such that $\widehat{\alpha} \subset \mathcal{N}_\theta(\widehat{\overline{\Phi}\Phi\alpha}, n)$ and $\widehat{\overline{\Phi}\Phi\alpha} \subset \mathcal{N}_\theta(\widehat{\alpha}, n')$.*

Proof. By Proposition 2.11, there exists n_0 such that for every x in α, $d(x, \overline{\Phi}\Phi x) \leqslant n_0\theta(x)$. Hence $\alpha \subset \mathcal{N}_\theta(\overline{\Phi}\Phi\alpha, n_0)$. Morever, by Equation (2.3) and Remark 2.3, there is n_1 depending on L and θ such that

$$d(\widehat{\gamma}(t), \gamma(t) \leqslant n_1\theta(\max(\|\gamma(t)\|, \|\widehat{\gamma}(t)\|)$$

for all t. It follows that

$$\alpha \subset \mathcal{N}_\theta(\overline{\Phi}\Phi\alpha, n_0 + n_1).$$

By Lemma 2.1, since $d(x, \overline{\Phi}\Phi x) \leqslant n_0\kappa(x)$ for all x in α, there are D_1, D_2 such that $\mathsf{D}_1\kappa(x) \leqslant \kappa(\overline{\Phi}\Phi x) \leqslant \mathsf{D}_2\kappa(x)$ for every x on α. Finally, $\overline{\Phi}\Phi\alpha$ is a (L', θ) ray where L' depends on L and Φ. So applying again Equation (2.3) and Remark 2.3, there exists n_1' such that

$$d(\widehat{\overline{\Phi}\Phi\gamma}(t), \overline{\Phi}\Phi\gamma(t)) \leqslant n_1\theta(\max(\|\overline{\Phi}\Phi\gamma(t)\|, \|\widehat{\overline{\Phi}\Phi\gamma}(t)\|).$$

Setting $n' = n_0 + n_1'$ finishes the proof. □

There exist several degree of closeness between κ-rays. The first is the $\sim$ relation defined earlier.

Definition 2.14 (κ-fellow travelling rays)**.** Given two rays α and β (which are frequently either quasi-geodesic rays or θ-rays in this paper) we say, α and β *κ-fellow travel* each other if there exists n and for all $t > 0$, we have

$$d(\alpha(t), \beta(t)) \leq n \cdot \kappa(t).$$

Further, we say that α and β κ-track each other if there exists n_1 such that

$$d(\alpha_r, \beta_r) \leq n_1 \cdot \kappa(r).$$

for all r.

Note that if α and β κ-fellow travel each other, then they κ-track each other, however the converse is not true.

3. θ-SBE invariance of the κ-Morse quasi-geodesic rays, when $\theta = O(\kappa)$

In this section we establish what happens to κ-Morse quasi-geodesic rays under (L, θ) maps. We prove that they behave well in the sense that they are sent to images sublinearly tracking sublinearly Morse geodesic rays, provided that κ dominates θ.

Precisely, given two sublinear functions κ, θ, we say κ *dominates* θ if there exists constants C_1, C_2 and some t_0 such that for all $t > t_0$,

$$\kappa(t) \geq C_1\theta(t) + C_2.$$

Therefore in the results proven, we show frequently a ray is $(\kappa + \theta)$-Morse, which implies it is κ-Morse if $\theta \preceq \kappa$ and it is θ-Morse if $\kappa \preceq \theta$.

3.1. κ-Morse rays.

Assume that α is a κ-Morse (L, θ)-ray in a proper metric space. Then we can establish that a quasi-geodesic ray that tracks α sublinearly is itself κ-Morse.

Proposition 3.1. *Let α be an (L, θ)-ray that is κ-Morse with its Morse gauge m_α, and let $\beta \sim \alpha$ be a (q, Q)-quasigeodesic ray that tracks α sublinearly. Then β is κ-Morse.*

Proof. The proof is similar to that of [QRT20, Lemma 3.4] where β is also (q, Q)-quasigeodesic but α is a κ-Morse quasigeodesic instead of a κ-Morse (L, θ)-ray. First, assume that β is continuous. We will see in the end how to remove the assumption.

Define $\kappa'(r) := d_X(\alpha_r, \beta_r)$. By definition of $\sim$, the function κ' is sublinear. Let us now prove that β is κ-Morse. Let $r > 0$ and let β' be a $(\mathsf{q}', \mathsf{Q}')$-quasi-geodesic ray such that

$$d_X(\beta'_R, \beta) \leq \kappa'(R)$$

for some sufficiently large R. Let p_R be a nearest point projection of β'_R to β; by construction and by triangle inequality, we have

$$\|p_R\| \leq 2\|\beta'_R\| = 2R.$$

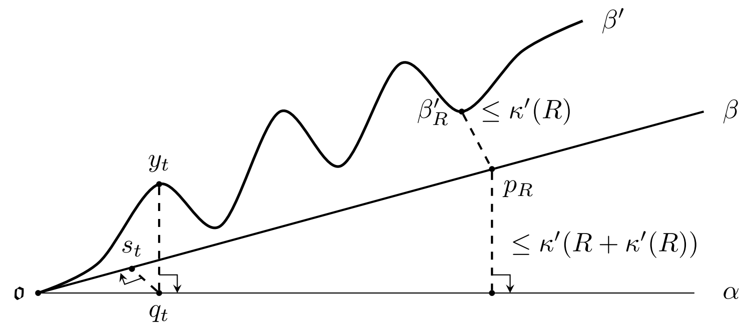

FIGURE 3. The setup in the proof of Proposition 3.1.

Then, by the triangle inequality,

$$\begin{aligned} d_X(\beta'_R, \alpha) &\leq d_X(\beta'_R, p_R) + d_X(p_R, \alpha) \\ &\leq \kappa'(R) + m_\alpha(\mathsf{q}, \mathsf{Q}) \cdot \kappa(\|p_R\|) \quad \beta \text{ is a quasi-geodesic ray and hence is in a } \kappa\text{-neighbourhood of } \alpha. \\ &\leq \kappa'(R) + m_\alpha(\mathsf{q}, \mathsf{Q}) \cdot \kappa(2R). \end{aligned}$$

Since $\kappa''(R) := \kappa'(R) + m(\mathsf{q}, \mathsf{Q}) \cdot \kappa(2R)$ is also a sublinear function, and since α is κ-Morse, this implies that

$$\beta'|_r \subseteq \mathcal{N}_\kappa(\alpha, m_\alpha(\mathsf{q}', \mathsf{Q}')). \tag{3.1}$$

Let y_t be any point on β' with $\|y_t\| = t \leq r$. By construction and triangle inequality, if q_t is a nearest point projection of y_t to α, we have

$$\|q_t\| \leq 2\|y_t\| = 2t.$$

Now, if q is any point on α and s is a nearest point projection of q to β, by the triangle inequality and the Morse property,

$$\|q\| \geq \|s\| - d_X(s, q) \geq \|s\| - m_\alpha(\mathsf{q}, \mathsf{Q}) \cdot \kappa(\|s\|).$$

Moreover, again by construction and triangle inequality, $\|s\| \leq 2\|q\|$, hence by concavity

$$d_X(s, q) \leq m_\alpha(\mathsf{q}, \mathsf{Q}) \cdot \kappa(\|s\|) \leq 2m_\alpha(\mathsf{q}, \mathsf{Q}) \cdot \kappa(\|q\|).$$

Thus, let s_t be a nearest point projection of q_t to β, the above estimate yields

$$\begin{aligned} d_X(q_t, s_t) &\leq 2m_\alpha(\mathsf{q}, \mathsf{Q}) \cdot \kappa(\|q_t\|) \\ &\leq 4m_\alpha(\mathsf{q}, \mathsf{Q}) \cdot \kappa(t) \end{aligned}$$

hence, putting everything together,

$$\begin{aligned} d_X(y_t, \beta) &\leq d_X(y_t, q_t) + d_X(q_t, s_t) \\ &\leq m_\alpha(\mathsf{q}', \mathsf{Q}') \cdot \kappa(t) + 4m_\alpha(\mathsf{q}, \mathsf{Q}) \cdot \kappa(t) \end{aligned}$$

which, by setting $m_\beta(\mathsf{q}', \mathsf{Q}') := m_\alpha(\mathsf{q}', \mathsf{Q}') + 4m_\alpha(\mathsf{q}, \mathsf{Q})$, proves the claim. □

Building on these results we are now ready to establish the map on $\partial_\kappa X$ that is induced by an SBE.

Lemma 3.2. *If α is a (q, Q)-κ-Morse quasi-geodesic ray in X, and Φ is an (L, θ)-sublinear bi-Lipschitz equivalence between two proper metric spaces X and Y. Then $\overline{\Phi}\Phi\alpha$ is $(\kappa+\theta)$-Morse.*

Proof. Consider $\alpha, \overline{\Phi}\Phi\alpha$. By Proposition 2.11, every point x of α is $\theta(x)$ away from the point $\overline{\Phi}\Phi(x)$. Consider a (q, Q)-quasi-geodesic γ that is sublinearly tracking $\overline{\Phi}\Phi\alpha$. Let this tracking function be denoted κ'.

Then, there exists a constant C such that any point y on γ is distance at most $C(\kappa'(y)+\theta(y))$ from α. Indeed, $d(y, \overline{\Phi}\Phi\alpha_{\|y\|}) \leqslant \kappa'(y)$ and then by Proposition 2.11, $d(y,\alpha) \leqslant \kappa'(y) + \widehat{\theta}(y)$ where $\widehat{\theta} \leqslant C\theta$ for some C by (2.2).

Since α is κ-Morse and $C(\kappa'(y)+\theta(y))$ is a sublinear function, by Proposition 3.1 we have that γ is in a κ-neighbourhod of α and γ is κ-Morse.

Therefore we have that γ is in $\mathcal{N}_\kappa(\alpha, m)$ for some m. Since α is θ-tracking $\overline{\Phi}\Phi\alpha$, we have that γ is $m\cdot\kappa+\theta$ tracking $\overline{\Phi}\Phi\alpha$. Thus we have shown that any quasi-geodesic ray that sublinearly tracks $\overline{\Phi}\Phi\alpha$ with rate $m\cdot\kappa+\theta$. Thus $\overline{\Phi}\Phi\alpha$ is $(\kappa+\theta)$-Morse. □

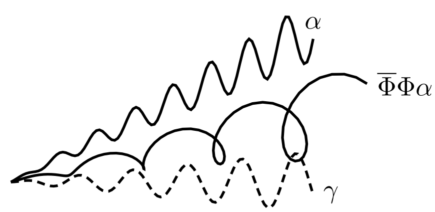

FIGURE 4. Proof of Lemma 3.2.

Proposition 3.3. *If α is a (q, Q)-κ-Morse quasi-geodesic ray in X, and Φ is an (L, θ)-sublinear bi-Lipschitz equivalence between two proper metric spaces X and Y. Then $\Phi\alpha$ is $(\kappa+\theta)$-Morse.*

Proof. Consider a (q, Q)-quasi-geodesic ray γ that sublinearly tracks $\Phi\alpha$ and let the tracking function be κ'. Consider the image $\overline{\Phi}\gamma$. $\overline{\Phi}\gamma$ is distance $\theta+\kappa'$ from $\overline{\Phi}\Phi\alpha$. Since $\overline{\Phi}\Phi\alpha$ is θ-tracking α we have that $\overline{\Phi}\gamma$ is $2\theta+\kappa'$-tracking α. By Proposition 3.1, $\overline{\Phi}\gamma$ is κ-Morse. Therefore, since α is a quasi-geodesic that sublinearly tracks $\overline{\Phi}\gamma$, we have that α and $\overline{\Phi}\gamma$ are κ-close.

Now apply Φ to both α and $\overline{\Phi}\gamma$. $\Phi\alpha$ and $\Phi\overline{\Phi}\gamma$ are $\kappa+\theta$ apart. Since $\Phi\overline{\Phi}\gamma$ and γ are θ apart, then we have that $\Phi\alpha$ and γ are at most $\kappa+2\theta$ apart. This holds for every γ and thus we have that $\Phi\alpha$ is $(\kappa+\theta)$-Morse.

□

These two results guarantees that the image of a κ-Morse quasi-geodesic ray, under SBE, is an (L,θ)-ray that carries the κ-Morse property, i.e. any quasi-geodesic ray that sublinearly tracks this set tracks it with a uniformly controlled sublinear function. The next two results shows that the latter set, i.e. a κ-Morse (L,θ)-ray still has strong association with a geodesic ray and thus can be connected with a κ-Morse equivalence class in $\partial_\kappa X$ in Section 4.

Lemma 3.4. *Let α be a κ-Morse (L,θ)-ray. Then there exists a geodesic ray $\underline{a}$ such that $\underline{a}$ and α κ- track each other.*

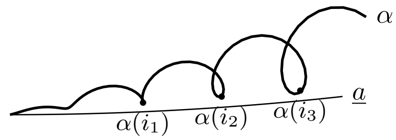

FIGURE 5. Proof of Lemma 3.4.

Proof. Since α is κ-Morse, let $\mathsf{m}_\alpha(\mathsf{q}, \mathsf{Q})$ be its κ-Morse gauges. Consider the geodesic segments connecting the basepoint and $\alpha(i)$, $i = 1, 2, 3 \ldots$.

Since α is κ-Morse, for every $r > 0$, there exists i_r such that

$$\alpha(i_r) \in \mathcal{N}_1(\alpha, 1) \implies [\mathfrak{o}, \alpha(r)] \in \mathcal{N}_\kappa(\alpha, m(1, 0)).$$

The space X is assumed to be proper. Thus by the Arzelá-Ascoli Theorem, for some extraction $\{i_n\}$, the subsequence $\{[\mathfrak{o}, \alpha(i_n)], n = 1, 2, 3 \ldots\}$ converges to a geodesic ray $\underline{a}$.

For each n, all but the first $n-1$ segments in this sequence has the property that the intersection of them with the ball of radius i_n is in the neighbourhood $\mathcal{N}_\kappa(\alpha, m(1, 0)$, thus their limit $\underline{a}$ is in the neighbourhood $\mathcal{N}_\kappa(\alpha, m(1, 0))$. Thus $\underline{a}$ and α κ-track each other. □

Note that we used the κ-Morse property in an essential way.

When X is Gromov hyperbolic, Lemma 3.4 is a consequence of [Pal20, Lemma 3.4].

Proposition 3.5. *If α is an (L, θ)-ray that is κ-Morse, then α is in a κ-neighbourhood of a unique κ-Morse geodesic ray, up to sublinear tracking. Thus a κ-Morse (L, θ)-ray is associated with a unique equivalence class $\boldsymbol{a} \in \partial_\kappa X$.*

Proof. By Lemma 3.4, there exists a geodesic ray $\underline{a}$ such that $\underline{a}$ and α κ-tracks each other. Then Proposition 3.1 implies that $\underline{a}$ is κ-Morse. Since every κ-Morse geodesic ray is in a unique equivalence class $\mathbf{a} \in \partial_\kappa X$, we get that α is associated with $\mathbf{a} \in \partial_\kappa X$. Suppose α is in a κ-neighbourhood of another κ-Morse geodesic ray $\underline{a}'$, we have that $\underline{a}'$ and $\underline{a}$ κ-track each other and thus $\underline{a}' \in \mathbf{a}$. □

4. SBE ON SUBLINEARLY MORSE BOUNDARIES

In this section we look at the induced map of an given SBE $\Phi \colon X \to Y$ on $\partial_\kappa X$. To begin with, we need the following basic observation about (L, θ)-ray and geodesics that are in sublinear neighbourhoods of each other. The proof is identical to the analogous claim in [QRT20] regarding quasi-geodesic rays and geodesic rays:

Lemma 4.1. *Let β be a (L, θ)-ray and $\underline{a}$ be a geodesic ray, both based at $\mathfrak{o} \in X$. Suppose that*

$$\beta \subseteq \mathcal{N}_\kappa(\underline{a}, m)$$

for some function κ and some constant m. Then we also have

$$\underline{a} \subseteq \mathcal{N}_\kappa(\beta, 2m).$$

Proof. The proof is the same as that of [QRT20, Lemma 3.1]; we reproduce it here for completeness. First, assume for simplicity that β is a continuous θ-ray. Let $y \in \underline{\alpha}$ be a point and let $r := \|y\|$. Let $z \in \beta$ be a point such that $\|z\| = r$ and let q be a nearest point projection of z to $\underline{a}$. By assumption,

$$d_X(z, q) \leq m \cdot \kappa(r).$$

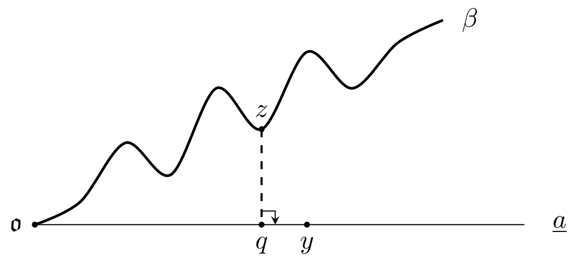

FIGURE 6. $\|y\| = \|z\|$ and $q \in \pi_{\underline{a}}(z)$ as in the proof of Lemma 4.1.

On the other hand,

$$\begin{aligned} d_X(y,q) &= |\|y\| - \|q\|| && \text{since } \underline{a} \text{ is geodesic} \\ &= |\|z\| - \|q\|| \\ &\le d_X(z,q) && \text{by the triangle inequality.} \end{aligned}$$

Therefore we have

$$\begin{aligned} d_X(y,\beta) \le d_X(y,z) &\le d_X(y,q) + d_X(q,z) \\ &\le 2d_X(z,q) \\ &\le 2m \cdot \kappa(r) \end{aligned}$$

which completes the proof. Now, β may not be continous. However, by Lemma 2.4 there is a continuous θ-ray that θ-fellow travels with β. □

Let $\Phi\colon X \to Y$ be a (L,θ)-SBE; we are now ready to define the induced map $\Phi_\star$.

Given a (q,Q)-quasi-geodesic ray $\zeta\colon [0,\infty) \to X$ in X, we assume without loss of generality that the image of ζ is a continuous path. Let $\Phi\zeta$ be the $(L\mathsf{q},\theta)$-ray in Y constructed from the composition of ζ and Φ.

Proposition 4.2. *Assume that κ dominates θ, and let Φ be a (L,θ)-sublinear bi-Lipschitz equivalence from X to Y, where X and Y are proper geodesic metric spaces. Two κ-Morse quasi-geodesics α and β in X κ-fellow travel each other if and only if $\Phi\alpha$ and $\Phi\beta$ κ-fellow travel each other in Y.*

Proof. If α and β in X κ-fellow travel each other with multiplicative constant n, then at radius r, the distances between points of $\Phi\alpha$ and $\Phi\beta$ is apart by

$$L(n \cdot \kappa(r)) + \theta(r) \le (Ln+1)\kappa(r)$$

thus $\Phi\alpha$ and $\Phi\beta$ κ-fellow travel each other in Y. By Proposition 2.11, there exists an inverse, $\overline{\Phi}$, that is also an (L,θ)-SBE. Thus we have that if $\Phi\alpha$ and $\Phi\beta$ are $n'\kappa(r)$-tracking, then $\overline{\Phi}\Phi\alpha$ and $\overline{\Phi}\Phi\beta$ are $(Ln'+1)\kappa(r)$-tracking each other by the preceding argument. Furthermore, α,β are $\theta(r)$ tracking $\overline{\Phi}\Phi\alpha$ and $\overline{\Phi}\Phi\beta$, respectively. Thus α,β are tracking each other with distance at most

$$(Ln'+1)\kappa(r) + \theta(r) + \theta(r) \le (Ln'+3)\kappa(r).$$ □

Definition 4.3. It follows from Proposition 4.2 that two quasi-geodesics ζ and ξ in X κ–fellow travel each other if and only if $\Phi\zeta$ and $\Phi\xi$ κ-fellow travel each other in Y. Also by Corollary 3.3, the property of being in Morse-neighbourhood of a Morse geodesic ray is preserved under an (L,θ)-SBE. Hence, $[\zeta] \in \partial_\kappa X$ if and only if $[\Phi\zeta] \in \partial_\kappa Y$. We write $\mathbf{a} = [\zeta]$ and $\mathbf{b} = [\Phi\zeta]$ and we thus define

$$\Phi_\star(\mathbf{a}) = \mathbf{b}.$$

Theorem 4.4. *Consider proper geodesic metric spaces X and Y, let $\Phi\colon X \to Y$ be a (L,θ)-sublinear bi-Lipschitz equivalence between X and Y and let $\overline{\Phi}$ denote its inverse as in Proposition 2.11. Let $\Phi_\star$ be defined by Definition 4.3. Then for every sublinear function κ that dominates θ, $\Phi_\star\colon \partial_\kappa X \to \partial_\kappa Y$ is a homeomorphism.*

Proof. By Definition 4.3, $\Phi_\star$ defined as above gives a bijection between $\partial_\kappa X$ and $\partial_\kappa Y$. We need to show that $(\Phi_\star)^{-1}$ is continuous. Then, the same argument applied in the other direction will show that $\Phi_\star$ is also continuous which means $\Phi_\star$ is a homeomorphism.

Let $\mathcal{V}$ be an open set in $\partial_\kappa X$, $\mathbf{b}_X \in \mathcal{V}$ and $\mathcal{U}_\kappa(\mathbf{b}_X, \mathsf{r})$ be a neighborhood of $\mathbf{b}_X$ that is contained in $\mathcal{V}$. Let $\mathbf{b}_Y = \Phi_\star(\mathbf{b}_X)$. We need to show that there is a constant r' such that, for every point $\mathbf{a}_Y \in \mathcal{U}_\kappa(\mathbf{b}_Y, \mathsf{r}')$, we have

$$(\Phi_\star)^{-1}(\mathbf{a}_Y) = \mathbf{a}_X \in \mathcal{U}_\kappa(\mathbf{b}_X, \mathsf{r}).$$

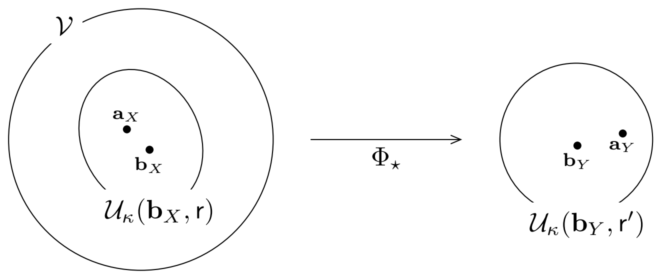

FIGURE 7. $\Phi_\star$ is a homeomorphism.

First, we find the upper bound for the constants q', Q' where

(1) $\mathsf{m}_{b_X}(\mathsf{q}, \mathsf{Q})$ is small compared to r, and
(2) If ζ is a (q, Q)–quasi-geodesic ray in X then $\Phi\zeta$ is in a $n \cdot \kappa$-neighborhood of a κ Morse geodesic ray (Lemma 3.6).

We obtain this by using Theorem A.4 and let $\mathsf{q}' = k\mathsf{q}$ and $\mathsf{Q}' = \mathsf{q}K + \kappa(\mathsf{q}r)$ be constants (depending on $\mathsf{q}, \mathsf{Q}, L$ and K).

Next, let b_Y be the representative geodesic ray in $\mathbf{b}_Y$ and let m_{b_X} and m_{b_Y} be their κ-Morse gauges respectively. By Lemma 4.1, there is a constant n_1 depending on L, κ and $\mathsf{m}_{b_Y} \cap B(\mathfrak{o}, r)$ such that

$$\Phi b_X \subset \mathcal{N}_\kappa(b_Y, \mathsf{n}_1).$$

Next, we let

$$\mathsf{n} = L\big(n + \mathsf{n}_1\big)(L+1) + 1$$

and let $\mathsf{R} = \mathsf{R}(b_X, \mathsf{r}, \mathsf{n}, \kappa)$ as in Definition 2.7 and choose r' such that:

(1) $\mathsf{r}' \geq L\,\mathsf{R} + \kappa(r)$, and
(2) n is small compare to r'.

Now we are ready to consider any given $\alpha \in \mathbf{a}_X$ be a (q, Q)–quasi-geodesic in X, where

(1) q, Q is such that $\mathsf{m}_{b_X}(\mathsf{q}, \mathsf{Q})$ is small compared to r; and
(2) $\Phi\alpha$ is a ray that is in $\mathcal{U}_\kappa(\mathbf{b}_Y, \mathsf{r}')$.

By the choice of r', n is small compared to r'. Hence,

$$\Phi\alpha|_{\mathsf{r}'} \subset \mathcal{N}_\kappa\big(b_Y, n\big)$$

Pick $x \in \alpha_X|_{\mathsf{R}}$. Then $\Phi x \in \Phi\alpha|_{\mathsf{r}'}$ and we have

$$\begin{aligned} d_X(x, b_X) &\leq L(d_Y(\Phi(x), \Phi b_X) + \theta(x) \\ &\leq L\Big(d_Y(\Phi(x), b_Y) + \mathsf{n}_1 \cdot \kappa(\Phi x)\Big) + \theta(x) \\ &\leq L\big(n + \mathsf{n}_1\big) \cdot \kappa(\Phi x) + \theta(x) \\ &\leq L\big(n + \mathsf{n}_1\big) \cdot \kappa(\Phi x) + \kappa(x) \end{aligned}$$

We also have

$$\kappa(\Phi x) \leq L\kappa(x) + \theta(x) \leq (L+1)\kappa(x)$$

Combine the preceding inequalities we have

$$\begin{aligned} d_X(x, b_X) &\leq L\big(n + \mathsf{n}_1\big) \cdot (L+1)\kappa(x) + \kappa(x) \\ &\leq \Big(L\big(n + \mathsf{n}_1\big)(L+1) + 1\Big)\kappa(x) \end{aligned}$$

imply that

$$\alpha|_{\mathsf{R}} \subset \mathcal{N}_\kappa(b_X, \mathsf{n}).$$

Now, Definition 2.7 implies that

$$\alpha|_{\mathsf{r}} \subset \mathcal{N}_\kappa(b_X, \mathsf{m}_{b_X}).$$

Therefore, $\mathbf{a}_X \in \mathcal{U}_\kappa(\mathbf{b}_X, \mathsf{r})$ and

$$(\Phi_\star)^{-1}\mathcal{U}_\kappa(\mathbf{b}_Y, \mathsf{r}') \subset \mathcal{U}_\kappa(\mathbf{b}_X, \mathsf{r}).$$

But $\mathcal{U}_\kappa(\mathbf{b}_Y, \mathsf{r}')$ contains an open neighborhood of $\mathbf{b}_Y$, therefore, $\mathbf{b}_Y$ is in the interior of $\Phi\mathcal{V}$. This finishes the proof. □

5. Application: random walk on groups

Random walks. Let G be a countable group, and let μ be a probability measure on a symmetric generating set of G. We consider the *step space* $(G^{\mathbb{N}}, \mu^{\mathbb{N}})$, whose elements we denote as (g_n). The *random walk driven by* μ is the G-valued stochastic process (w_n), where for each n we define the product

$$w_n := g_1 g_2 \dots g_n.$$

We denote as $(\Omega, \mathbb{P})$ the *path space*, i.e. the space of sequences (w_n), where $\mathbb{P}$ is the measure induced by pushing forward the measure $\mu^{\mathbb{N}}$ from the step space. Elements of Ω are called *sample paths* and will be also denoted as ω. Finally, let $T : \Omega \to \Omega$ be the left shift on the path space.

Background on boundaries. Let us recall some fundamental definitions from the boundary theory of random walks. For more extensive background, see [Kai00]. Let $(B, \mathcal{A})$ be a measurable space on which G acts by measurable isomorphisms; a measure ν on B is *μ-stationary* if $\nu = \int_G g_\star \nu \; d\mu(g)$, and in that case the pair (B, ν) is called a *(G, μ)-space*. Recall that a *μ-boundary* is a measurable (G, μ)-space (B, ν) such that there exists a T-invariant, measurable map **bnd** : $(\Omega, \mathbb{P}) \to (B, \nu)$, called the *boundary map*.

Moreover, a function $f : G \to \mathbb{R}$ is *μ-harmonic* if $f(g) = \int_G f(gh) \; d\mu(h)$ for any $g \in G$. We denote by $H^\infty(G, \mu)$ the space of bounded, μ-harmonic functions. One says a μ-boundary is the *Poisson boundary* of (G, μ) if the map

$$\Phi : H^\infty(G, \mu) \to L^\infty(B, \nu)$$

given by $\Phi(f)(g) := \int_B f \; dg_\star \nu$ is a bijection. The Poisson boundary (B, ν) is the maximal μ-boundary, in the sense that for any other μ-boundary (B', ν') there exists a G-equivariant, measurable map $p : (B, \nu) \to (B', \nu')$. The result of this section concerns the shape of all sample paths.

Theorem 5.1. *Let G be the mapping class group $Mod(S)$ of a finite type surface, or let G be a relative hyperoblic group. Let μ be a probability measure on G with finite first moment with respect to the metric d, such that the semigroup generated by the support of μ is a non-amenable group. Then there exists a constant A such that almost every sample path (w_n) is such that $(w_n \mathfrak{o})$ is a (A, κ)-ray. Moreover, one can take $\kappa(r) = \log(2 + r)$.*

Proof. By Theorem C in [QRT20],

$$\limsup_{n \to \infty} \frac{d_w(w_n, \gamma_\omega)}{\log n} < +\infty.$$

Thus there exists a C such that

$$\limsup_{n \to \infty} d_w(w_n, \gamma_\omega) \leq C \log n$$

By [MT18], weakly hyperbolic groups have positive drift on their respective associated hyperbolic spaces. And by the Distance Formula (Masur-Minsky [MM00]), distance (to the origin) in the random walk on the group is coarsely bounded by the distance to the origin in the associated curve graph d_S. Thus random walks on a mapping class groups have positive drifts ([MT18]) That is to say, there exists an A such that for n large enough

$$d(w_n \mathfrak{o}, \mathfrak{o}) \geq An.$$

Let p_n denotes the nearest point projection of w_n to γ_ω. We have by triangle inequality

$$d(\mathfrak{o}, p_n) \geq d(w_n \mathfrak{o}, \mathfrak{o}) - d(w_n \mathfrak{o}, p_n)) \geq An - C \log n.$$

We have that as $n \to \infty$

$$An \geq \lim d(\mathfrak{o}, p_n) \lim \geq An - C \log n = An = d(\mathfrak{o}, \gamma(An)),$$

where the last equality comes from the fact that γ is unit speed. Therefore

$$d(w_n \mathfrak{o}, p_n) = C \log n = d(w_n \mathfrak{o}, \gamma(An)).$$

Next we construct a map on γ such that for each $i \in \mathbb{N}$ and $t \in [i - \frac{1}{2}, i + \frac{1}{2})$ we define:

$$\Phi(\gamma(At)) := w_i.$$

We need to show that this is a sublinear bi-Lipschitz equivalence. For any given t, t', assume that $t \in [i - \frac{1}{2}, i + \frac{1}{2})$ and $t' \in [j - \frac{1}{2}, j + \frac{1}{2})$ and also assume without loss of generality that $j - i$, we have that

$$\begin{aligned} d(\Phi(\gamma(At), \gamma(At')) = d(w_i, w_j) &\le d(w_i, \gamma(Ai)) + |j - i|A + d(w_j, \gamma(Aj)) \\ &\le A|j - i| + 2C \log w_j \\ &\le A|t - t' + 1| + 2C \log w_j \\ &\le |t - t'| + A_2 C \log w_j \end{aligned}$$

On the other hand we have

$$\begin{aligned} d(\Phi(\gamma(At), \gamma(At')) &= d(w_i, w_j) \\ &\ge A|j - i| - d(w_i, \gamma(Ai)) - d(w_i, \gamma(Ai)) \\ &\ge A|j - i| - 2C \log j \\ &\ge A|t - t'| - A - 2C \log w_j. \end{aligned}$$

Thus Φ is an SBE and the image of the geodesic γ_ω contains the sample path and is an $(A, \log n)$-ray. The proof for G is a relative hyperbolic group is identical, using also the facts that relative hyperbolic group is weakly hyperbolic and there is also a distance formula that is similar to that of the mapping class group [Sis13]. □

More generally, aside from the aforementioned two groups, the conclusion can be applied to a wider range of countable groups with a compact boundary. We combine the same argument as Theorem 5.1 and apply Theorem 6 in [Tio15] to obtain the following:

Theorem 5.2. *Let G be a countable group acting via isometries on a proper, geodesic, metric space (X, d) with a non-trivial, stably visible compactification X. Let G be a finitely generated group, and let (X, d) be a Cayley graph of G. Let μ be a probability measure on G with finite first moment with respect to d, such that the semigroup generated by the support of μ is a non-amenable group. Then there exists a constant A and a sublinear function κ such that almost every sample path (w_n) is such that $(w_n \mathfrak{o})$ is a (A, κ)-ray.*

Example of such compactifications includes but are not limited to:

(1) the hyperbolic compactification of Gromov hyperbolic spaces;
(2) the end compactification of Freudenthal and Hopf [Hop44];
(3) the Floyd compactification (Section 3.2, [Tio15]);
(4) the visual compactification of a large class of CAT(0) spaces (Section 3.4, [Tio15]).
(5) the redirecting compactification of asymptotically tree-graded spaces ([QRT22]).

Appendix A. SBE and κ-contracting quasi-geodesic rays

In this section we check a phenomenon that is independent of the rest of the paper, that is, if an (L, θ)-ray sublinearly tracks a κ-Morse set, then it tracks that κ-Morse set uniformly, i.e. with a well-defined sublinear function that is related to κ and not the function that governs the tracking. As always let X is a proper geodesic metric space. We define below κ-projections and κ-weakly contracting according to [QRT20, Definition 5.3].

Definition A.1 (κ-projection, κ-weakly contracting)**.** Let $Z \subseteq X$ a closed subset, and let κ be a sublinear function with properties as prescribed in §2.1. A map $\pi_Z \colon X \to \mathcal{P}(Z)$ is a *κ-projection* if there exist constants D_1, D_2 such that for any points $x \in X$ and $z \in Z$,

$$\operatorname{diam}_X(\{z\} \cup \pi_Z(x)) \leq D_1 \cdot d_X(x, z) + D_2 \cdot \kappa(x). \tag{A.1}$$

Further, we say that Z is *κ-weakly contracting* with respect to π_Z if there are constants $C_1, C_2 > 0$ such that for all $x, y \in X$,

$$d(x, y) \leqslant C_1 d(x, Z) \Longrightarrow \operatorname{diam}(\pi_Z(x) \cup \pi_Z(y)) \leqslant C_2 \kappa(x).$$

Remark A.2. Cornulier has defined the notion of a (non set-valued) $O(\kappa)$-retraction ([Cor19, Definition 2.5], there called $O(v)$-retraction) in reformulating the main theorem of [Cor11]. Though the two notions look similar, they do not coincide. An $O(\kappa)$-retraction in Cornulier's sense is only asked to satisfy $d_X(x, z) + D_2 \cdot \kappa(\max(\|x\|, \|z\|))$ instead of the right-hand side of Equation (A.1).

Lemma A.3. *Let π_Z be a κ-projection on Z. For any $x \in X$*

$$\operatorname{diam}_X(\{x\} \cup \pi_Z(x)) \leq (D_1 + 1) \cdot d_X(x, Z) + D_2 \cdot \kappa(x).$$

Proof. See [QRT20, Lemma 5.2]. □

Let κ be a concave sublinear function and let Z be a closed subspace of X. Let π_Z be a κ-projection onto Z and suppose that Z is κ-weakly contracting with respect to π_Z. Then, (L, θ)-rays that end sublinearly close to Z stay in a κ-neighbourhood of Z whose constants only depends on Z, L, θ. Specifically,

Theorem A.4 ((L, θ)-rays that sublinearly track a κ-contracting geodesic ray uniformly track the κ-contracting geodesic ray; compare [QRT20, Theorem A.1].)**.** *Assume that κ dominates θ. Let Z be a κ-weakly contracting subset of a proper geodesic space X. Then there is a function $n_Z(L, \theta)$ such that, for every $r \geq L\theta(r)$ and every sublinear function κ', there is an $R = R(Z, r, \kappa', \theta) > 0$ where the following holds: Let $\eta \colon [0, \infty) \to X$ be a (L, θ)-ray. Let t_r be the first time $\|\eta(t_r)\| = r$ and let t_R be the first time $\|\eta(t_R)\| = R$. Then*

$$d_X\big(\eta(t_R), Z\big) \leq \kappa'(R) \quad \Longrightarrow \quad \eta([0, t_r]) \subset \mathcal{N}_\kappa\big(Z, n_Z(L, \theta)\big).$$

Proof. The proof originates in [QRT19, Theorem 3.16]. We shall reproduce its reinstalment in [QRT20, Appendix], though not all of it. The main change being that η is now a (L, θ)-ray rather than a (q, Q)-geodesic.

Let C_1, C_2, D_1, D_2 be the constants which appear in the definitions of κ-projection and κ-weakly contracting (Definition A.1).

Note that the condition of being κ-weakly contracting becomes weaker as C_1 gets smaller, hence we can assume that $C_1 \leq 1/2$. We first set

$$m_0 := \max\left\{\frac{L(LC_2 + q + 1) + 1}{C_1}, \frac{2C_2(D_1 + 1)}{L} + 1\right\}, \qquad m_1 := L(C_2 + 1)(D_1 + 1). \tag{A.2}$$

Claim A.5. Consider a time interval $[s, s']$ during which η is outside of $\mathcal{N}_\kappa(Z, m_0)$. Then there exists a constant $\mathfrak{A}$ depending only on $\{C_1, C_2, D_1, D_2, L\}$, such that

$$|s' - s| \leq m_1\big(d_X\big(\eta(s), Z\big) + d_X\big(\eta(s'), Z\big)\big) + \mathfrak{A} \cdot \kappa(\eta(s')). \tag{A.3}$$

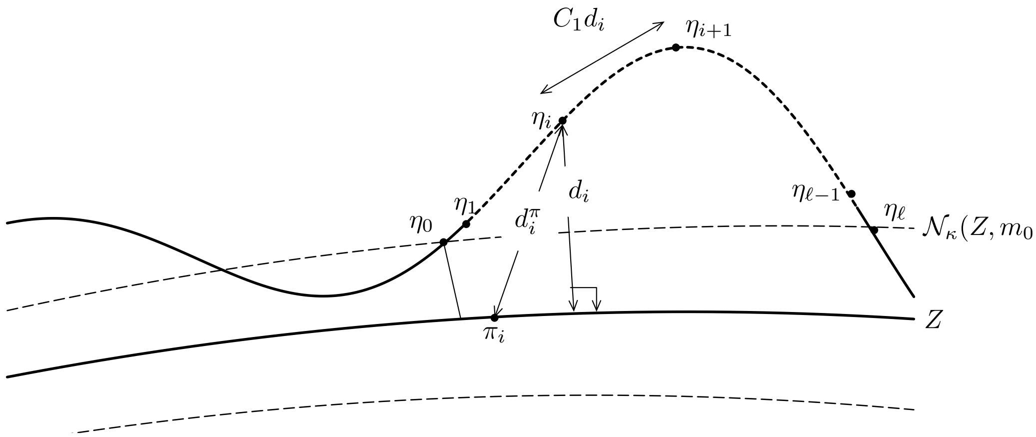

FIGURE 8. Proof of Claim A.5.

Proof of Claim A.5. Let

$$s = t_0 < t_1 < t_2 < \cdots < t_\ell = s'$$

be a sequence of times such that, for $i = 0, \ldots, \ell - 2$, we have t_{i+1} is a first time after t_i where

$$d_X\big(\eta(t_i), \eta(t_{i+1})\big) = C_1 d_X(\eta(t_i), Z) \quad \text{and} \quad d_X\big(\eta(t_{\ell-1}), \eta(t_\ell)\big) \leq C_1 d_X(\eta(t_{\ell-1}), Z). \tag{A.4}$$

To simplify the notation, we define

$$\eta_i := \eta(t_i), \qquad r_i := \|\eta(t_i)\|$$

and moreover, we pick some $\pi_i \in \pi_Z(\eta_i)$ and let

$$d_i^\pi := d_X(\eta_i, \pi_i), \qquad d_i := d_X(\eta_i, Z).$$

Note that, by assumption

$$d_i^\pi \geq d_i = d_X(\eta_i, Z) \geq m_0 \cdot \kappa(r_i). \tag{A.5}$$

The following estimate is obtained inside the proof of [QRT20, Claim A.3], using all the assumption $C_1 \leqslant 1/2$. Though η is actually a (q, Q)-quasigeodesic rather than a (L, θ)-ray there, the quasigeodesicity is not used for this estimate, so we do not reproduce the proof.

Claim A.6. We have the inequality $d_{\ell-1}^\pi \leq 2(D_1 + 1)\, d_\ell^\pi + D_2 \cdot \kappa(\eta_{\ell-1})$.

Now, since Z is κ-weakly contracting with respect to π_Z, by Definition A.1 we get

$$d_X\big(\pi_0, \pi_\ell\big) \leq \sum_{i=0}^{\ell-1} d_X\big(\pi_i, \pi_{i+1}\big) \leq \sum_{i=0}^{\ell-1} C_2 \cdot \kappa(r_i).$$

But η is (L,θ)-ray, hence,

$$\begin{aligned} |s'-s| &\leq L\, d_X(\eta_0,\eta_\ell)+\theta(\eta_\ell) \\ &\leq L\left(d_0^\pi + d_X(\pi_0,\pi_\ell)+d_\ell^\pi\right)+\theta(\eta_\ell) \\ &\leq L\, C_2\left(\sum_{i=0}^{\ell-1}\kappa(r_i)\right)+L\,(d_0^\pi+d_\ell^\pi)+\theta(\eta_\ell). \end{aligned} \tag{A.6}$$

On the other hand,

$$|s'-s| = \sum_{i=0}^{\ell-1}|t_{i+1}-t_i| \geq \frac{1}{L}\sum_{i=0}^{\ell-1}\left(d_X(\eta_i,\eta_{i+1})-\theta(\eta_\ell)\right). \tag{A.7}$$

Meanwhile, for $i=0,\dots,\ell-2$, we have $d_X(\eta_i,\eta_{i+1}) = C_1 d_X(\eta_i,Z)$. Furthermore, we have by the triangle inequality,

$$d_X(\eta_{\ell-1},\eta_\ell)+d_\ell^\pi+d_X(\pi_{\ell-1},\pi_\ell)\geq d_{\ell-1}^\pi \geq d_X(\eta_{\ell-1},Z),$$

which gives

$$d_X(\eta_{\ell-1},\eta_\ell)\geq d_X(\eta_{\ell-1},Z)-d_\ell^\pi - C_2\cdot\kappa(r_{\ell-1}). \tag{A.8}$$

In the last inequality, we used that $d_X(\pi_{\ell-1},\pi_\ell)\leqslant C_2\cdot\kappa(r_{\ell-1})$ since Z is κ-weakly contracting with respect to π_Z. Hence, together with Equation (A.5) and using $C_1\leq 1$ we have

$$\begin{aligned} |s'-s| &\geq \frac{1}{L}\sum_{i=0}^{\ell-1}\left(C_1 d_X(\eta_i,Z)-\theta(\eta_\ell)\right)-\frac{d_\ell^\pi + C_2\cdot\kappa(r_{\ell-1})}{L} && \text{by Equations (A.7) and (A.8)} \\ &\geq \frac{1}{L}\sum_{i=0}^{\ell-1}\left(C_1 m_0\cdot\kappa(r_i)-\theta(\eta_\ell)\right)-\frac{d_\ell^\pi}{L}-C_2\frac{\kappa(r_{\ell-1})}{L} && \text{by Equation (A.5)} \\ &\geq \left(\frac{C_1 m_0 - 1}{L}\right)\sum_{i=0}^{\ell-1}\kappa(r_i)-\frac{d_\ell^\pi}{L}-C_2\frac{\kappa(r_{\ell-1})}{L} && \text{since } \kappa \text{ dominates } \theta. \end{aligned}$$

Combining the above inequality with Equation (A.6) we get

$$\begin{aligned} L\,(d_0^\pi+d_\ell^\pi)+\theta(\eta_\ell)+\frac{d_\ell^\pi}{L}+C_2\frac{\kappa(r_{\ell-1})}{L} \geq |s-s'| &\geq \left(\frac{C_1 m_0-1}{L}-L\,C_2\right)\sum_{i=0}^{\ell-1}\kappa(r_i) \\ &\geq (L+1)\sum_{i=0}^{\ell-1}\kappa(r_i), \end{aligned} \tag{A.9}$$

where in the last step we plugged in the definition of m_0 from (A.2).

Applying Claim A.5 to its last term, the left side of Equation (A.9) is also bounded above:

$$L\,(d_0^\pi+d_\ell^\pi)+\theta(\eta_\ell)+\frac{d_\ell^\pi}{L}+C_2\frac{\kappa(r_{\ell-1})}{L}\leq L\,(d_0^\pi+d_\ell^\pi)+\theta(\eta_\ell)+\frac{d_\ell^\pi}{L}+C_2\frac{d_\ell^\pi}{m_0 L}$$

Let the right-hand side above be labelled $\star$, we then have

$$\begin{aligned}
\star &\le L\,(d_0^\pi + d_\ell^\pi) + \theta(\eta_\ell) + \frac{d_\ell^\pi}{L} + \frac{C_2}{m_0 L}[2(D_1+1)d_\ell^\pi + D_2\kappa(\eta_{\ell-1})] && \text{by Claim A.6}\\
&= L\,(d_0^\pi + d_\ell^\pi) + \theta(\eta_\ell) + \frac{d_\ell^\pi}{L} + \frac{C_2}{m_0 L}2(D_1+1)d_\ell^\pi + \frac{C_2 D_2}{m_0 L}\kappa(\eta_{\ell-1})\\
&= L\,(d_0^\pi + d_\ell^\pi) + \frac{d_\ell^\pi}{m_0} + \frac{d_\ell^\pi}{L} + \frac{C_2}{m_0 L}2(D_1+1)d_\ell^\pi + \frac{C_2 D_2}{m_0 L}\kappa(\eta_{\ell-1})\\
&= L\,(d_0^\pi + d_\ell^\pi) + \left(\frac{1}{m_0} + \frac{1}{L} + \frac{C_2(D_2+1)}{m_0 L}\right) d_\ell^\pi + \frac{C_2 D_2}{m_0 L}\kappa(\eta_{\ell-1})\\
&\le L\,(d_0^\pi + d_\ell^\pi) + d_\ell^\pi + \frac{C_2 D_2}{m_0 L}\kappa(\eta_{\ell-1}) && \text{by Definition of } m_0 \text{ in Equation (A.2)}\\
&\le (L+1)(d_0^\pi + d_\ell^\pi) + \frac{C_2 D_2}{m_0 L}\kappa(\eta_{\ell-1}).
\end{aligned}$$

Plugging this inequality into Equation (A.9) and dividing by $L+1$ on both sides, we get

$$\begin{aligned}
\sum_{i=0}^{\ell-1}\kappa(r_i) &\le d_0^\pi + d_\ell^\pi + \frac{C_2 D_2}{m_0\, L(L+1)}\cdot\kappa(\eta_{\ell-1})\\
&\le (D_1+1)(d_0+d_\ell) + D_2(\kappa(\eta_0)+\kappa(\eta_\ell)) + \frac{C_2 D_2}{m_0\, L(L+1)}\cdot\kappa(\eta_{\ell-1})
\end{aligned}$$

where we recall $d_i = d_X(\eta_i, Z)$, and the last inequality comes from Lemma A.3 (note the difference between d_i and $d_i^\pi = d_X(\eta_i, \pi_i)$).

Recall that by construction $s' > t_i$ and we can take s' to be big enough such that $s' > \theta(s')$:

$$\begin{aligned}
\big|\|\eta_i\| - \|\eta(s')\|\big| &\le d(\eta_i, \eta(s'))\\
&\le L|s' - t_i| + \theta(\|s'\|)\\
&\le L|s'| + \theta(s')\\
&\le (L+1)|s'|\\
&\le (L+1)\big(L\|\eta(s')\| + \theta(\|s'\|)\big)\\
&\le (L+1)^2\|\eta(s')\|
\end{aligned}$$

Thus we have $\|\eta_i\| \le ((L+1)^2+1)\|\eta(s')\|$. Hence $\kappa(\eta_i) \le ((L+1)^2+1)\cdot\kappa(\eta(s'))$; thus, to shorten the preceding expression,

let $\mathfrak{A}$ be a constant, depending on $\{C_1, C_2, D_1, D_2, L, \theta, \kappa\}$, such that

$$L(C_2+1)D_2(\kappa(\eta_0)+\kappa(\eta_\ell)) + \theta(\eta_\ell) + \frac{C_2^2 D_2}{m_0(q+1)}\cdot\kappa(\eta_{\ell-1}) \le \mathfrak{A}\cdot\kappa(\eta(s')).$$

By Equation (A.6) and the definition $m_1 = L(C_2+1)(D_1+1)$ from (A.2),

$$|s' - s| \le m_1(d_0 + d_\ell) + \mathfrak{A}\cdot\kappa(\eta(s')).$$

This proves Claim A.5. □

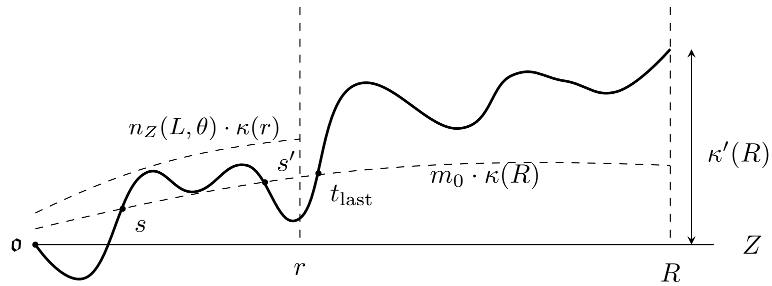

Figure 9. Proof of Theorem A.4

Now let t_{last} be the last time η is in $\mathcal{N}_\kappa(Z, m_0)$ and consider the path $[\eta([t_{\text{last}}), \eta(t_R)]_\eta$. Since this path is outside of $\mathcal{N}_\kappa(Z, m_0)$, by Claim A.5 we have

$$|t_R - t_{\text{last}}| \leq m_1\left(d_X(\eta(t_{\text{last}}), Z) + d_X(\eta(t_R), Z)\right) + \mathfrak{A}\cdot\kappa(R).$$

But

$$\begin{aligned} d_X(\eta(t_{\text{last}}), Z) &\leq m_0\cdot\kappa(\eta(t_{\text{last}})) && \text{by the choice of } t_{\text{last}} \\ &\leq m_0\cdot\kappa(R) && \text{since } \kappa \text{ is nondecreasing} \end{aligned}$$

and we have by assumption $d_X(\eta(t_R), Z) \leq \kappa'(R)$. Therefore,

$$|t_R - t_{\text{last}}| \leq m_0\, m_1\cdot\kappa(R) + m_1\cdot\kappa'(R) + \mathfrak{A}\cdot\kappa(R). \tag{A.10}$$

Since η is a (L,θ)-ray, we obtain $R = d_X(\eta(0), \eta(t_R)) \leq L t_R + \theta(t_R)$, hence

$$t_R \geq \frac{R - \theta(t_R)}{L}.$$

Since m_0 and m_1 are given and κ, κ' and θ are sublinear, there is a value of R depending on m_0, m_1, r, $\mathfrak{A}$, κ, κ' and θ such that

$$m_0\cdot m_1\cdot\kappa(R) + m_1\cdot\kappa'(R) + \mathfrak{A}\cdot\kappa(R) \leq \frac{R-\theta(t_R)}{L} - r.$$

For any such R, we then have

$$t_{\text{last}} \geq t_R - \frac{R-\theta(t_R)}{L} + r \geq r.$$

We show that $\eta([0, t_{\text{last}}])$ stays in a larger κ-neighborhood of Z. Consider any other subinterval $[s, s'] \subset [0, t_{\text{last}}]$ where η exits $\mathcal{N}_\kappa(Z, m_0)$. By taking $[s, s']$ as large as possible, we can assume $\eta(s), \eta(s') \in \mathcal{N}_\kappa(Z, m_0)$. In this case,

$$d_X(\eta(s), Z) \leq m_0\cdot\kappa(\eta(s)) \qquad \text{and} \qquad d_X(\eta(s'), Z) \leq m_0\cdot\kappa(\eta(s')).$$

Again applying Claim A.5, we get

$$|s' - s| \leq m_0\, m_1\cdot\big(\kappa(\eta(s)) + \kappa(\eta(s'))\big) + \mathfrak{A}\cdot\kappa(\eta(s')) \tag{A.11}$$

and thus

$$\begin{aligned} d_X(\eta(s'), \eta(s)) &\leq L\, m_0\, m_1\cdot\big(\kappa(\eta(s)) + \kappa(\eta(s'))\big) + q\,\mathfrak{A}\cdot\kappa(\eta(s')) + \theta(\eta(s')) \\ &\leq (2L\, m_0\, m_1 + L\,\mathfrak{A} + 1)\cdot\max\big(\kappa(\eta(s)), \kappa(\eta(s'))\big). \end{aligned}$$

as κ dominates θ. Applying Lemma 2.1 we obtain

$$\kappa(\eta(s')) \leq m_2 \cdot \kappa(\eta(s))$$

for some m_2 depending on q, θ and κ. Therefore, by plugging this inequality back into Equation (A.11), we have for any $t \in [s, s']$

$$|t - s| \leq (m_0\, m_1(1 + m_2) + \mathfrak{A}\, m_2) \cdot \kappa(\eta(s)) = m_3 \cdot \kappa(\eta(s)). \tag{A.12}$$

with $m_3 = (m_0\, m_1(1 + m_2) + \mathfrak{A}\, m_2)$. As before, this implies

$$d_X(\eta(t), \eta(s)) \leq L\, m_3 \cdot \kappa(\eta(s)) + \theta(\eta(s)) \leq (L\, m_3 + 1) \cdot \kappa(\eta(s)),$$

where the last inequality comes from the fact that $\theta \leq \kappa$. Applying the Estimation Lemma again, we have

$$\kappa(\eta(s)) \leq m_4 \cdot \kappa(\eta(t)), \tag{A.13}$$

for some m_4 depending on q, θ and κ.

Now, for any $t \in [s, s']$ we have

$$\begin{aligned} d_X(\eta(t), Z) &\leq d_X(\eta(t), \eta(s)) + r_0 \\ &\leq L\,|t - s| + \theta(s) + m_0 \cdot \kappa(\eta(s)) \\ \text{(Equation (A.12))} \quad &\leq (L\, m_3 + 1 + m_0) \cdot \kappa(\eta(s)) \\ \text{(Equation (A.13))} \quad &\leq (L\, m_3 + 1 + m_0)\; m_4 \cdot \kappa(\eta(t)). \end{aligned}$$

Now setting

$$n_Z(L, \theta) = (L\, m_3 + 1 + m_0)\; m_4 \tag{A.14}$$

we have the inclusion

$$\eta([s, s']) \subset \mathcal{N}_\kappa\big(Z, n_Z(L, \theta)\big) \qquad \text{and hence} \qquad \eta([0, t_{\text{last}}]) \subset \mathcal{N}_\kappa\big(Z, n_Z(L, \theta)\big).$$

□

There are strong connections between κ-Morse and κ-contracting quasi-geodesic rays. In particular we have the following two facts:

(F1). [QRT19] Let X be a proper, complete CAT(0) space. The a (q, Q)-quasi-geodesic ray is κ-Morse if and only if it is κ-contracting.

(F2). [QRT20] Let X be a proper, geodesic space and let α be a (q, Q)-quasi-geodesic ray. If α is κ-contracting then it is κ-Morse. If α is κ-Morsre then it is κ'-contracting for some other κ'.

References

[Cho] Inhyeok Choi *Limit laws on Outer space, Teichmmüller space and CAT(0) spaces* in preparation.

[CK00] C. B. Croke and B. Kleiner, *Spaces with nonpositive curvature and their ideal boundaries*, Topology 39 (2000), no. 3, 549–556.

[Cor08] Yves Cornulier. *Dimension of asymptotic cones of Lie groups.* J. Topology, **1** (2), 343–361, 2008.

[Cor11] Yves Cornulier. *Asymptotic cones of Lie groups and cone equivalences* Illinois J. Math., 55 (1), 237–259, 2011.

[Cor19] Yves Cornulier. *On sublinear bilipschitz equivalence of groups.* Ann. Sci. Éc. Norm. Supér. (4), 52(5):1201–1242, 2019.

[Dru02] Cornelia Druţu *Quasy-isometry invariants and asymptotic cones*, Int. J. Alg. Comp. 12 (2002), 99-135.

[DK18] Cornelia Druţu and Michael Kapovich. *Geometric group theory*, volume 63 of *American Mathematical Society Colloquium Publications.* American Mathematical Society, Providence, RI, 2018. With an appendix by Bogdan Nica.

[GQR22] Ilya Gekhtman, Yulan Qing and Kasra Rafi *QI-invariant model of Poisson boundaries of* CAT(0) *groups* arXiv:2208.04778

[Gro87] Mikhael L. Gromov. *Hyperbolic groups.* Essays in group theory, volume 8 of *Math. Sci. Res. Inst. Publ.*, pages 75–263. Springer, New York, 1987.

[Gro93] Mikhael L. Gromov. *Asymptotic invariants of infinite groups.* Geometric group theory, Vol. 2 (Sussex, 1991), volume 182 of *London Math. Soc. Lecture Note Ser.*, pages 1–295. Cambridge Univ. Press, Cambridge, 1993.

[Hop44] Hopf, H., *Enden offener Räume und unendliche diskontinuierliche Gruppen,* Comment. Math. Helv. 16 (1944), 81–100.

[IZ] Merlin Incerti-Medici and Abdul Zalloum *Sublinearly Morse boundaries from the view point of combinatorics* arXiv:2101.01037

[Kai00] V. Kaimanovich, *The Poisson formula for groups with hyperbolic properties*, Ann. of Math. (2) 152 (2000), no. 3, 659–692.

[Mah10] J. Maher, *Linear progress in the complex of curves*, Trans. Amer. Math. Soc. 362 (2010), no. 6, 2963–2991.

[Mah12] J. Maher, *Exponential decay in the mapping class group*, J. Lond. Math. Soc. (2) 86 (2012), no. 2, 366–386.

[MT18] J. Maher and G. Tiozzo, Random walks on weakly hyperbolic groups, J. Reine Angew. Math. 742, (2018), 187–239.

[MM00] H. Masur and Y. Minsky, *Geometry of the complex of curves. II. Hierarchical structure*, Geom. Funct. Anal. 10 (2000), no. 4, 902–974.

[MQZ21] Devin Murray, Yulan Qing and Abdul Zalloum *Sublinearly Morse geodesics in CAT(0) spaces: lower divergence and hyperplane characterization* to appear in Algebraic & Geometric Topology.

[MS20] P. Mathieu and A. Sisto, *Deviation inequalities for random walks*, Duke Math. J. 169 (2020), no. 5, 961–1036.

[Mor24] H. M. Morse, *A fundamental class of geodesics on any closed surface of genus greater than one*, Trans. Amer. Math. Soc. 26 (1924), 25–60.

[NQ22] Hoang Thanh Nguyen, Yulan Qing *Sublinearly Morse Boundary of CAT(0) admissible groups* arXiv:2203.00935

[Pal20] Gabriel Pallier. Large-scale sublinearly Lipschitz geometry of hyperbolic spaces. *J. Inst. Math. Jussieu*, 19(6):1831–1876, 2020.

[Pau96] Frédéric Paulin. Un groupe hyperbolique est déterminé par son bord. *J. London Math. Soc. (2)*, 54(1):50–74, 1996.

[Qin16] Y. Qing, *Geometry of Right-Angled Coxeter Groups on the Croke-Kleiner Spaces*, Geom. Dedicata 183 (2016), no. 1, 113–122.

[QRT19] Y. Qing, K. Rafi and G. Tiozzo, *Sublinearly Morse boundary I: CAT(0) spaces*, Advances in Mathematics

[QRT20] Yulan Qing, Kasra Rafi, and Giulio Tiozzo. *Sublinearly Morse Boundary II: Proper geodesic spaces* arXiv:2011.03481

[QRT22] Yulan Qing, Kasra Rafi, and Giulio Tiozzo. *Compactification of Asymptotically Tree-graded Spaces*, in preparation

[QT21] Y. Qing and G. Tiozzo, *Excursions of generic geodesics in right-angled Artin groups and graph products*, Int. Math. Res. Not. IMRN 2021 (22), 16910–16937

[RV18] K. Rafi and Y. Verberne, *Geodesics in the mapping class groups*, preprint arXiv:1810.12489.

[Ser83] C. Series, *Martin boundaries of random walks on Fuchsian groups*, Israel J. Math. 44 (1983), 221–242.

[Sis13] A. Sisto, *Projections and relative hyperbolicity*, Enseign. Math. (2) 59 (2013), no. 1-2, 165–181.

[Sis17] A. Sisto, *Tracking rates of random walks*, Israel J. Math. 220 (2017), 1–28.

[Sis18] A. Sisto, *Contracting elements and random walks*, J. Reine Angew. Math. 742 (2018), 79–114.

[ST19] A. Sisto and S. Taylor, *Largest projections for random walks and shortest curves for random mapping tori*, Math. Res. Lett. 26 (2019) no. 1, 293–321.

[Tio15] G. Tiozzo, *Sublinear deviation between geodesics and sample paths*, Duke Math. J. 164 (2015), no. 3, 511–539.

[Zal21] Abdul Zalloum *Convergence of sublinearly contracting horospheres* Accepted at Geometric Dedicata.

Sorbonne université, IMJ-PRG, 75252 Paris Cedex 05, France
Email address: gabriel@pallier.org

Shanghai Center for Mathematical Sciences, Fudan University, China
Email address: yulan.qing@gmail.com