Knight's Tours for Cubes and Boxes

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Abstract

The study of Hamiltonian tours of knights on chessboards dates back more than a thousand years — and Euler found knight's tours as early as 1759 — and knight's tours have now been been fairly completely studied not only for rectangular boards but also for boards on other surfaces such as the cylinder, the torus, the Möbius strip, and the Klein bottle. In this paper we study knight's tours on the surface of cubes and boxes. In particular, we show that for any values of a, b, and c the box of dimension $a \times b \times c$ has a knight's tour on its surface. We also introduce a 3-dimensional knight's move that is a generalization of the standard 2-dimensional knight's move and exhibit a Hamiltonian tour using this 3-dimensional knight's move that visits every cell of an $8 \times 8 \times 8$ cube.

1 Introduction

> Can a knight start at a square on an empty chessboard and visit every other square of the chessboard exactly once and return to the square on which it began?

This puzzle is known as the *Knight's Tour Problem* and is very, very old. One solution for an ordinary 8×8 chessboard was found in 1759 by Leonard Euler [2] and is shown in Figure 1. In this solution Euler had his knight begin at square 1 and visit every square in the lower half of the chessboard — this would be called an *open* tour — before jumping to the upper half of the board and then symmetrically repeating the exact same moves before linking back to square 1. The elegantly simple idea that enabled Euler to link an open tour of the lower half of the chessboard with an open tour of the upper half of the chessboard is an idea that has been used over and over again through the years to solve various knight's tour problems and will be used repeatedly in what follows.

On the other hand, not all chessboards have knight's tours. For example, a 7×7 chessboard could not have a knight's tour for the simple reason

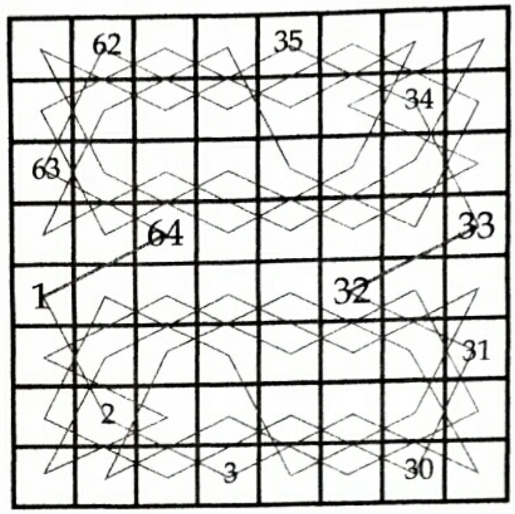

Figure 1: a knight's tour found by Euler

that a knight always goes from a black square to a white square to a black square to a white square and on and on; and so, in a knight's tour, the number of black squares and the number of white squares must be equal; therefore, the total number of squares on the board must be even in order to have a knight's tour. In other words, no board with an odd number of squares can have a knight's tour.

Surprisingly, there are other chessboards that don't have knight's tours either. For example, on a 4×4 chessboard a knight's tour is also impossible. In fact, the 5×6 and the 3×10 chessboards are the smallest rectangular boards that have knight's tours. In 1991, almost 250 years after Euler found his solutions to the Knight's Tour Problem and more than 1000 years after the idea of a knight's tour first emerged, Allen Schwenk wrote an article [4] in *Mathematics Magazine* in which he describes precisely which rectangular chessboards have knight's tours and which do not.

Now we will place a knight on a 3-dimensional chessboard such as a cube or, more generally, a box and ask exactly the same question as in the original Knight's Tour Problem: *can the knight visit each square once and return to the square on which it began*? For example, in Figure 2, we see a knight shown on the surface of a $2 \times 2 \times 2$ cube and claim that such a knight's tour is possible — we urge you to try this yourself. As usual, a knight makes a move either by moving two squares forward and then one square sideways, or by moving one square forward and then two squares sideways; moreover, a knight passes over an edge of the cube, much as a ladybug would, without even noticing the edge. An unfolded picture of the cube is helpful for recording moves.

What about cubes or boxes of other sizes? For example, you might

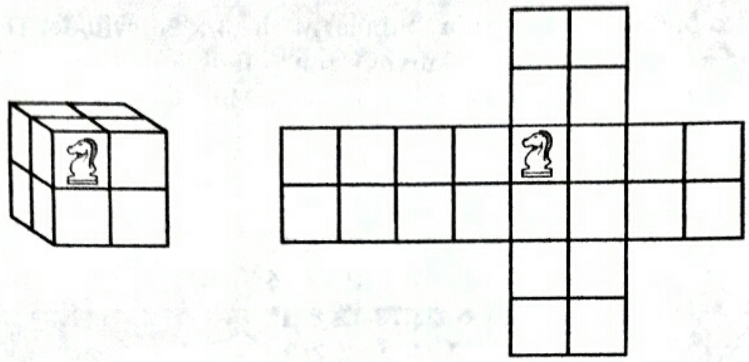

Figure 2: a $2 \times 2 \times 2$ chessboard

think that finding a knight's tour on the surface of an $8 \times 8 \times 8$ cube would be difficult. Actually, it is quite easy using Euler's idea of linking together previously known tours. All we need to do is use a known tour for each of the six faces of the $8 \times 8 \times 8$ cube and then carefully arrange for the knight to jump from face to face, touring each face in turn. The famous British puzzlist H. E. Dudeney used exactly this approach over 200 years ago to find a knight's tour on the surface of an $8 \times 8 \times 8$ cube [1].

2 Which Boxes Have Knight's Tours?

For rectangular boards, as we have seen, there are numerous boards that do not have knight's tours. It is quite surprising, then, that we have the following theorem for boxes.

Theorem 1 *All boxes have a knight's tour on their surface.*

Proof. In other words, for any values of a, b, and c the box of dimension $a \times b \times c$ has a knight's tour on its surface. There are no exceptions. The basic strategy for the proof is very simple. Rather than thinking of the box as being composed of six rectangular chessboards as we did for the $8 \times 8 \times 8$ box, we will in general think of a box as a cylinder with two ends; in other words, we think of a box as consisting of a single cylindrical chessboard (going around the side) together with a rectangular chessboard at the top and another rectangular chessboard at the bottom. Since it is known exactly which rectangular chessboards have tours [4] and also exactly which cylindrical chessboards have tours [6, 7], almost all boxes can be handled by joining together a tour of the top, a tour of the bottom, and a tour of the 'cylindrical' part of the box.

As an easy first example of this basic strategy, let's look at the $3 \times 6 \times 7$ box and orient the box so that the top is a 6×7 rectangle. This rectangle is known to have a knight's tour. Similarly, the 3×26 cylinder that forms the side of the box also has a knight's tour. Individual tours of the top, the bottom, and the 3×26 cylinder are shown in Figure 3.

1	40	3	30	33	38	35
4	27	42	39	36	29	32
41	2	5	28	31	34	37
20	23	26	7	10	13	16
25	6	21	18	15	8	11
22	19	24	9	12	17	14

52	55	58	61	64	67	70	73	76	1	4	7	10	13	16	19	22	25	28	31	34	37	40	43	46	49
47	50	53	56	59	62	65	68	71	74	77	2	5	8	11	14	17	20	23	26	29	32	35	38	41	44
54	57	60	63	66	69	72	75	78	3	6	9	12	15	18	21	24	27	30	33	36	39	42	45	48	51

1	40	3	30	33	38	35
4	27	42	39	36	29	32
41	2	5	28	31	34	37
20	23	26	7	10	13	16
25	6	21	18	15	8	11
22	19	24	9	12	17	14

Figure 3: the $3 \times 6 \times 7$ box

It is now possible, and quite easy, to link these three tours together into a single tour. Begin on the top at square 1 and tour the top until you get to square 14. Then jump to square 11 on the side of the box and do that tour in order until completely finished with the 'cylindrical' side back at square 10. Then jump back to the top at square 15 and complete the tour of the top. In the same way, the tour of the bottom rectangle can now be added by linking squares 35-17 and 36-18.

There are cases, however, when this basic idea fails for the simple reason that you don't have individual tours of the top, the bottom, or the 'cylindrical' side of the box to start with. Completing a full proof, then, consists of cataloging these special cases and making sure each of them can be dealt with. As a good example of one of these special cases, let's look at the $3 \times 3 \times 4$ box and orient the box so that the top is a 3×4 rectangle. In this case, it is the top — and hence also the bottom — that doesn't have a tour. However, as shown in Figure 4, there is an *open* tour for the 3×4 board that begins at square 1 and ends at square 12 and so we can link the three individual tours together at 1-37 and 12-38 at the top and at 1-41 and 12-42 at the bottom to produce a single knight's tour for the $3 \times 3 \times 4$ box.

This last example illustrates a general trend in the complete proof that it is with small boxes where there are complications and this quite naturally

					10	5	8						
					7	2	11						
					4	9	6						
					1	12	3						
28	31	34	37	40	1	4	7	10	13	16	19	22	25
23	26	29	32	35	38	41	2	5	8	11	14	17	20
30	33	36	39	42	3	6	9	12	15	18	21	24	27
					3	12	1						
					6	9	4						
					11	2	7						
					8	5	10						

Figure 4: the $3 \times 3 \times 4$ box

leads to numerous special cases that have to be considered. But these small special cases never present any real difficulty because a knight has considerable freedom of motion near corners — this is perhaps the key difference between knight's tours on boxes and regular chessboards! The details of all of these special cases can be found in [3].

3 Three-dimensional Knight's Tours

In this section we will now allow a knight to move *within* a cube or box rather than merely move on its 2-dimensional surface. In other words, rather than move from one square to another square on the surface of a box, the knight can now move from one *cell* to another *cell* within a box with the Hamiltonian objective being to visit each cell exactly once and to return to the cell within which the knight began the tour. In fact, Ian Stewart did precisely such a knight's tour for the $8 \times 8 \times 8$ cube [5]. This is not nearly as hard as you might first think, since one can imitate Dudeney's method for touring the surface of a cube by touring the cube *one layer at a time* and then linking these individual tours together by having the knight begin, say, at layer 1, tour that entire layer, jump to layer 3, tour that layer, and so on, jumping to layers 5, 7, 8, 6, 4, and 2 in that order before finishing back at the beginning cell in layer 1.

Ian Stewart's knight's tour of the cube is impressive to be sure, and yet, it still has a completely 2-dimensional feel to it simply because the knight tours each 8×8 layer or 'board' in the cube separately using the standard 2-dimensional move of two squares forward, one square sideways. We therefore introduce a *3-dimensional knight's move*:

1 cell forward, 2 cells sideways, and 4 cells in a third direction so that the knight actually makes use of all three dimensions in each move.

This, of course, is not the only possible option for a three-dimensional knight's move. After all, an early version of chess in fourteenth century Persia used a piece known as a camel that moved three cells forward and one cell sideways. So, perhaps a $1-2-3$ move for a three-dimensional knight might make good historical sense. Such a move, however, would quickly lead to a serious problem in terms of knight's tours. If we imagine coloring the cells of a cube or box alternately black or white so that adjacent cells are always colored differently, then we see that a knight making a series of $1-2-3$ moves would always land in cells of the same color as the cell in which it started. Therefore, such a knight could only hope to visit cells of one color. Avoiding this parity problem is the main reason for our choice of $1-2-4$ for the 3-dimensional knight's move, although moves such as $1-3-3$ would accomplish that as well, but we greatly prefer $1-2-4$ as the natural generalization of the knight's move because it also so nicely preserves the fundamental nature of the quirky L-shaped move of the knight in two dimensions.

Now, we present a 3-dimensional knight's tour of an $8 \times 8 \times 8$ cube using this 3-dimensional knight's move. First, we label the cells of the cube with coordinates in the natural way so that, say, 111 is the cell in the front upper left corner of the cube and 888 is the cell in the back lower right corner of the cube. We begin the tour at cell 111:

111-325-441-235-311-435-241-125-546-622-816-732-573-657-863-787-548-624-818-734-
528-714-838-644-883-677-553-767-683-567-753-877-636-512-726-842-421-345-131-215-
331-145-221-415-836-642-526-712-333-147-223-417-258-464-388-174-368-484-278-154-
212-426-342-136-515-721-845-631-715-831-645-521-142-226-412-336-177-253-467-383-
144-228-414-338-124-318-434-248-487-273-157-363-287-163-357-473-232-116-322-446-
825-741-535-611-735-541-625-811-432-246-122-316-737-543-627-813-654-868-784-578-
764-888-674-558-616-822-746-**532-151**-365-481-275-351-475-281-165-586-662-856-772-
533-617-823-747-588-664-858-774-568-754-878-684-843-637-513-727-643-527-713-837-
676-552-766-882-461-385-171-255-371-185-261-455-876-682-566-752-373-187-263-457-
218-424-348-134-328-444-238-114-252-466-382-176-555-761-885-671-755-871-685-561-
182-266-452-376-137-213-427-343-184-268-454-378-164-358-474-288-447-233-117-323-
247-123-317-433-272-156-362-486-865-781-575-651-775-581-665-851-472-286-162-356-
777-583-667-853-614-828-744-538-724-848-634-518-656-862-786-**572-731**-615-531-745-
821-442-326-112-236-477-353-167-283-367-153-277-483-244-438-314-128-334-418-224-
148-387-463-257-173-332-416-222-146-525-641-835-711-635-841-725-511-132-346-422-
216-158-274-488-364-178-384-468-254-413-227-143-337-716-522-646-832-411-225-141-
335-211-135-341-425-846-722-516-632-873-757-563-687-763-557-673-887-648-834-718-
524-738-814-628-544-783-867-653-577-736-812-626-542-121-245-431-315-231-445-321-
115-536-742-826-612-554-678-884-768-574-788-864-658-817-623-547-733-312-126-242-
436-815-621-**545-166**-282-476-855-661-585-771-655-571-785-861-482-366-152-276-437-
313-127-243-327-113-237-443-284-478-354-168-374-458-264-188-347-423-217-133-372-
456-262-186-565-681-875-751-675-881-765-551-172-386-462-256-118-234-448-324-138-
344-428-214-453-267-183-377-756-562-686-872-451-265-181-375-251-175-381-465-886-
762-556-672-833-717-523-647-723-517-633-847-688-874-758-564-778-854-668-584-743-
827-613-537-776-852-666-582-161-285-471-355-271-485-361-155-576-782-866-652-514-
638-844-728-534-748-824-618-857-663-587-773-**352-111** .

The method for constructing this 3-dimensional knight's tour is as follows. Each octant of the $8 \times 8 \times 8$ cube is itself a $4 \times 4 \times 4$ cube. We begin by finding an ordinary '2-dimensional' knight's tour of a $4 \times 4 \times 4$ cube using the standard knight's move of two squares forward, one square sideways, as shown in Figure 5.

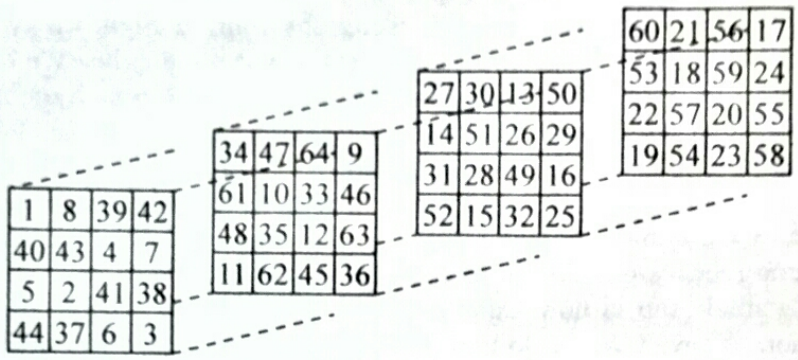

Figure 5: a 2-D knight's tour of the $4 \times 4 \times 4$ cube

Next, we augment this '2-dimensional' tour by thinking of this $4 \times 4 \times 4$ cube as starting as the front upper left octant of the $8 \times 8 \times 8$ cube. Then, every time we are about to make an ordinary L-shaped 2-dimensional knight's move in one of the eight $4 \times 4 \times 4$ octants of the $8 \times 8 \times 8$ cube according to the tour from Figure 5, instead we augment that 2-dimensional move by also moving 4 cells in a direction orthogonal to the L-shaped move and this will put us in exactly the same relative position in one of the other octants of the $8 \times 8 \times 8$ cube. We will begin the tour to illustrate how this works.

Begin the tour at cell 111 and think about what the first eight positions would be using just the '2-dimensional' move from Figure 5. These all would stay not only in this same octant but in the front plane of this octant:

$$111 - 321 - 441 - 231 - 311 - 431 - 241 - 121 \,.$$

Now, the augmentation to the 3-dimensional knight's move is easy, we simply make use of the dimension that previously was unchanging. So, here are the augmented moves:

$$111 - 325 - 441 - 235 - 311 - 435 - 241 - 125 \,.$$

Notice how we are bouncing back and forth between two different octants in the $8 \times 8 \times 8$ cube, but effectively are still doing the tour in Figure 5.

Let's do one more move to make sure the pattern is clear. The next move in Figure 5 is from cell 121 to cell 142, that is, to a new layer in the $4 \times 4 \times 4$ cube. So, now it is the first coordinate that needs to change by 4 in the augmented move. But, we also need to remember where the knight actually is right now, namely at cell 125. Thus, the knight moves from cell 125 to cell 546. This puts the knight into a new octant, but of course in the same relative position in this octant as at square 9 in Figure 5.

That's almost all there is to it, but we need to say what eventually happens to the knight on this tour. It turns out that the knight does return to cell 111, but only after using the tour in Figure 5 twice and visiting exactly one fourth of the cells in the $8 \times 8 \times 8$ cube. We can then arrange to have the knight visit another fourth of the cells by starting in a different octant and cell 151 is a good place to start; another fourth by starting at cell 511, and another fourth by starting at cell 551. In other words, we can produce four disjoint tours that together visit every cell of the $8 \times 8 \times 8$ cube by augmenting the tour from Figure 5 and starting in the same relative position in four different octants.

The final step is now Euler's simple step of linking these four tours together. All we have to do is to scan through these four individual tours and look for convenient places to form the links. It requires four links to join four cycles into a single cycle and these are shown in bold in the 3-dimensional knight's tour listed above.

It would be quite straightforward to apply this method and the tour in Figure 5 to produce a 3-dimensional knight's tour for any box having dimension $4a \times 4b \times 4c$ for any value of a, b, and c.

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