BOUNDED GEODESIC IMAGE THEOREM FOR THE NON-PERIPHERAL CURVE GRAPH

YULAN QING AND ALEXANDRA SQUIRES

Abstract. We prove that under mild assumptions on the surface, the Bounded Geodesic Image Theorem holds for the graph of non-peripheral curves associated with a surface of infinite type. Specifically, suppose that an infinite type surface Σ is stable and of sufficiently high end-complexity. Let $Y \subset \Sigma$ (compact or otherwise) be a non-peripheral subsurface that is disjoint from at least one non-peripheral curve. Then there exists a uniform $M > 0$ such that any geodesic segment in the graph of non-peripheral curves on Σ that cuts Y projects to a set of diameter bounded above by M in the non-peripheral curve graph of Y.

1. Introduction

The curve graph [Har81, MM98-I] has played a central role in the large-scale study of mapping class groups and Teichmüller spaces. In the finite-type setting, Masur and Minsky introduced subsurface projections and established the bounded geodesic image theorem, one of the foundational tools of the hierarchical approach to mapping class groups. After this seminal result, Webb [Web13], among others, gave a short proof of the bounded geodesic image theorem using the hyperbolicity of the curve graph, and emphasized that geodesics in the ambient curve graph have uniformly bounded projection to any proper subsurface that every vertex of the geodesic cuts.

For infinite-type surfaces, one seeks analogous coarse-geometric structures, but the classical curve graph is often too small from the large-scale perspective: in many infinite-type cases the usual curve graph has finite diameter, so it cannot detect any interesting unbounded geometry of the mapping class group. This difficulty has led to the search for alternative hyperbolic graphs adapted to big mapping class groups, such as the Sep_2-curve graph [DFV18], the grand arc graph [BNV23] and the Bestvina-Bromburg-Fujiwara complex of curve graphs of images of a nondisplaceable subsurface [HQR22], among many others. One such recent construction is the non-peripheral curve graph [BQR26] $C_{\mathrm{np}}(\Sigma)$, whose vertices are essential simple closed curves that cannot be pushed via homeomorphism out of every compact subsurface and whose edges given by disjointness. For a stable, infinite-type surface Σ with CB-generated $\mathrm{Map}(\Sigma)$ and sufficiently large end-complexity $\zeta(\Sigma)$, this graph is connected, has infinite diameter, is Gromov hyperbolic, and admits an action on $\mathrm{Map}(\Sigma)$ with unbounded orbits.

In this paper, we establish the bounded geodesic image theorem for non-peripheral curve graphs of infinite-type surfaces. Beyond its intrinsic interest, this theorem is a basic ingredient in hierarchical structures [BHS21], distance formulas [MM98-I, MM98-II], and the study of quasi-geodesics, and it provides a precise way to say that geodesics which remain visibly engaged with a subsurface cannot make unbounded progress when projected to that subsurface.

In the infinite-type setting, the relevant subsurfaces for a proper generalization of Bounded Geodesic Image Theorem should be non-peripheral and should be disjoint from at least one non-peripheral curve (i.e. the subsurface is proper in a non-peripheral sense). Furthermore, to construct the non-peripheral graph in a subsurface, we only consider mapping classes in Σ that acts on Y(Definition 3.4). The associated non-peripheral graph we denote as $C_{\mathrm{np}}(Y^\Sigma)$. Our main theorem is the following:

Theorem 1.1 (Bounded Geodesic Image Theorem for Non-Peripheral Curve Graphs)**.** *Let Σ be a stable, infinite-type surface such that $\zeta(\Sigma) \geq 5$. Let $Y \subset \Sigma$ be a subsurface of Σ such that Y is non-peripheral in Σ and Y is disjoint from a non-peripheral curve. where at least one boundary component of Y is a non-peripheral curve in Σ and Y. In addtion, if Y is infinite type, $\zeta(Y) \geq 5$. which are we using?*

Then there exists a constant $M = M(\Sigma)$ such that if

$$g = (\gamma_1, ...\gamma_n)$$

is a geodesic in $C_{\mathrm{np}}(\Sigma)$ such that every vertex γ_i cuts Y, then the subsurface projection of g to Y produces a set in $C_{\mathrm{np}}(Y^\Sigma)$ that has diameter at most M.

The end complexity $\zeta(\cdot)$ of an infinite-type surface is introduced in [BQR26] and recalled in Definition 2.6. The assumptions on the end-complexity of S and Y and the complexity of finite type subsurface is to ensure that the associated graphs are connected and δ-hyperbolic.

From the definition of $C_{\mathrm{np}}(Y^\Sigma)$ We note the following cases for the curve graphs of the subsurfaces we may consider:

- If Y is annular, then let $C_{\mathrm{np}}(Y^\Sigma)$ denote the annular arc graph of Y.
- If Y is compact and and $3g(Y) - 3 + b \geq 5$, then $C_{\mathrm{np}}(Y^\Sigma) = C(Y)$.
- If Y is a surface of infinite type, then suppose $\zeta(Y) \geq 5$ and consider $C_{\mathrm{np}}(Y^\Sigma)$ as defined in 3.5.

Secondly, we note that in order to construct $C_{\mathrm{np}}(Y^\Sigma)$, we do not isolate Y as an abstract surface but instead consider all homeomorphisms in $\mathrm{Map}(\Sigma)$ and whether the curves of Y can be moved to infinity in Y by these homeomorphisms (See Definition 3.4). Consequently, when Y is a compact subsurface, $C_{\mathrm{np}}(Y^\Sigma)$ is precisely the curve graph of Y (see 3.6). Therefore our main theorem can be viewed as an infinite-type analogue of the Masur–Minsky bounded geodesic image theorem, with the Masur-Minsky curve graph replaced by $C_{\mathrm{np}}(\Sigma)$. In the finite type setting, the bounded geodesic image theorem expresses a fundamental form of hierarchical control. In our setting, it shows that although $C_{\mathrm{np}}(\Sigma)$ is built from curves on an infinite-type surface, its geodesics still obey strong subsurface constraints once one restricted to the non-peripheral part of the topology.

The proof combines two inputs. As classically shown in [MM98-I], bounded projection is ultimately a consequence of the incompatibility between geodesicity in the ambient graph and large projection to a proper subsurface. We adapt the beautiful and short proof of [?], where any curve can be surgered to become an α-β loop. The surgeries we use are different as our resulting curves need to be non-peripheral. But the key idea remains the same: via these surgeries we create a coarse projection from any point in the curve graph to a bounded-constant quasi-geodesic segment with endpoints α and β. The second ingredient is [BQR26] which argues that non-peripheral curves and subsurfaces provide the correct replacement curve graph in the setting of big mapping class groups, and proves hyperbolicity of $C_{\mathrm{np}}(\Sigma)$.

1.1. **Acknowledgments.** We thank Richard Webb for his generous help throughout the project and we thank Kasra Rafi for helpful conversations. Yulan Qing is supported by Simons Foundation grant [SFI-MPS-TSM-00014066, Y.Q.]

2. Preliminaries

Throughout the paper, all surfaces are assumed to be connected and orientable unless explicitly stated otherwise. A *surface* is a second-countable Hausdorff 2-manifold. For a surface Σ, we write $g(\Sigma)$ for its genus, $E(\Sigma)$ for its end space, and $E^g(\Sigma) \subseteq E(\Sigma)$ for the subspace of ends accumulated by genus. Assuming that the surface is without boundary, by Richards' classification theorem [RI63], the homeomorphism type of Σ is determined by the triple

$$\big(g(\Sigma), E(\Sigma), E^g(\Sigma)\big).$$

The mapping class group of Σ, denoted $\mathrm{Map}(\Sigma)$, is the group of isotopy classes of orientation-preserving homeomorphisms of Σ.

A surface S is said to be of *finite type* if its fundamental group $\pi_1(S)$ is finitely generated. If S has finite type, its complexity is

$$\xi(S) = 3g(S) - 3 + p,$$

where p is the number of punctures of S. In this paper, we will customarily use Σ to denote an infinite-type surface and consider its *end-complexity* $\zeta(\Sigma)$ [BQR26] which we recall in Definition 2.6.

A *subsurface* $Y \subseteq \Sigma$ is a closed, connected subset of Σ whose boundary consists of pairwise-disjoint, essential, simple closed curves. In particular, no boundary component of Y is an infinite arc. A priori, Y itself can be either a surface of finite type or infinite type and with a finite or infinite number of boundary curves. For a subsurface $Y \subseteq \Sigma$, then the inclusion $Y \hookrightarrow \Sigma$ induces an embedding of end spaces, and we denote its image by $E(Y) \subseteq E(\Sigma)$.

2.1. **Curve graphs and non-peripheral curves.** Let Σ be a surface. A simple closed curve on Σ is *essential* if it does not bound a disk, a once-punctured disk, or a boundary component.

For a finite-type subsurface $Y \subseteq \Sigma$, we write $\mathcal{C}(Y)$ for the usual curve graph of Y: its vertices are isotopy classes of essential simple closed curves in Y, and two vertices are joined by an edge if they admit disjoint representatives.

The geometric intersection number between free homotopy classes a and b of simple closed curves on a surface S is defined to be the minimal number of intersection points between each pair of representative curves $\alpha \in a$ and $\beta \in b$:

$$i(a, b) := \min\{|\alpha \cap \beta| : \alpha \in a, \beta \in b\}.$$

We sometimes employ a slight abuse of notation by writing $i(\alpha, \beta)$ for the intersection number between the homotopy classes of simple closed curves α and β.

Definition 2.1 (Minimal position)**.** In practice, one computes the geometric intersection number between two homotopy classes a and b by finding representatives α and β that realize the minimal intersection in their homotopy classes, so that

$$i(a, b) = |\alpha \cap \beta|.$$

When this is the case, we say that α and β are in *minimal position*.

Definition 2.2 (General position)**.** We say a collection of essential, simple closed curves $\{\gamma_i\}$ is *in general position* if they are pairwise in minimal position and have no triple points. I.e. for distinct i, j, k, we have that $\gamma_i \cap \gamma_j \cap \gamma_k = \emptyset$.

We now recall the class of curves relevant to the coarse geometry of big mapping class groups.

Definition 2.3 (Non-peripheral curve)**.** An essential simple closed curve α on Σ is *non-peripheral* if there exists a compact subsurface $K_\alpha \subseteq \Sigma$ such that

$$f(\alpha) \cap K_\alpha \neq \varnothing$$

for every $f \in \mathrm{Map}(\Sigma)$.

We can say that K_α is an *anchor* for the non-peripheral curve α. Ap riori, K_α depends on the curve α. However, we can establish that this dependance is not necessary.

Lemma 2.4 (Universal anchor)**.** *There exists a compact set K_0 such that for any non-peripheral curve α we have that $\forall f \in \mathrm{Map}(\Sigma)$*

$$K_0 \cap f(\alpha) \neq \emptyset$$

.

In fact, one can take K_0 to be big enough such that the associated

$$V_{K_0} = \{\phi \in \mathrm{Map}(\Sigma) : \phi|_{K_0} = id$$

is a Coursely Bounded (CB)-generating set. We will assume K_0 to be the surface associated with a fixed CB-generating set from now on and we refer to K_0 as the *universal anchor.* See [MR23] for a description of CB-generating sets.

An equivalent formulation for the definition of non-peripheral that we will employ is often simpler to check:

Lemma 2.5 (Sufficient conditions for non-peripheral curves)**.** *A separating, essential curve α is non-peripheral if $\Sigma \smallsetminus \alpha$ has two connected components and neither component contains exactly one maximal end. (See* [BQR26]*)*

End-complexity of surfaces of infinite type. If S is a finite type surface, we then denote the surface as $S_{g,n}$ where g is the number of geni and n can be thought of the number of boundary components are punctures or as the number of equivalence classes of maximal ends. Recall the *complexity* of the surface S is defined as

$$\chi(S) = 3g - 3 + n.$$

However, when Σ is of infinite type, $3g-3+n$ is frequently infinite. In this case we use instead the following characterization of complexity for infinite-type surfaces.

Definition 2.6 (End-complexity)**.** We now recall $\zeta(\Sigma)$ [BQR26, Section 2.5, Definition 2.3], which measures the "end-complexity" of Σ in a way that is tailored to the CB setting. Concretely, $\zeta(\Sigma)$ is defined using the minimal anchor surface (a nondisplaceable surface that defines a CB-generating set) K_0 [MR23] where this stabilizer of K_0 is a CB subgroup of $\mathrm{Map}(\Sigma)$.

One should think of $\zeta(\Sigma)$ as an analogue of the finite-type complexity parameter "number of boundary components" (or punctures), which strongly influences the coarse geometry of $\mathrm{Map}(S)$ for finite-type surfaces. We show that $\zeta(\Sigma)$ predicts several coarse-geometric features of $\mathrm{Map}(\Sigma)$.

Definition 2.7. We say a clopen subset $X \subset E(\Sigma)$ is *small* if there exist $A \in \mathcal{A}$ and $g \in \mathrm{Map}(\Sigma)$ such that $g(X) \subset A$. Since every $P \in \mathcal{P}$ fits inside some $A \in \mathcal{A}$, if $g(X) \subset P$ then X is also small.

We now construct the combinatorial graph for this set of curves.

Definition 2.8 (Non-peripheral curve graph)**.** We denote the *non-peripheral curve graph* of Σ as by $\mathcal{C}_{np}(\Sigma)$. Its vertices are isotopy classes of non-peripheral, essential, simple closed curves, and two vertices are joined by an edge if they admit disjoint representatives.

The following result provides the hyperbolicity of the ambient graph in which we work.

Theorem 2.9. [BQR26, Theorem A] *Suppose that Σ is a stable, infinite-type surface and that either $E(\Sigma)$ contains at least five maximal ends or Σ has finite positive genus at least two. Then $\mathcal{C}_{np}(\Sigma)$ is connected, has infinite diameter, and is Gromov hyperbolic.*

Furthermore, we list more condition that yields non-peripheral curves.

Lemma 2.10. *(*[BQR26, Lemma 4.4]*) Let Σ be a connected, orientable surface of infinite type.*

(1) If α is a separating *peripheral curve then for some component Σ' of $\Sigma - \alpha$ the clopen set $E(\Sigma') \subset E(\Sigma)$ is small (in the sense of Definition 2.7).*
(2) If α is non-separating, *then α is peripheral if and only if Σ has infinite genus.*

3. The Bounded Geodesic Image Theorem

An *essential subsurface* is a subsurface that is not homeomorphic to a disk. Furthermore, we recall the analogous definition of a non-peripheral subsurface form [BQR26].

Definition 3.1. A subsurface Y is *non-peripheral* if there exists a compact subset $K \subset \Sigma$ such that for any $g \in \mathrm{Map}(\Sigma)$

$$g(Y) \cap K = \emptyset.$$

Let Σ be a connected, stable surface of infinite type with $\xi(S) \geq 5$. Assume $\mathrm{Map}(\Sigma)$ is CB-generated. Let Y be any infinite-type, essential, proper subsurface of Σ such that

$*$ *Y is non-peripheral and is disjoint from a non-peripheral curve.*

Furthermore, we assume that $\zeta(Y) > 5$, otherwise $C_{\mathrm{np}}(Y^{\Sigma})$ is not guaranteed to be connected.

Another observation we have is, if a curve is peripheral in Y then it is peripheral in $C_{\mathrm{np}}(\Sigma)$, as an homeomorphism of Y can be realized as homemorphism of Σ. In general we note that $C_{\mathrm{np}}(Y^{)} \not\subset C_{\mathrm{np}}(\Sigma)$, i.e. there may be curves non-peripheral in Y by not non-peripheral in Σ. See the following example.

Example 3.2. The $C_{\mathrm{np}}(\Sigma)$ of a flute surface is an empty graph. The subsurface that is one-side flute with a boundary component has non-empty non-peripheral curve graph.

3.1. Subsurface projection in surfaces of infinite type.

Suppose $\alpha \cap Y \neq \emptyset$. It follows that α intersects only finitely many components of ∂Y since α is contained in a compact subsurface of Y. We now recall subsurface projection from Masur-Minsky [MM98-II].

Definition 3.3 (Subsurface projection)**.** Let S be an orientable surface, let $Y \subset S$ be an essential subsurface, and let α be an essential simple closed curve on S.

Non-annular case. Assume that Y is not an annulus. Put α and ∂Y in minimal position and call $\partial Y = \{\gamma_1, \gamma_2, ...\}$.

- If α can be isotoped disjoint from Y, define

$$\pi_Y(\alpha) = \varnothing.$$

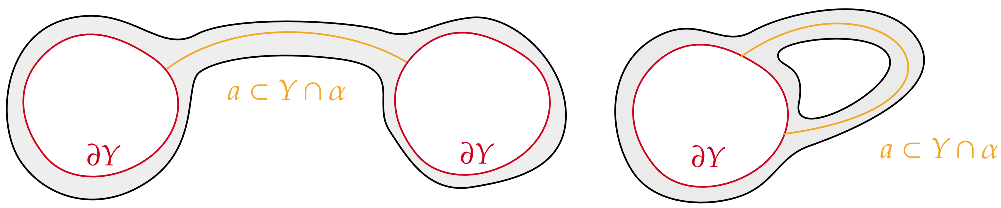

FIGURE 1. These are the two cases of the subsurface projection surgery. An arc $a \subset Y \cap \alpha$ can intersect one component of ∂Y twice or two components of ∂Y once each. We take the regular neighborhood of a unioned with the boundary components it has endpoints on and consider all non-peripheral and essential curves in the neighborhood. $\pi_Y(\alpha)$ is the set of all such curves for all arcs a.

- If $\alpha \subset Y$ then define $\pi_Y(\alpha) = \alpha$.
- Otherwise, consider the arcs of $\alpha \cap Y$. For each component a of $\alpha \cap Y$, a can either have two endpoints on one boundary component γ_i or have one endpoint on two boundary components γ_i and γ_j, since an arc that enters Y must exit Y and can either do so through the same boundary component that it entered through or a different boundary component. Then,

$$N(a \cup \gamma_i) \text{ or } N(a \cup \gamma_i \cup \gamma_j)$$

  are respectively closed regular neighborhoods of a and the boundary component(s) it has endpoints on in Y. Define $\pi_Y(\alpha)$ to be the set of isotopy classes of essential boundary components of this neighborhood, taken over all such essential arcs $a \subset \alpha \cap Y$.

Annular case. Assume that Y is an annulus with core curve γ. Let

$$p_Y \colon \widetilde{S}_Y \to S$$

be the annular cover corresponding to the subgroup $\langle \gamma \rangle \leq \pi_1(S)$. Compactify $\widetilde{S}_Y$ to a closed annulus. If α has lifts joining the two boundary components of this compactified annulus, define $\pi_Y(\alpha)$ to be the set of isotopy classes of all such lifts, viewed as vertices of the arc graph of the annulus Y. If no such lift exists, define

$$\pi_Y(\alpha) = \varnothing.$$

Definition 3.4 (A subgroup of the mapping class group)**.** For an essential subsurface $Y \subset \Sigma$ we define the set

$$\Gamma(Y) := \{\phi|_Y : \phi \in \mathrm{Map}(\Sigma), \phi(Y) = Y\}.$$

$\Gamma(Y)$ is a subset of $\mathrm{Map}(\Sigma)$. It can be verified that $\Gamma(Y)$ is a subgroup of $\mathrm{Map}(\Sigma)$ and a subgroup of $\mathrm{Map}(Y)$. To note its distinction from $\mathrm{Map}(Y)$ we write this group also as

$$\mathrm{Map}(Y^{\Sigma}) := \Gamma(Y).$$

The motivation for making this definition is to provide a more tolerant qualification for what makes a non-peripheral curve in Y by keeping track of the fact that Y is a subsurface of Σ.

Definition 3.5 (Non-peripheral curves on a subsurface)**.** For an essential subsurface $Y \subset \Sigma$, an essential, simple closed curve $\alpha \subset Y$ is *non-peripheral in the subsurface* Y if there exists a compact subset $K_\alpha \subset Y$ such that for any $f \in \mathrm{Map}(Y^{\Sigma})$,

$$K_\alpha \cap f(\alpha) \neq \emptyset$$

We denote the curve graph of curves contained in Y that are non-peripheral in Y^{Σ} as $C_{\mathrm{np}}(Y^{\Sigma})$.

Lemma 3.6. *If Y is compact, then $C_{\mathrm{np}}(Y^{\Sigma})$ is isomorphic to the curve graph $C(Y)$.*

Proof. For a curve on Y, its image under elements of $\mathrm{Map}(Y^{\Sigma})$ is contained in Y by Definition 3.4, which is itself compact. Thus every curve is non-peripheral in Y^{Σ} and therefore $C(Y) = C_{\mathrm{np}}(Y^{\Sigma})$. □

Given Definition 3.5, we can now claim the following:

Lemma 3.7. *If α is non-peripheral in Σ and $\alpha \in Y$, then α is non-peripheral in Y^{Σ}.*

Proof. By way of contradiction, if α is peripheral in Y^{Σ} then there exists a sequence of elements $\{g_i\} \in \mathrm{Map}(\Sigma)$ that restricts to mapping classes of Y and also maps α outside of larger and larger compact subset of Y. Thus they map α outside of larger and larger compact subset of Σ and thus α is peripheral in Σ. □

Note that $|\pi_Y(\alpha)| < \infty$ since α is a closed curve and therefore is itself a compact subsurface of Σ. This implies that α cannot intersect infinitely many components of ∂Y and that α cannot intersect any one component of ∂Y infinitely many times. So, consider the finite number of curves formed in the subsurface projection $\pi_Y(\alpha)$ by connecting $\alpha \cap Y$ with either of the arcs formed along the boundary curve where α cuts into Y. As discussed in Definition 3.3 there are two cases for the surgery that produces an element of $\pi_Y(\alpha)$. The following two lemmas show that in both cases, at least one of the curves produced will be non-peripheral. I.e., $\pi_Y(\alpha) \neq \emptyset$ for α intersecting at least one non-peripheral boundary curve of Y.

Lemma 3.8. *Suppose that $Y \subset \Sigma$ satisfies the assumptions of Theorem 1.1. That is, Y is an essential and non-peripheral subsurface in Σ and Y is also disjoint from a non-peripheral curve. Furthermore, $\zeta(Y) \geq 5$. Let α be a non-peripheral curve in Σ. Then at least one projection curves α_Y is non-peripheral in Y^{Σ}.*

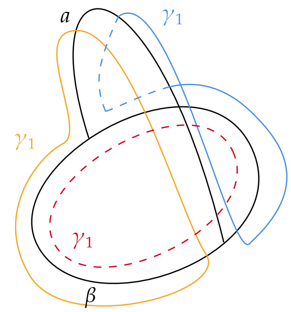

FIGURE 2. The three holed sphere given by a regular neighborhood of $a \cup \beta$ will have three boundary components γ_1, γ_2, and γ_3 shown in red, yellow, and blue. Whether $\Sigma - (a \cup \beta)$ has one, two, or three components depends on whether β is separating in Σ and whether a is separating in Y.

Proof. Consider α a non-peripheral curve on Σ and suppose $a \subset Y \cap \alpha$ has two endpoints on one component of $\beta \in \partial Y$.

Case 1: Assume α is non-separating. β is a priori either separating or non-separating. We address these possibilities. Suppose β is non-separating. Then the neighborhood of $a \cup \beta$ is a thrice punctured sphere. The boundary curves of this three punctured sphere bound three components of Σ. However, two out of three of these components must be connected to one another or β will be separating. Thus one of the resulting projection curve α_Y is non-separating in Σ. Since α is also non-separating and non-peripheral , we conclude that α_Y is non-peripheral in Σ and thus non-peripheral in Y^{Σ}.

Otherwise, β is separating. Then the end complexity of both complimentary components of β are ≥ 1 by Lemma 2.10. If a is a separating arc in Y, one of the surgery curves will be separating and bound components of end complexity ≥ 1. Also $\zeta(Y) \geq 5$ and thusβ bounds components of end complexity ≥ 1. Thus α_Y is non-peripheral in Σ and consequently non-peripheral in Y^{Σ}. If α is a non-separating arc in Y, then the projection curves are non-peripheral and by Lemma 2.10, the non-separating curves of a surface are either all non-peripheral or all peripheral. Since α itself is non-peripheral and non-separating , the projection curves are also non-peripheral .

Case 2: Now assume that α is separating.

If β is separating, then by Lemma 2.10 β bounds more than one equivalent class of Cantor-type ends or more than one singular end. One of the arcs of the surgery bound the same ends as β and therefore will be non-peripheral in Y. If β is non-separating and non-peripheral , then at least one of the surgery curves will be non-separating for the same reason as in case 1. The fact β

is non-peripheral implies all the non-separating curves are non-peripheral (Lemma 2.10 (2)), thus the projection curve is non-peripheral . On the other hand, we now address the case β is non-separating and peripheral, and α is non-peripheral and separating. Assume also that α contains at least one subarc that starts and ends on β. We claim the following:

Claim 3.9. *Suppose β is non-separating and peripheral, and α is non-peripheral and separating. Assume also that α contains at least one subarc that starts and ends on β. Then either a surgery curve is separating and non-peripheral , or α contains at least two subarcs that start and end on β and their partitions of β do not nest.*

Proof. Let α_1 be the first subarc of α with endpoints on β. Let P_1 be the curve that is the union of α_1 and a segment of β between the endpoints of α_1. Let

$$P_1' := \text{union of } \alpha_1 \text{ and the other segment of } \beta \text{ between the endpoints of } \alpha_1.$$

If P_1 is separating, then P_1' is also separating as they differ by a compact region. And by construction either P_1 or P_1' separating the boundary β from the rest of Y and is thus non-peripheral in Y^Σ and we are done. Otherwise P_1 is not separating in Y. This implies there exists a pair of x, x' points in a bounded neighborhood of β can be connected to each other by a path that does not intersect α_1. These points are on both sides of α which contradicts the assumption that α is separating, thus there exists another arc α_2 with endpoints on β such that the associated curve P_2 is also non-separating. □

In the first case we then have a non-peripheral surgery curve in Y^Σ; in the second case, one add α_2 to P_1 to create a separating curve that separates β from the rest of Y and is thus non-peripheral . This second surgery curve is distance one from the projection curve.

Suppose otherwise that α is a non-peripheral curve on Σ and suppose $a \subset (Y \cap \alpha)$ is an arc with endpoints on two boundary components γ_i and γ_j. Then the subsurface projection given by case 2(the dumbbell case) is non-peripheral in Y^Σ as it separates the boundary curves γ_i and γ_j from the rest of Y. Since $\mathrm{Map}(Y^\Sigma)$ fixes γ_i and γ_j, γ' is non-peripheral in Y^Σ.

□

The core of the proof is generalization the idea of α-β loops from [Web13] to the setting of non-peripheral curve graphs. α-β loops are a set of curves whose intersection pattern with β is closely related to that of α. The key idea is to establish the following: when exist their projection to a subsurface Y via a boundary component $\beta \in \partial Y$ is boundedly close to α_Y.

An intersection of a pair of oriented curves (α, β) can be assigned $\{+, -\}$ depending on the orientation from the outgoing ray of α to the outgoing ray of β is either clockwise or counterclockwise.

Definition 3.10 (α-β loop)**.** [Web13] We say that γ is an (α, β)-*loop* if for each arc $b \subset \beta - \alpha$ we have $|\gamma \cap b| \leq 2$ with equality only if $\gamma \cap \beta$ have opposite sign.

Lemma 3.11. *Let Y be a subsurface of Σ and let γ_1, γ_2 be curves on Σ. Suppose that γ_1 and γ_1 cut Y and γ_1 misses γ_2. Then $d_{C_{\mathrm{np}}(Y^{\Sigma})}(\gamma_1, \gamma_2) \leq 1$.*

Proof. Since γ_1 and γ_2 cut $Y, \pi_Y(\gamma_1)$ and $\pi_Y(\gamma_2)$ are defined. And, γ_1 misses γ_2 implies that $\pi_Y(\gamma_1)$ misses $\pi_Y(\gamma_2)$. So $d_{C_{NP}}(\gamma_1, \gamma_2) \leq 1$. □

Note that as a result if every curve in the geodesic from α to β cuts Y we then have that $d_{C_{\mathrm{np}}(Y^{\Sigma})}(\alpha, \beta) \leq d(\alpha, \beta)$.

3.2. **Surgery.** Suppose that γ, α, β are in general position. We shall describe a surgery process on γ to construct an (α, β)-loop which will be written γ'. If γ is an (α, β)-loop then we set $\gamma' = \gamma$. If γ is not an (α, β)-loop then let c be a minimal (with respect to inclusion) connected subarc $c \subset \gamma$ such that there exists an arc $b \subset \beta - \alpha$ with either

- $c \cap b$ is a pair of points with same sign
- $c \cap b$ has cardinality at least 3

Since c is minimal we have that c has endpoints on b, b is the unique arc with properties described above, and $|c \cap b| \leq 3$, that is, only take the first three intersection along γ. Thus, each arc $b' \subset \beta - \alpha$ such that $b' \neq b$, we have $|c \cap b'| \leq 2$ with equality only if $c \cap b'$ have opposite sign.

In what follows, we write $N = N(\beta)$ to denote a closed regular neighborhood of β. We now describe how to construct γ', in each case of how c intersects b.

Case 1: $|c \cap b| = 2$ and $c \cap b$ have same sign. Write $R \subset N - \alpha$ to denote the rectangle with $b \subset R$. Let $\{p_1, p_2, p_3, p_4\} = c \cap \partial R$ where p_i is adjacent to p_{i+1}. Connect p_1 to p_3 by an arc $a \subset R$ that intersects b once and intersects c only at the endpoints of a. We let γ' be the simple closed curve $a \cup (c - R)$.

Case 2: $|c \cap b| = 3$ with alternating signs of intersection with respect to some order on b. In this case the surgery takes place in a few cases which will be described.

Case 3: $|c \cap b| = 3$ with non-alternating signs of intersection. We define γ' in a similar fashion as Case 1.

In all of these cases, the intersection number between $|c \cap b|$ is reduced by one. Applying the surgeries finitely many times will result in an α-β loop.

Lemma 3.12. *In Case 2, where $|c \cap b| = 3$ and each pair of adjacent intersections are of opposite sign, there exists a curve γ' such that γ' has fewer intersection with β, is an α-β loop and is a non-peripheral curve.*

Proof. Consider the bigon A formed by β_1 with γ_1 and the bigon B formed by β_2 with γ_2. It is established in [Web13, Lemma 2.2, Case 2] that A and B are not bigons. Furthermore, we have the following cases to consider.

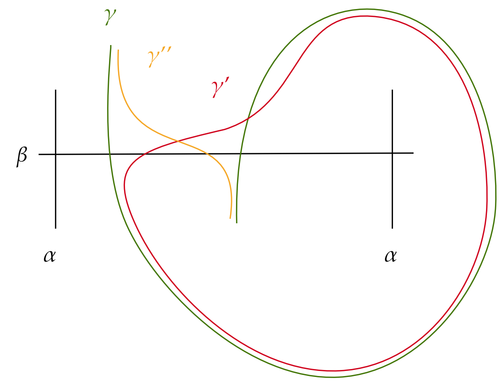

FIGURE 3. Case 1 Surgery: γ', the resulting (α, β)-loop, is shown by the red line, and the curve γ'' is used to show that γ' is non-peripheral.

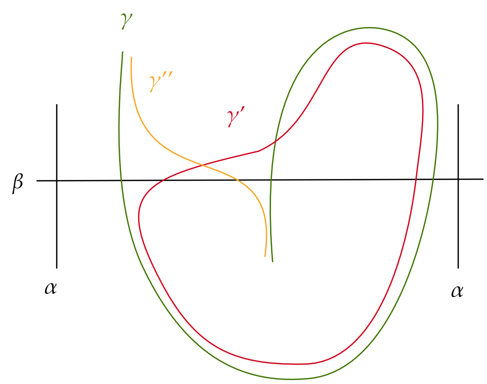

FIGURE 4. Case 3 Surgery: γ', the resulting (α, β)-loop, is shown by the red line, and the curve γ'' is used to show that γ' is non-peripheral.

- Either A or B is a separating curve such that the "inside" surface associated with A or B is a compact subsurface. Without loss of generality assume A is separating and the associated "inside" subsurface $\S_A$ is compact. Thus it contains a finite number of handles or punctures only. In this case consider the curve γ' that differs from γ only by S_A and since γ is non-peripheral , γ' is non-peripheral.
- Both A and B are non-separating curves. In this case, since A is non-separating, γ is non-separating (any point on the right side of γ can be

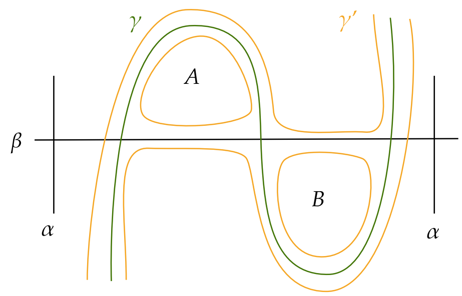

FIGURE 5. In Case 2, one of the four orange curves serve as a suitable choice for γ' depending on the separating property of the curves bounding region A and B.

connected to any point on the left side of γ). Therefore by Lemma **??**, A is also non-peripheral and let $\gamma' = A$.

- If not the first two cases, then both A and B are boundaries of subsurfaces Σ_A and Σ_B both with nonempty end spaces. In this case, consider γ' and γ'', both differ from γ by only Σ_A or Σ_B. If γ is non-separating then γ' is non-separating. This is because both $(\gamma, \gamma'$ together bound a subsurface Σ_B and thus are homologous. Otherwise suppose γ is separating. Denote the two end spaces resulting from $\Sigma - \gamma$ by (E_1, E_2). Consequently, the two end sets of γ' are $(E_1 - E_A, E_2 + E_A)$, the two end sets of γ'' are $(E_2 - E_B, E_1 + E_B)$.
  Now consider the pair $(E_1 - E_A, E_2 + E_A)$ that is associated with γ'. If $E_1 - E_A$ is not small we are done by Lemma 2.10. Thus suppose otherwise that $E_1 - E_A$ contains no more than the maximal equivalence class of a maximal end. Since γ is non-peripheral , E_1 contains strictly more than one equivalence class of maximal ends. Thus E_A contains at least part of a new equivalence class of maximal end that is different from the class found in $E_1 - E_A$. Then consider γ'' whose ends are $E_1 + E_A$ which contains strictly more than one class.

□

Lemma 3.13. *In each of the three cases, there exists a curve γ' in minimal position with β and with α. Furthermore, γ' is essential and an (α, β)-loop. If α, β, γ are all nonperipheral curves, then γ' is nonperipheral.*

Proof. First we verify the claim for Cases 1 and 3. In Case 1 the curve γ intersects this segment of $\beta - \alpha$ three times and in Case 3, γ intersects other segments of $\beta - \alpha$ an arbitrary number of times before returning to this segment of $\beta - \alpha$.

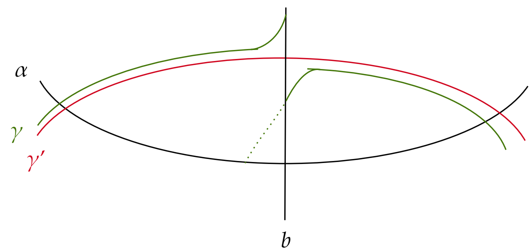

FIGURE 6. Cases 1 and 3: The $(\alpha-\beta)$-loop γ' is in minimal position with α

.

The following argument does not depend on the difference between these two cases: Any arc of $\gamma' - \beta$ is isotopic in $S - \beta$ to some arc of $\gamma - \beta$. Therefore, γ' and β cannot share a bigon since γ and β do not. If γ' is nonessential, it is isotopic to some boundary component $\omega \in \partial S$. But β intersects γ' twice, and since they are in minimal position, β must intersect ω at least twice, a contradiction. We now show that γ' and α do not share a bigon in all the cases of the surgery process described above, via contradiction.

Case 1: Pick an innermost bigon B between the pair γ', α. We must have the arc $a \subset \partial B$, otherwise γ and α share a bigon. We have $a \cap b \neq \emptyset$, and since γ' and β do not share a bigon, we must have one endpoint of b in ∂B. Let $\{p\} = B \cap c \cap b$, and $c_\alpha \subset \gamma - \alpha$ be the arc with $p \in c_\alpha$. Now γ and β do not share a bigon, and neither does γ intersect itself, thus c_α is contained with the disc $B \cup T$, where T is the triangle region adjacent to B cobounded by γ', γ and β. We conclude that c_α and α cobound a bigon, contradicting γ and α do not share a bigon.

To show that γ' is non-peripheral, we define γ'' in the following way: Define $c' = \gamma - c$. Connect p_2 to p_4 by an arc a' that intersects c only at the endpoints of a'. We let γ'' be the simple closed curve $a' \cup c'$. We note that $\gamma = D_{\gamma''}(\gamma')$. Since γ is assumed to be Non-peripheral and this property is preserved by homeomorphism, we have that γ' must be non-peripheral.

Case 2: since the property of non-peripheral is established in Lemma 3.9 it is left to show that any of the resulting curves of Lemma 3.9 are in minimal position. Since every possible option for γ' is constructed from arcs of $\gamma \cup \beta$ this follows from γ and β being in minimal position with α.

□

Corollary 3.14. *Suppose γ_1, γ_2 are disjoint curves and suppose $\gamma_1, \gamma_2, \alpha, \beta$ are in general position (Definition 2.2). Then the (α, β)-loops γ_1', γ_2' constructed by the* *surgery method above satisfy* $i(\gamma_1', \gamma_2') \leq 4$. *Furthermore, if* γ_1 *is disjoint from* α *or* β, *then* γ_1' *is disjoint from* α *or* β, *respectively.*

Proof. Let $\gamma_1' = c_1 \cup a_1$ with the surgery rectangle R_1 and $\gamma_2' = c_2 \cup a_2$ with the surgery rectangle R_2. Then $c_1 \cap c_2 = \emptyset$ since they follow γ_1 and γ_2 which miss each other. When $R_1 \cap R_2 = \emptyset$ we have that $a_1 \cap a_2 = \emptyset$, and that $|a_1 \cap c_2| \leq 2$ and $|a_2 \cap c_1| \leq 2$ by the minimality of the chosen c_1 and c_2 arcs. When $R_1 = R_2$ we have that $a_1 \cap c_2 = a_2 \cap c_1 = \emptyset$ and that $|a_1 \cap a_2| \leq 4$. □

Therefore we have established a set of quasi-geodesics with bounded constant from any vertex to any other vertex in $C_{\text{np}}(\Sigma)$:

Corollary 3.15. *Given any pair* $\alpha, \beta \in C_{\text{np}}(\Sigma)$, *there exists a* $(4, 0)$ *quasi-geodesic from* α *to* β *that consists of* α-β *loops in* $C_{\text{np}}(\Sigma)$.

Proof. Suppose α, β does not NP-fill Σ, then there exists a another non-peripheral curve $\gamma \in \Sigma$ such that

$$\gamma \cap \alpha = \emptyset \text{ and } \gamma \cap \beta = \emptyset.$$

Therefore the path $\alpha - \gamma - \beta$ is a geodesic segment of length 2 with α-β loops, trivially. Otherwise, suppose $\alpha, \beta \in C_{\text{np}}(\Sigma)$ NP-fills Σ. Start with a geodesic $\gamma_0, \ldots, \gamma_m$ of curves from α' to β, such that this collection of curves is sensible. Using the surgery process on each γ_i with $1 \leq i \leq m-1$, we obtain a sequence $\gamma_1', \ldots, \gamma_{m-1}'$ of (α, β)-loops.

By Lemma 2.3, we have

$$i(\gamma_i', \gamma_{i+1}') \leq 4 \quad \text{for each } i,$$

and therefore

$$d_S(\gamma_i', \gamma_{i+1}') \leq 4 \quad \text{and} \quad d_S(\gamma_i', \gamma_j') \leq 4|i-j| \quad \text{for all } i, j.$$

If for some $i > j$ we have

$$i - j > d_S(\gamma_i', \gamma_j'),$$

then we connect γ_i' and γ_j' with a geodesic and apply the surgery process to each of its vertices using α and β again. Repeating this procedure, we obtain the required quasi-geodesic of (α, β)-loops. □

Lemma 3.16. *Let* α' *be a component of a multi-curve* α *on* Σ *and let* β *be a curve on* Σ. *Then there exists a* $(4, 0)$*-quasi-geodesic* $\alpha' = \gamma_0, \gamma_1, ..., \gamma_n = \beta$ *with* γ_i *a* (α, β)*-loop for every* $0 < i < n$.

Proof. Note that if α' and β are disjoint then $\alpha' - \beta$ is the desired quasi-geodesic. Next, if α' and β do not fill Σ then we have that there exists a non-peripheral curve c that is disjoint from both α' and β. Then $d(\alpha, \beta) = 2$ and $\alpha - c - \beta$ is a geodesic (and therefore a quasi-geodesic) between them. And since c is disjoint from β it is vacuously a (α, β) -loop

So now assume that α' and β fill Σ. Start with a geodesic $\gamma_0, ..., \gamma_m$ of curves from α' to β, such that this collection of curves is sensible. Using the surgeries in

Section 3.2 on each γ_i with $1 \leq i \leq n-1$, we obtain a sequence $\gamma_1', ..., \gamma_{m-1}'$ of (α, β)-loops. We have $i(\gamma_i', \gamma_{i+1}') \leq 4$ for each i by the previous lemma, therefore $d_S(\gamma_i', \gamma_{i+1}') \leq 4$ and $d_S(\gamma_i', \gamma_j') \leq 4|i-j|$ for each i, j. If for some $i > j$ we have $i - j > d_S(\gamma_i', \gamma_j')$, then we connect γ_i' and γ_j' with a geodesic and surger each vertex of it using α, β again. Repeating this process, we obtain the required quasigeodesic of (α, β)-loops. ☐

Definition 3.17. Let Y be a non-peripheral subsurface. We say that a pair of non-peripheral curves α_1, α_2 *NP-fills* Y if there does not exist a curve $\gamma \in Y$, *gamma* is non-peripheral in Y^Σ and $\gamma \cap \alpha_i = \emptyset$.

Note that an $\alpha, \beta-$loop can also be defined for a multicurve $A = \{\alpha_1, ...\alpha_n\}$ where $\forall b \subset \beta \smallsetminus A$, $i(\gamma, b) \leq 2$ with equality if and only if the two intersections have opposite sign. And, if $A = \partial Y$, then we note that the arcs of $Y \cap \gamma$ are comprised of a subset of the arcs b. We then have the following lemma:

Lemma 3.18. *Let Y be a subsurface of Σ. Let γ be a $(\partial Y, \beta)$-loop that cuts ∂Y. Then $d_{C_{\mathrm{np}}(Y^\Sigma)}(\gamma, \beta) \leq 2$ if Y is non-annular and $d_{C_{\mathrm{np}}(Y^\Sigma)}(\gamma, \beta) \leq 5$ otherwise.*

Proof. If Y is non-annular, then any pair of arcs in the projection $\pi_Y(\gamma, \beta)$ will intersect at most twice by Definition 2.1, so one can consider a closed regular neighborhood of the arcs to prove the required bound on distance. (because intersection number is an upper bound for curve graph distance).

If Y is annular, then suppose for contradiction that

$$d_{C_{\mathrm{np}}(Y^\Sigma)}(\gamma, \beta) \geq 6.$$

Then there exist arcs $\delta^* \in \pi_Y(\gamma)$ and $\epsilon^* \in \pi_Y(\beta)$ with

$$|\delta^* \cap \epsilon^*| \geq 5.$$

Following [MS13, Theorem 10.1, Claim], after we isotope the triangles cobounded by ∂Y, β, and γ into Y (this retains minimal position), we have that

$$|\delta^* \cap \epsilon^* \cap Y'| \geq 3,$$

where Y' is the homeomorphic lift of Y in the cover whose only generators of non-trivial loops are the core curves in lifts of Y.

Therefore there exists an arc of $\beta - \partial Y$ which intersects γ at least two times with the same sign, contradicting γ a $(\partial Y, \beta)$-loop. ☐

Next we establish a classic lemma that says projecting a geodesic segment in $C_{\mathrm{np}}(\Sigma)$ to a subsurface with one non-peripheral boundary component that is exactly the starting vertex of the geodesic segment, we get a finite diameter projection of the geodesic segment to $C_{\mathrm{np}}(Y^\Sigma)$.

Lemma 3.19. *There exists $D = D(\delta)$ such that for any subsurface Y, component $\alpha \subset \partial Y$, and geodesic in $C_{\mathrm{np}}(\Sigma)$ with $\alpha = \gamma_0, \gamma_1, ..., \gamma_n = \beta$ with $n \geq 3$, we have $d_{C_{\mathrm{np}}(Y^\Sigma)}(\gamma_i, \beta) \leq D$ whenever $i \geq 2$.*

Proof. We proceed by using Lemma 3.16 to construct a (4,0)-quasi-geodesic Q of $(\partial Y, \beta)$-loops from α to β. For each i, we have γ_i is D'-close to Q where $D' = D'(\delta)$. For an explicit D', we can take $D' = D'' + 2$, where D'' is the largest integer with $D'' \leq \delta\lceil \log_2(26D'')\rceil$. Using Lemma 3.11, we can take $D = 2D' + B$, where B is the bound provided in Lemma 3.18. □

Now we are ready to prove the main theorem using the thin triangle property of Gromov hyperoblic spaces.

Theorem 3.20. *Given a surface Σ let Y be a (proper) subsurface with a boundary component that is a non-peripheral curve. Let $g = (\gamma_i)$ is a geodesic in $C_{\text{np}}(\Sigma)$ such that γ_i cuts Y for all i. There exists $M = M(\delta)$ independent of the choice of Y such that $d_{C_{\text{np}}(Y^\Sigma)}(g) \leq M$.*

Proof. Take $M = 4\delta + 2D + 4$, where D is defined as in Lemma 3.19. Fix $i < j$. We shall show that $d_{C_{\text{np}}(Y^\Sigma)}(\gamma_i, \gamma_j) \leq M$. Let α be the non-peripheral curve in ∂Y. Let P be a geodesic from α to γ_i and Q be a geodesic from α to γ_j. Let $I = N_{\delta+1}(\alpha) \cap g$.

We have two cases. if $I = \emptyset$ then let γ_k be a vertex on the geodesic such that γ_m is δ-close to P for $m < k$ and γ_m is δ-close to Q for $m > k$. Then since geodesic triangles are δ slim, γ_{k-1} is within distance δ of P and γ_{k+1} is within distance δ of Q. So $d_{C_{\text{np}}(Y^\Sigma)}(\gamma_i, \gamma_j) \leq 2D + 2\delta + 2 \leq M$.

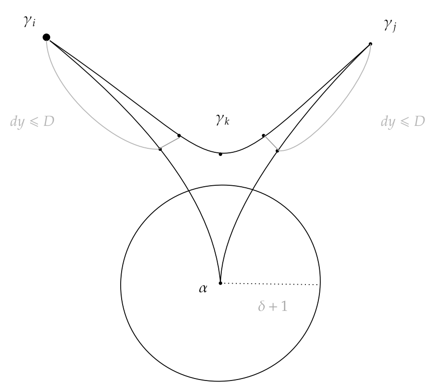

FIGURE 7. Case 1 of Theorem 3.20: When $N_{\delta+1} \cap g = \emptyset$ then we find a bound on $d_{C_{NP}(Y)}\gamma_i, \gamma_j)$ using the bound D from Lemma 3.19 and the δ-hyperbolicity.

If $I \neq \emptyset$ then the path between γ_i' and P or the path between γ_j' and Q could miss Y. To remedy this we find a path outside of I. there exists $g' = (\gamma_{i'}, ..., \gamma_{j'})$ a geodesic of length at most $2\delta + 2$ such that $I \subset g' \subset g$.

Let $i'' = \max\{i, i'-1\}$ and $j'' = \min\{j, j'+1\}$. Since geodesic triangles are δ-slim, we have either $\gamma_{i''}$ is δ-close to P and $\gamma_{j''}$ is δ-close to Q, or, there exists adjacent vertices of $g - g'$ with one δ-close to P and the other δ-close to Q.

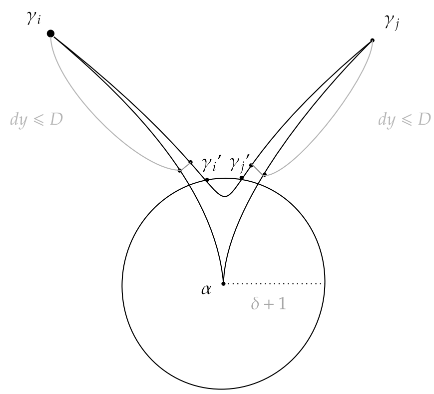

FIGURE 8. Case 2 of Theorem 3.20: When $N_{\delta+1} \cap g \neq \emptyset$ then we find a bound on $d_{C_{NP}(Y)}(\gamma_i, \gamma_j)$ using the bound D from Lemma 3.19, the δ hyperbolicity, and Lemma 3.11

□

REFERENCES

[BH09] Martin R. Bridson and André Häfliger. *Metric Spaces of Non-Positive Curvature*, Springer, 2009.

[BHS21] J Behrstock, M Hagen, A Sisto, Hierarchically hyperbolic spaces, I: Curve complexes for cubical groups *Geometry & Topology* 21 (3), 1731-1804

[BK02] Benakli, Nadia and Kapovich, Ilya. *Boundaries of hyperbolic groups*, *arXiv preprint math/0202286*

[BNV23] A. Bar-Natan and Y. Verberne, The grand arc graph, *Math. Z.* 305 (2023), no. 20.

[BQR26] Assaf Bar-Nartan, Yulan Qing and Kasra Rafi. The non-peripheral curve graph and divergence in big mapping class groups *arXiv:2603.21560* preprint.

[DFV18] M. Durham, F. Fanoni, and N. Vlamis, Graphs of curves on infinite-type surfaces with mapping class group actions, *Ann. Inst. Fourier* (Grenoble) 68 (2018), no. 6, 2581–2612.

[Gro87] M. Gromov, Hyperbolic groups, *In Gersten, Steve M. (ed.). Essays in group theory*, Mathematical Sciences Research Institute Publications. 8. New York: Springer. 75–263.

[Har81] W. J. Harvey, Boundary structure of the modular group, *Riemann Surfaces and Related Topics: Proceedings of the 1978 Stony Brook Conference* Ann. of Math. Stud. 97, Princeton, 1981.

[HQR22] C. Horbez, Y. Qing, and K. Rafi, *Big mapping class groups with hyperbolic actions: classification and applications*, J. Inst. Math. Jussieu (2022).

[MM98-I] Howard Masur and Yair Minsky Geometry of the Complex of Curves I: Hyperbolicity, *arXiv:math/9804098*, 1998.

[MM98-II] Howard Masur and Yair Minsky Geometry of the complex of curves II: Hierarchical structure, *arXiv:math/9807150*, 1998.

[MR23] K. Mann and K. Rafi, *Large scale geometry of big mapping class groups*, Geom. Topol. **27** (2023), 2237–2296.

[MS13] Howard Masur and Saul Schleimer, The geometry of the disk complex, *J. Amer. Math. Soc.* 26 (2013), no. 1, 1–62. MR2983005

[RI63] Ian Richards On the Classification of Noncompact Surfaces, *Trans. Amer. Math. Soc.*, 106, 259–269, 1963

[Web13] Richard C.H. Webb A Short Proof of the Bounded Geodesic Image Theorem, *arxiv.org/abs/1301.6187*, 2013.

Department of Mathematics, University of Tennessee at Knoxville, Knoxville, TN, USA
Email address: yqing@utk.edu

Department of Mathematics, University of Tennessee at Knoxville, Knoxville, TN, USA
Email address: asquires@vols.utk.edu