# FIRST PASSAGE PERCOLATIONS PRESERVE SUBLINEARLY MORSE BOUNDARIES

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Abstract. Sublinearly Morse directions in proper geodesic spaces are defined by sublinearly Morse stability. In this paper we offer an alternative characterization for sublinearly Morse geodesic lines via middle recurrence. We then study first passage percolation (FPP) on proper geodesic graphs of bounded degree. We associate an i.i.d. collection of random passage times to each edge. Under suitable conditions on the passage time distribution, we prove that sublinearly Morse boundaries are invariant under first passage percolation.

## 1. Introduction

First passage percolation is a model of random perturbation of a given geometry. In this paper, we shall restrict to the simplest model, where random i.i.d lengths are assigned to the edges of a fixed graph.

Let us briefly recall how first passage percolation is defined. We consider a connected non-oriented graph $X$, whose set of vertices (resp. edges) is denoted by $V$ (resp. E). For every function $\omega : E \to (0, \infty)$, we equip $V$ with the weighted graph metric $d_\omega$, where each edge $e$ has weight $\omega(e)$. In other words, for every $v_1, v_2 \in V, d_\omega(v_1, v_2)$ is defined as the infimum over all path $\gamma = (e_1, \dots, e_m)$ joining $v_1$ to $v_2$ of $|\gamma|_\omega := \sum_{i=1}^{m} \omega(e_i)$. Observe that the graph metric on $V$ corresponds to the case where $\omega$ is constant equal to 1 , we shall simply denote it by $d$. We will now consider a probability measure on the set of all weight functions $\omega$. We let $\nu$ be a probability measure supported on $[0, \infty)$. Our model consists in choosing independently at random the weights $\omega(e)$ according to $\nu$. More formally, we equip the space $\Omega = [0, \infty)^E$ with the product probability that we denote by $P$.

A famous open problem in percolation theory is whether with positive probability, first passage percolation on $\mathbb{Z}^2$ admits a bi-infinite geodesic. In his Saint-Flour course from 84 ', Kesten attributes this question to Furstenberg (see [??]). Licea and Newman[??] made partial progress on this problem, which is still open, and mentioned that the conjecture that there are no such geodesics arose independently in the physics community studying spin glass. Wehr and Woo [16] proved absence of two sided infinite geodesic in a half plane, assuming the lengths distribution is continuously distributed with a finite mean. The picture spaces with some negaive curvature is drastically different. In [BT17] it is established that if $X$ has a bi-infinite Morse geoddesic line, then $X_\omega$ contains a bi-infinite geodesic line almost surely. However, in [BJQ25] it is shown that first passage percolation do not

preserve the Morseness of geodesic lines. Naturally we consider weaker notions of Morse: sublinearly Morse. In [JQ25] it is established that if $X$ has a bi-infinite *sublinearly Morse* geodesic line, then $X_\omega$ contains a bi-infinite geodesic line almost surely. Therefore, the immediate motivating question for this project is:

**Question 1.1.** *Are sublinearly Morseness of sets almost surely preserved under first passage percolations?*

In this paper we answer the question positively:

**Theorem A.** *Let $X$ be a graph with uniformly bounded degree and $\partial_\kappa X \neq \emptyset$. Assume that the edge length distribution has finite expectation and also $\nu(\{0\}) = 0$. Then let $\omega$ be induced graphs with weight functions. Then for almost every $\omega$, such the first passage percolation induces a homeomorphism on the kappa-Morse boundaries: $\partial_\kappa X \simeq \partial_\kappa X_\omega$.*

This result greatly opens up the possibility of analysis into the interaction between sublinearly Morse sets and first passage percolations. We give several natural and immediate questions that can be investigated.

Since first passage percolations do not preserve quasi-geodesics, this paper depends on characterizations of sublinear Morseness using path of bounded slope. This notion is established for Morse geodesic rays and is utilized in [DMS09] and improved in [ADT17]. We first develop an proper analogue of such a notion in the context of sublinear Morseness:

**Definition 1.2.** We say that a path (geodesic line or quasi-geodesic) $\beta$ is *middle third recurrent* if for every $C \geq 1$ there is a constant $c$ (depends on $C$ and $\beta$) such that the following holds: for any path $p$ with endpoints $a, b \in \beta$ satisfying $\ell(p) \leq Cd(a, b)$, intersects the $c \cdot \kappa$-neighborhood of the middle third of $\beta[a, b]$ for some sublinear function $\kappa$. That is,

$$p \cap \mathcal{N}_\kappa(\beta_{\frac{1}{3}[a,b]}, c) \neq \emptyset.$$

Where, $\beta_{\frac{1}{3}[a,b]} := \left\{ x \in \beta : \min\{d(x, a), d(x, b)\} \geq \frac{1}{3} \cdot d(a, b) \right\}$.

We are able to establish the path characterization of sublinearly Morse sets:

**Theorem B.** *A geodesic ray is sublinearly Morse if and only if it has the middle third recurrent property.*

We remark that it is communicated to the authors during the preparation of this document that one of the directions of this characterization, Proposition 3.4, is also in the process of being established by Raghunath Jaganathan Vennila as part of his Master's thesis.

**History in the boundary consturction.** The sublinearly Morse boundary is a geometric boundary construction generalizing the Gromov boundary. It is always quasi-isometrically invariant, metrizable, and often provides a group-invariant topological model for suitable random walks. By contrast, the Morse boundary, constructed in [Cor18, CS15], frequently fails to be metrizable and does not always realize the Poisson boundary.

Recent work also highlights strong genericity properties for sublinearly Morse directions. In [GQR22], genericity with respect to the Patterson–Sullivan measure was established for actions admitting a strongly contracting element. The stationary measure results of [GQR22] were later claimed in a different context by Choi [Choi22] using a pivoting method of Gou"ezel [Gou22]. Following [Yan22], genericity on the horofunction boundary was proved for all proper statistically convex-cocompact actions [QY24]. Most recently, Wen studied genericity of sublinearly Morse directions in higher-rank symmetric spaces [Wen25].

**Further investigations.** There are several immediate directions of interest as a result of this study. We record them here and offer also existing related results that may be useful in pursuing these questions.

(1) *Are hyperbolic graphs mapped to hyperbolic graphs by first passage percolation if the support is finite?*

A recent preprint by Dominic Bair and the authors of this paper [BJQ25] shows that, assuming the passage time distribution have infinite support, then the hyperbolicity of the graphs are almost surely not preserved. We ask further if the infinite support assumption can be removed or weakened.

(2) *Does first passage percolation induced a Hölder map on the associated sublinearly Morse boundaries?*

Theorem B establishes that a first passage percolation of suitable assumption induced a homemophism between the sublinearly Morse boundaries. Given that sublinearly Morse boundaries are metrizable topological spaces, we ask if stronger observations can be made. It is established in [DV15] that radial maps induced Holder map on Gromov boundaries and [BJQ25] showed that first passage percolation exhibits certain radial property.

(3) *With suitable assumptions, is a generic first passage percolation boundedly close or sublinearly close to a sublinear biLipschtiz equivalence (SBE)?*

Sublinear biLipschtiz equivalence are studied by Drutu to classify quasi-isometry rigidity of Lie groups. It is studied in the context of sublinearly Morse boundary by the second-named author and Gabriel Pallier in [PQ24]. This question is posed by Gabriel Pallier.

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## 2. PRELIMINARIES

### 2.1. Quasi-isometry and quasi-isometric embeddings.

**Definition 2.1** (Quasi Isometric embedding)**.** Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces. For constants $\mathsf{k} \geq 1$ and $\mathsf{K} \geq 0$, we say a map $f\colon X \to Y$ is a $(\mathsf{k}, \mathsf{K})$–*quasi-isometric embedding* if, for all points $x_1, x_2 \in X$

$$\frac{1}{\mathsf{k}} d_X(x_1, x_2) - \mathsf{K} \leq d_Y\big(f(x_1), f(x_2)\big) \leq \mathsf{k}\, d_X(x_1, x_2) + \mathsf{K}.$$

If, in addition, every point in $Y$ lies in the $\mathsf{K}$–neighbourhood of the image of $f$, then $f$ is called a $(\mathsf{k}, \mathsf{K})$–quasi-isometry. When such a map exists, $X$ and $Y$ are said to be *quasi-isometric*.

A quasi-isometric embedding $f^{-1}\colon Y \to X$ is called a *quasi-inverse* of $f$ if for every $x \in X$, $d_X(x, f^{-1}f(x))$ is uniformly bounded above. In fact, after replacing $\mathsf{k}$ and $\mathsf{K}$ with larger constants, we assume that $f^{-1}$ is also a $(\mathsf{k}, \mathsf{K})$–quasi-isometric embedding,

$$\forall x \in X \quad d_X\big(x, f^{-1}f(x)\big) \leq \mathsf{K} \qquad \text{and} \qquad \forall y \in Y \quad d_Y\big(y, f\, f^{-1}(x)\big) \leq \mathsf{K}.$$

A *geodesic ray* in $X$ is an isometric embedding $\beta\colon [0, \infty) \to X$. We fix a base-point $\mathfrak{o} \in X$ and always assume that $\beta(0) = \mathfrak{o}$, that is, a geodesic ray is always assumed to start from this fixed base-point.

**Definition 2.2** (Quasi-geodesics)**.** In this paper, a *quasi-geodesic ray* is a continuous quasi-isometric embedding $\beta\colon [0, \infty) \to X$ starting from the basepoint $\mathfrak{o}$.

The additional assumption that quasi-geodesics are continuous is not necessary for the results in this paper to hold, but it is added for convenience and to make the exposition simpler.

If $\beta\colon [0, \infty) \to X$ is a $(\mathsf{q}, \mathsf{Q})$–quasi-isometric embedding, and $f\colon X \to Y$ is a $(\mathsf{k}, \mathsf{K})$–quasi-isometry then the composition $f \circ \beta\colon [t_1, t_2] \to Y$ is a quasi-isometric embedding, but it may not be continuous. However, one can adjust the map slightly to make it continuous (see Definition 2.2 [**?**]) such that $f \circ \beta$ is a $(\mathsf{kq}, 2(\mathsf{kq} + \mathsf{kQ} + \mathsf{K}))$–quasi-geodesic ray.

Similar to above, a *geodesic segment* is an isometric embedding $\beta\colon [t_1, t_2] \to X$ and a *quasi-geodesic segment* is a continuous quasi-isometric embedding

$$\beta\colon [t_1, t_2] \to X.$$

**Notation**. In this paper, we will use $\alpha, \beta \ldots$ to denote quasi-geodesic rays. If the quasi-geodesic constants are $(1, 0)$, we use $\alpha_0, \beta_0, \ldots$ to signify that they are in fact geodesic rays. Meanwhile, we use $[\alpha], [\beta], \ldots$ to denote the $\kappa$-equivalence classes of quasi-geodesic rays (see Definition 2.9), and we also use $\mathsf{a}, \mathbf{b}, \ldots$ to denote $\kappa$-equivalence classes without referring an element in each class. In contrast, we use $\alpha(\infty)$ to denote equivalence classes of $\alpha$ in the visual boundary (see Definition **??**). Furthermore, let $\alpha$ be a (quasi-)geodesic ray $\alpha\colon [0, \infty) \to X$, if $x_1, x_2$ are points on $\alpha$, then the segment of $\alpha$ between $x_1$ and $x_2$ is denoted $[x_1, x_2]_\alpha$. If a segment

is presented without a subscript, for example $[y_1, y_2]$, then it is a geodesic segment between the two points. Let $\beta$ be a quasi-geodesic ray. Define

$$\|x\| := d(\mathfrak{o}, x).$$

For $\mathsf{r} > 0$, let $t_{\mathsf{r}}$ be the first time where $\|\beta(t)\| = \mathsf{r}$ and define:

$$\beta_{\mathsf{r}} := \beta(t_{\mathsf{r}}) \qquad \text{and} \qquad \beta|_{\mathsf{r}} := \beta[0, t_{\mathsf{r}}] = [\beta(0), \beta_{\mathsf{r}}]_\beta \tag{1}$$

which are points and segments in $X$, respectively.

2.1.1. *Sublinearly Morse geodesic lines.*

Let $\kappa\colon [0,\infty) \to [1,\infty)$ be a sublinear function that is monotone increasing and concave. That is

$$\lim_{t\to\infty} \frac{\kappa(t)}{t} = 0.$$

The assumption that $\kappa$ is increasing and concave makes certain arguments cleaner; otherwise, they are not really needed. One can always replace any sub-linear function $\kappa$, with another sub-linear function $\overline{\kappa}$ so that

$$\kappa(t) \le \overline{\kappa}(t) \le \mathsf{C}\,\kappa(t)$$

for some constant $\mathsf{C}$ and $\overline{\kappa}$ is monotone increasing and concave. For example, define

$$\overline{\kappa}(t) = \sup\Big\{\lambda\kappa(u) + (1-\lambda)\kappa(v) \;\Big|\; 0 \le \lambda \le 1,\ u, v > 0,\ \text{and } \lambda u + (1-\lambda)v = t\Big\}.$$

**Lemma 2.3.** *Suppose $\kappa$ is any sublinear function. For any $D > 0$, there exists $D_1$, $D_2 > 0$ depending on $D$ and $\kappa$ such that, for $x, y \in X$, if $d(x,y) \le D \cdot max\{\kappa(\|x\|), \kappa(\|y\|)\}$ then $D_1 \cdot \kappa(\|x\|) \le \kappa(\|y\|) \le D_2 \cdot \kappa(\|x\|))\|$.*

*Proof.* The proof is identical to the proof of Lemma 3.2 [QRT22]. Since $\kappa$ is sublinear, there exists a constant $R \ge 0$ such that for every $t > 0$, $\kappa(t) \le \frac{t}{2D} + R$. If $\|x\| \le \|y\|$ then,

$$\|y\| - \|x\| \le d(x,y) \le D \cdot \max\{\kappa(\|x\|), \kappa(\|y\|)\} \le \frac{1}{2}(\|x\| + \|y\|) + 2RD.$$

Hence, $\|y\| \le 3\|x\| + 2RD$. This implies,

$$\kappa(\|x\|) \le \kappa(\|y\|) \le (3 + 2RD) \cdot \kappa(\|x\|).$$

Similarly for $\|x\| \le \|y\|$, we have

$$(3 + 2RD)^{-1} \cdot \kappa(\|x\|) \le \kappa(\|y\|) \le \kappa(\|x\|).$$

Therefore,

$$(3 + 2RD)^{-1} \cdot \kappa(\|x\|) \le \kappa(\|y\|) \le (3 + 2RD) \cdot \kappa(\|x\|)$$

□

**Definition 2.4** ($\kappa$–neighborhood)**.** For a closed set $Z$ and a constant $\mathsf{n}$ define the $(\kappa, \mathsf{n})$–neighbourhood of $Z$ to be

$$\mathcal{N}_\kappa(Z, \mathsf{n}) = \Big\{x \in X \;\Big|\; d_X(x, Z) \le \mathsf{n} \cdot \kappa(x)\Big\}.$$

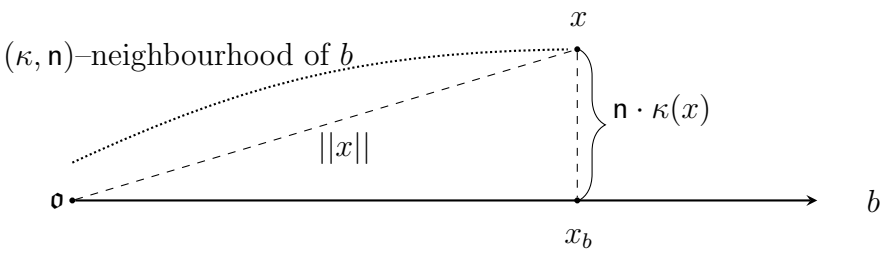



FIGURE 1. A $\kappa$-neighbourhood of a geodesic ray $b$ with multiplicative constant $\mathsf{n}$.

**Definition 2.5** ($\kappa$-Morse I, $\kappa$-Morse II)**.** Let $Z \subseteq X$ be a closed set, and let $\kappa$ be a concave sublinear function. We say that $Z$ is *$\kappa$-Morse* if one of the following equivalent (see Proposition 3.10 [QRT23]) condition holds:

I There exists a proper function $\mathsf{m}_Z : \mathbb{R}^2 \to \mathbb{R}$ such that for any sublinear function $\kappa'$ and for any $r > 0$, there exists $R$ such that for any $(\mathsf{q}, \mathsf{Q})$-quasi-geodesic ray $\beta$ with $\mathsf{m}_Z(\mathsf{q}, \mathsf{Q})$ small compared to $r$, if

$$d_X(\beta_R, Z) \leq \kappa'(R) \qquad \text{then} \qquad \beta|_r \subset \mathcal{N}_\kappa\big(Z, \mathsf{m}_Z(\mathsf{q}, \mathsf{Q})\big)$$

II There is a function

$$\mathsf{m}'_Z \colon \mathbb{R}^2_+ \to \mathbb{R}_+$$

so that if $\beta \colon [s, t] \to X$ is a $(\mathsf{q}, \mathsf{Q})$–quasi-geodesic with end points on $Z$ then

$$[s, t]_\beta \subset \mathcal{N}_\kappa\big(Z, \mathsf{m}'_Z(\mathsf{q}, \mathsf{Q})\big).$$

*Remark* 2.6. By taking the maximum function of $\mathsf{m}_Z, \mathsf{m}'_Z$, we may and will always assume that both conditions hold for the same $\mathsf{m}_Z$, which we refer to as the *kappa*-Morse gauge. Further,

$$\mathsf{m}_Z(\mathsf{q}, \mathsf{Q}) \geq \max(\mathsf{q}, \mathsf{Q}). \tag{2}$$

**Definition 2.7.** Let $(X, d_X)$ be a proper geodesic metric space and $Z \subseteq X$ a closed subset, and let $\kappa$ be a concave sublinear function. A map $\pi_Z \colon X \to \mathcal{P}(Z)$ is a *$\kappa$-projection* if there exist constants $D_1, D_2$, depending only on $Z$ and $\kappa$, such that for any points $x \in X$ and $z \in Z$,

$$\operatorname{diam}_X(\{z\} \cup \pi_Z(x)) \leq D_1 \cdot d_X(x, z) + D_2 \cdot \kappa(x).$$

A $\kappa$-projection differs from a nearest point projection by a uniform multiplicative error and a sublinear additive error. In particular, the nearest point projection is a $\kappa$-projection. Indeed, for $z \in Z$ and $w \in \pi_Z(x)$, we have

$$d(z, w) \leq d(z, x) + d(x, w) \leq 2d(z, x).$$

**Theorem 2.8.** *Let $(X, \mathfrak{o})$ be a proper geodesic metric space with a fixed base point and let $\alpha$ be a quasi-geodesic ray in $X$. Let $\pi$ be any $\kappa$-projection from $X$ to $\alpha$ in the sense of Definition 2.7. Then if $\alpha$ is $\kappa$-weakly Morse, then it is $\kappa'$-weakly contracting with respect to $\pi$ for some sublinear function $\kappa'$.*

**Definition 2.9** (sublinear equivalence classes in $\partial_\kappa X$)**.** Let $\beta$ and $\gamma$ be two quasi-geodesic rays in $X$. If $\beta$ is in some $\kappa$–neighbourhood of $\gamma$ and $\gamma$ is in some $\kappa$–neighbourhood of $\beta$, we say that $\beta$ and $\gamma$ $\kappa$–*fellow travel* each other. This defines an equivalence relation on the set of quasi-geodesic rays in $X$ (to obtain transitivity, one needs to change $\mathsf{n}$ of the associated $(\kappa, \mathsf{n})$–neighbourhood).

We denote the equivalence class that contains $\beta$ by $[\beta]$. We say two quasi-geodesic rays $\alpha, \beta$ *sublinearly fellow-travel* if as $r \to \infty$,

$$\lim_{r\to\infty} \frac{d(\alpha_r, \beta_r)}{r} = 0.$$

It is established in [QRT23] that suppose $\alpha$ is a $\kappa$-Morse quasi-geodesic ray and suppose $\beta$ is a sublinearly Morse quasi-geodesic ray that sublinearly fellow travels $\alpha$ then $\beta$ $\kappa$-fellow travels $\alpha$. Thus it is a generalization of Gromov hyperbolicity in the sense that in these set of directions sublinear fellow traveling implies uniform sublinearly fellow travel.

**Definition 2.10** (Sublinearly Morse boundary)**.** Let $\kappa$ be a sublinear function as specified in Section 2.1.1 and let $X$ be a proper geodesic space.

$$\partial_\kappa X := \{ \text{ all } \kappa\text{-Morse quasi-geodesics } \}/\text{sublinearl-fellow traveling}$$

2.1.2. *Coarse cone topology on* $\partial_\kappa X$. We equip $X$ with a topology which is a coarse version of the visual topology. In visual topology, if two geodesic rays fellow travel for a long time, then they are "close". In this coarse version, if two geodesic rays and all the quasi-geodesic rays in their respect equivalence classes remain close for a long time, then they are close. Now we define it formally. First, we say a quantity $\mathsf{D}$ *is small compared to a radius* $\mathsf{r} > 0$ if

$$\mathsf{D} \leq \frac{\mathsf{r}}{2\kappa(\mathsf{r})}. \tag{3}$$

Recall that given a $\kappa$–Morse quasi-geodesic ray $\beta$, we denote its associated *kappa*-Morse gauge $\mathsf{m}_\beta(\mathsf{q}, \mathsf{Q})$. These are multiplicative constants that give the heights of the $\kappa$–neighbourhoods.

**Definition 2.11** (topology on $\partial_\kappa X$)**.** Let $\mathsf{a} \in X$ and $\alpha_0 \in \mathsf{a}$ be the unique geodesic in the class $\mathsf{a}$. Define $\mathcal{U}_\kappa(\mathsf{a}, \mathsf{r})$ to be the set of points $\mathbf{b}$ such that for any $(\mathsf{q}, \mathsf{Q})$-quasi-geodesic of $\mathbf{b}$, denoted $\beta$, such that $\mathsf{m}_\beta(\mathsf{q}, \mathsf{Q})$ is small compared to $\mathsf{r}$, satisfies

$$\beta|_{\mathsf{r}} \subset \mathcal{N}_\kappa\big(\alpha_0, \mathsf{m}_{\alpha_0}(\mathsf{q}, \mathsf{Q})\big).$$

Let the topology of $\partial_\kappa X$ be the topology induced by this neighbourhood system. The following fact shows that a $\kappa$-boundary is well defined with respect the associated group.

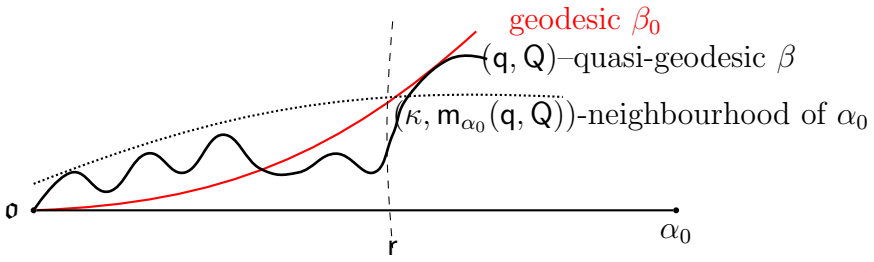



FIGURE 2. $\mathbf{b} \in \mathcal{U}_\kappa(\mathsf{a},\mathsf{r})$ because the quasi-geodesics of $\mathbf{b}$ such as $\beta, \beta_0$ stay inside the associated $(\kappa, \mathsf{m}_{\alpha_0}(\mathsf{q},\mathsf{Q}))$-neighborhood of $\alpha_0$ (as in Definition 2.4 ), up to distance $\mathsf{r}$.

2.2. **First passage percolation.** We consider a connected non-oriented graph $X$, whose set of vertices (resp. edges) is denoted by $V$ (resp. E). For every function $\omega : E \to (0,\infty)$, we equip $V$ with the weighted graph metric $d_\omega$, where each edge $e$ has weight $\omega(e)$. In other words, for every $v_1, v_2 \in V, d_\omega(v_1, v_2)$ is defined as the infimum over all path $\gamma = (e_1, \ldots, e_m)$ joining $v_1$ to $v_2$ of $|\gamma|_\omega := \sum_{i=1}^m \omega(e_i)$. Observe that the graph metric on $V$ corresponds to the case where $\omega$ is constant equal to 1 , we shall simply denote it by $d$. We consider a probability measure on the set of all weight functions $\omega$. We let $\nu$ be a probability measure supported on $[0,\infty)$. Our model consists in choosing independently at random the weights $\omega(e)$ according to $\nu$. More formally, we equip the space $\Omega = [0,\infty)^E$ with the product probability that we denote by $P$.

Let $d(\gamma, \mathfrak{o})$ denote the distance between $o$ and the set of vertices $\{\gamma(0), \gamma(1), \ldots\}$. That is

$$d(\gamma, \mathfrak{o}) := \inf\Big\{ d(v, \mathfrak{o}) : v \in \{\gamma(0), \gamma(1), \ldots\} \Big\}$$

The following two results (Proposition 2.12 and 2.14) along with the lemma 2.13 established upper and lower bounds for how distance can be perturbed under first passage percolation. Proposition 2.12 is established in [BT17]. Proposition 2.14 is a generalization of Lemma 2.5 in [BT17].

**Proposition 2.12.** [BT17] *Let $X$ be a connected graph, and let $\gamma$ be a self-avoiding path. Assume $0 < b = \mathbb{E}\omega_e < \infty$. Then for a.e. $\omega$, there exists $r_0 = r_0(\omega)$ such that for all $i \le 0 \le j$,*

$$\ell_\omega(\gamma([i,j]) \le 2b(j-i) + r_0$$

**Lemma 2.13** (Lemma 2.4 [BT17])**.** *Let $X$ be an infinite connected graph with bounded degree and assume that $\nu(0) = 0$. There exists an increasing function $\alpha : (0,\infty) \to (0,1]$ such that $\lim_{t\to 0} \alpha(t) = 0$, and such that for all finite path $\gamma$ and all $\epsilon > 0$,*

$$\mathbb{P}\big(\ell_\omega(\gamma) \le \epsilon \cdot \ell(\gamma)\big) \le \alpha(\epsilon)^{\ell(\gamma)}.$$

**Proposition 2.14.** *Let $X$ be an infinite connected graph with bounded degree, and let $\mathfrak{o}$ be some vertex of $X$. Assume $\nu(\{0\}) = 0$. Then for every constant $D$, there exists $c > 0$ such that for a.e. $\omega$, there exists $r_1 = r_1(\omega)$ such that for all finite path $\gamma$ such that $d(\gamma, o) \leq D \cdot \ell(\gamma)$, one has*

$$\ell_\omega(\gamma) \geq c(D) \cdot \ell(\gamma) - r_1$$

*Furthermore, $c(\cdot)$ is an decreasing function.*

*Proof.* Let $q$ be an upper bound on the degree of $X$, for simplicity, suppose $q \geq 2$ and let $n \geq 1$ be some integer. Suppose $\gamma$ is a path satisfying $d(o, \gamma) \leq D \cdot \ell(\gamma) = Dn$. Then the number of vertices in the ball $B(\mathfrak{o}, Dn)$ is,

$$|B(\mathfrak{o}, Dn)| \leq 1 + q \cdot \sum_{i=1}^{Dn-1} (q-1)^i \leq \sum_{i=0}^{Dn-1} (q+1)^i \leq (q+1)^{Dn+1}.$$

Now possible number of paths of length $n$, starting in the ball $B(\mathfrak{o}, Dn)$ is

$$P_{n,D} \leq q \cdot (q-1)^{n-1} \cdot (q+1)^{Dn+1} \leq (q+1)^{(D+1)n+1}$$

Suppose $\alpha(t)$ be the increasing function with $\lim_{t\to 0} \alpha(t) = 0$ in the Lemma 2.13. Now define a function $\beta : (0, 1] \to (0, \infty)$ such that,

$$\beta(t) := sup\{x \in (0, \infty) : \alpha(x) \leq t\}.$$

The function is well defined since for every $t$ there exists $\delta_t > 0$ such that $\alpha(x) \leq t$ wherever $x \in [0, \delta_t]$. Therefore, $[0, \delta_t] \subset \{x \in (0, \infty) : \alpha(x) \leq t\}$. Moreover, $\beta$ is non-decreasing and right continuous. Let, $C(D)$ be a function defined as

$$C(D) = \beta(\frac{1}{q^{D+1}})$$

As a function of $D$, $C(D)$ is decreasing. Moreover by definition, $\alpha(C(D) \leq \frac{1}{q^{D+2}}$. Let us consider the following event,

$A_{i,n,D} := \{$there exists some path $\gamma_i$ of length $n$, with $d(o, \gamma_i) \leq Dn$, and $\ell_\omega(\gamma_i) \leq c(D) \cdot \ell(\gamma_i)\}$.

By Lemma 2.13 $\mathbb{P}(A_{i,n,d}) \leq \alpha(C(D))^n \leq \frac{1}{q^{Dn+n}}$. There are up to $(q+1)^{(D+2)n}$ possible number of paths. Hence, the probability of at least one of these paths satisfies this property is,

$$\begin{aligned}
\mathbb{P}\left(\bigcup_{i=1}^{(q+1)^{(D+1)n+1}} A_{i,n,D}\right) &\leq \sum_{i=1}^{(q+1)^{(D+1)n+1}} P(A_{i,n,D}) \\
&\leq (q+1)^{(D+1)n+1} \cdot \frac{1}{(q+1)^{Dn+2n}} \\
&= \frac{1}{(q+1)^{n-1}}.
\end{aligned}$$

Thus $\sum_{n=1}^{\infty} \frac{1}{(q+1)^{n-1}} < \infty$. Therefore, by the Borel-Cantelli lemma, the probability that this event happens infinitely often is 0. □

**Corollary 2.15.** *Let $X$ be an infinite connected graph with bounded degree, and let $o$ be some vertex of $X$. Assume $\nu(\{0\}) = 0$. Let $\gamma : [0, n] \to X$ be any finite self avoiding path of length $n$, such that $d(\gamma, o) \leq D \cdot n$. Then the image of $\gamma$ in $X_\omega$, is a quasi-isometric image of $\gamma$ whose constants depends only on $\omega$, i.e. there exists constants $q(\omega), Q(\omega)$ such that,*

$$\frac{1}{q} \cdot \ell(\gamma) - Q(\omega) \leq \ell_\omega(\gamma) \leq q \cdot \ell(\gamma) + Q(\omega)$$

*Proof.* Let $Q(\omega) := \max\{r_1(\omega), r_0(\omega)\}$ and let $q := \max\{2b, \frac{1}{c(D)}\}$ then by Propositions 2.12, 2.14 we have the desired inequalities both sides. □

Now take the union of all $\kappa$-Morse boundaries and it is established in [QRT23] that the topology on them are consistent with each other and in fact there exists a natural topology on the set of all sublinearly Morse directions, of which all $\kappa$-boundaries are subspaces with subspace topology.

**Definition 2.16.** Let $\partial_s X$ denote the set of all sublinearly Morse quasi-geodesic rays, up to sublinear fellow travelling. We refer to this as the *Sublinearly Morse boundary.*

**Definition 2.17.** The topology on $\partial_s X$ is defined as follows: let $\beta$ be a $\kappa$ Morse sublinearly quasi-geodesic ray for some $\kappa$. An equivalence class $\mathsf{a} \in \partial_s X$ belongs to $U(\beta, r)$ if, for any $(q, Q)$-quasi-geodesic ray $\alpha \in \mathsf{a}$, where $m_\beta(q, Q)$ is small compared to $r$ with respect to $\kappa$, we have the inclusion

$$\alpha|r \ \subset N_\kappa(\beta, m_\beta(q, Q)).$$

An open neighborhood $\mathcal{V}$ of a point $\mathbf{b} \in \partial_s X$ contains all $\kappa'$-Morse quasi-geodesic rays that in the ball of radius $r$ around quasi-geodesic rays of $\mathbf{b}$, behaves as if they are a memeber of $\mathbf{b}$. It is established in ?? that with this topology the space $\partial_s X$ is metrizable and contains each $\partial_\kappa X$ as a topological subspace.

## 3. Middle-recurrence of sublinear lines

In this section we study the middle-recurrence property of sublinearly Morse bi-infinite lines and use it to charaterize sublinearly Morse direcitons and bi-infinite lines. Let $\gamma$ be a sublinearly Morse set. The middle recurrent property concerns the location of paths with endpoints on $\gamma$ as opposed to the location of quasi-geodesic segments with endpoints on $\gamma$.

By a *path* we mean the continuous image of an interval $[a, b]$ in $X$. In this paper, we assume all paths to be an embedding of an interval and thus are self-avoiding. This assumption makes the argument shorter and does not subtract from the generality for the following reason:??.

Given a path $\alpha$ with endpoints $x, y$, suppose $\ell(\alpha)$ be the arc-length of $\gamma$ then the *slope* of $\alpha$ is defined to be

$$sl(\alpha) := \frac{\ell(\alpha)}{d(x,y)}$$

where the arc-length of a continuous curve $\alpha : I \to X$ is defined as the infimum of the sum of the distances $d(\alpha(t_{i-1}), \alpha(t_i))$ over any partition $\{t_i\}_{i\geq 0}$ of $I$.

**Definition 3.1.** Let $\gamma$ be set. We say that a $\gamma$ has the *middle-recurrence property* if for every $C \geq 1$ there is a constant $c$ (depends on $C$ and $\beta$) such that the following holds: for any path $P$ with endpoints $a, b \in \beta$ satisfying $\ell(P) \leq Cd(a,b)$, intersects the $c \cdot \kappa$-neighborhood of the middle third of $\beta[a,b]$ for some sublinear function $\kappa$. That is,

$$P \cap \mathcal{N}_\kappa(\beta_{\frac{1}{3}[a,b]}, c) \neq \emptyset.$$

Middle recurrence is first studied by ?? and then corrected and further expanded by [ADT17]. The key correction is... Here we use an distance estimation lemma [ADT17, Lemma 3.3] that gives an upper bound of the distance between the end points of a path with fixed slope and other properties. Let $p$ be a path with end points $s$ and $e$.

**Lemma 3.2.** *Let $\beta$ be a $\kappa'$–radius-contracting quasigeodesic and suppose that $p$ is a path with the following properties:*

- *slope of $p$ is $D$*
- *$p$ is at distance at least $K$ from $\beta$*
- *The end points of $p$ are at distance exactly $K$ from $\beta$.*

*Let $|p| : d = d(s,e)$ denote the distance between the endpoints of $p$. Then*

$$\frac{\kappa(K)}{2K} \geq \frac{1 - (2K + \kappa'(K))/|p|}{D}$$

**Notation.** In the rest of this paper, if $\beta$ is a geodesic line, then the restriction of $\beta$ between two points $a, b \in \beta$ is notated as $\beta(a,b)$. Furthermore, if we are referring to a segment of this geodesic segment that starts at one third from the endpoint $a$ and ends at $\frac{2}{3}$ from the endpoint $a$ then we notate it as $\beta_{[\frac{1}{3},\frac{2}{3}]}[a,b]$. In general, if a geodesic segment $\alpha$ lies on a geodesic ray that emanates from the origin, then it is assumed that $\alpha_{[p,q]}$, where $0 \leq p, q \leq 1$, lies between the fractions $p$ and $q$ starting from the point closer to the origin.

**Proposition 3.3** ($\kappa$-Morse implies middle recurrence)**.** *Let $\beta$ be a bi-infinite geodesic line. Assume that $\beta$ is $\kappa$-Morse with Morse gauge $m(q,Q)$. Let $p$ be any simple path with endpoints $a, b \in \beta$ and such that $\ell(p) < Cd(a,b)$. Then there exists a sublinear neighborhood of $\beta$ depending only on $\beta$ and $C$ such that*

$$p \cap N_{\kappa'}(\beta(a,b)_{[\frac{1}{3},\frac{2}{3}]}, c(C,\beta)) \neq \emptyset.$$

*Here $\kappa'$ is the contracting sublinear function of $\beta$.*

*Proof.* To proceed, let $\kappa'$ denote the contracting function of $\beta$ [**?**]. By way of contradiction, suppose there does not exist any sublinear neighborhood of $\beta$ in which $p$ intersects the middle third. Then there exists a constant $c_1$ such that there exists a family of paths $\{p_i\}$ such that for arbitrarily large $t$, there exists a $p_i$ where $p_i(t)$ is outside of the $c_1$-linear neighborhood of the middle third of $p_i$. That is, Let

$$LN(\beta, c_1) := \{x \in X | \, d(x, \beta) \geq c_1 \cdot d(\mathfrak{o}, x)\}.$$

Given any such $p_i$, let $p_i'$ be a connected segment of $p_i$ that connects points where $p_i$ exits the $c_1$-linear neighborhood of $\beta$ and does not intersect the middle third. By construction, with respect to points on $p_i'$, $\beta$ is at least $\kappa$-radius-contracting.

Secondly, suppose the end points of $p_i'$ are $(s_i', e_i')$, then

$$d(s_i', e_i') \geq \frac{1}{3} d(s_i, e_i)$$

Thus

$$(p_i)' \leq 3s(p_i) \leq C.$$

Furthermore, we claim that for large enough $i$, $d(s_i, e_i)$ does not grow sublinearly with respect to $d(\mathfrak{o}, s_i)$, otherwise, since slope of $p_i$ are bounded, eventually $p_i$ will not be outside of the linear neighborhood of of $\gamma$, thus there exists a constant $c_2 > 0$ where

$$d(s_i, e_i) \geq c_2 d(\mathfrak{o}, s_i).$$

Therefore, a slight adaptation of the Lemma?? we get

$$\frac{\kappa(t)}{t} \geq \frac{1 - (c_2' t + \kappa(t))/d(s_i', e_i')}{3C}$$

However, as $i \to \infty$, if $d(s_i', e_i')$ grows at most sublinearly with respect to $d(s_i, e_i)$, then eventually there will mid-third recurrence. Thus by construction, there exists $c_2$ such that for all $i$ large enough, $d(s_i', e_i') \geq c_3 t$ and thus there exists a uniform constant $D > 0$ depending only on $C, c_1, c_2, c_3 \kappa \kappa'$ where

$$\frac{1 - (c_3 t + \kappa(t))/d(s_i', e_i')}{3C} \geq D.$$

Thus there exist a maximal $t$ for which the inequality holds, which is a contradiction as desired.

□

**Proposition 3.4** (Middle recurrence implies sublinearly Morse)**.** *Let $\gamma$ be a geodesic ray, and let $P$ be a self avoiding path with endpoints $a, b$ on $\gamma$ with the following properties:*

- *the slope of $P$: $sl(P) < C$*
- *There exists a family of constants $M \colon \mathbb{R}^2 \to \mathbb{R}^2$ and a sublinearly function $\kappa$ such that*

$$P \cap \mathcal{N}_\kappa(\gamma_{\frac{1}{3}[a,b]}, M(c_1, c_2)) \neq \emptyset$$

*for some $c_1, c_2 \in \mathbb{R}$.*

*Then $\gamma$ is sublinearly Morse.*

Before giving the main proof of the proposition, we establish the following lemma.

**Lemma 3.5.** *Given the setup of Proposition 3.4 (See Figure 3), let*

$$d := \max\{M(\frac{13q+2}{10}, 0), 1\}$$

*Let $\mathsf{p} : I \to X$ be a continuous $(q, q)$ quasi-geodesic with endpoints $a, b$ on $\gamma$. Suppose a sub-path $\mathsf{p}'$ is distance $d\kappa(t)$ from $\gamma$ and the endpoints of the sub path, denoted $a'$ and $b'$, are distance at least $12d \cdot \max\{\kappa(||b'||), \kappa(||a'||)\}$ apart, then*

$$\mathsf{p}' \cap \mathcal{N}_\kappa(\gamma, d) \neq \emptyset$$

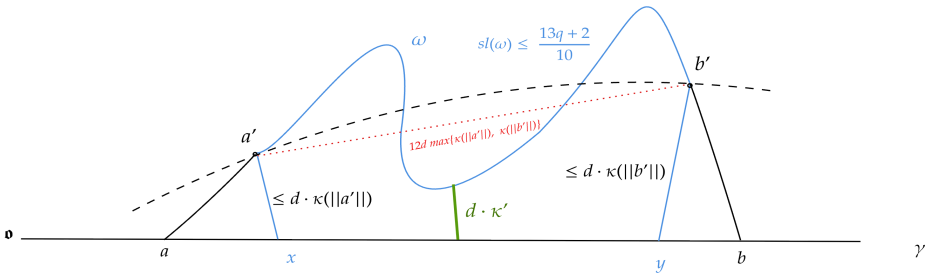



FIGURE 3. if $a', b'$ are $\kappa-$separated then a sub path of $\mathsf{p}$ intersects a $\kappa'$ neighborhood of $\gamma$.

*Proof.* Suppose, $A = A(a', b') = \max\{\kappa(||a'||), \kappa(||b'||)\}$. Let $x, y \in \gamma$ such that $d(x, a') \leq d \cdot \kappa(||a'||)$, $d(y, b') \leq d \cdot \kappa(||b'||)$. Let $\omega$ be the concatenation $[x, a'] \cdot \mathsf{p}' \cdot [b', y]$. From the given hypothesis, we have the following estimates:

$$d(x, y) \geq d(a', b') - d(a', x) - d(b'y) \geq 12d \cdot A(a', b') - d\kappa(||a'||) - d\kappa(||b'||) \geq 10d \cdot A(a', b')$$

Therefore we have,

$$d(x, y) \geq 10d \cdot A(a', b') \tag{4}$$

On other hand, since $d(a', b) \geq 12d \cdot A(a', b')$ we get,

$$d(a', b') - 2d \cdot A(a', b') \geq \frac{5}{6} d(a', b'). \tag{5}$$

Since $\mathsf{p}$ is $(q, q)$ quasi-geodesic we can assume that $\|\mathsf{p}'\| \leq q \cdot d(a', b') + q$.

Now we compute the slope of $\omega$:

$$\begin{aligned}
\frac{\ell(\omega)}{d(x,y)} &\leq \frac{\|\mathsf{p}'\| + d(a',x) + d(b',y)}{d(x,y)} \\
&\leq \frac{\|\mathsf{p}'\| + d\cdot\kappa(\|a'\|) + d\cdot\kappa(\|b'\|)}{d(x,y)} \\
&\leq \frac{\|\mathsf{p}'\|}{d(a',b') - d\cdot\kappa(\|a'\|) - d\cdot\kappa(\|b'\|)} + \frac{d\cdot\kappa(\|a'\|) + d\cdot\kappa(\|b'\|)}{d(x,y)} \\
&\leq \frac{q\cdot d(a',b') + q}{d(a',b') - 2d\cdot A(a',b')} + \frac{2d\cdot A(a',b')}{10d\cdot A(a',b')} \qquad \text{(by equation 4)} \\
&\leq \frac{q\cdot d(a',b') + q}{\frac{5}{6}d(a',b')} + \frac{1}{5} \qquad \text{(by equation 5)} \\
&\leq \frac{13q+2}{10}
\end{aligned}$$

Thus, $\omega$ has bounded slope. Hence, by middle recurrence, there is a point $c \in \mathsf{p}'$ that intersects the $\kappa$-neighborhood of the middle third of $\gamma_{[a,b]}$. □

**Proof of Proposition 3.4.** Let, $Q = \{t \in I : d(\mathsf{p}(t),\gamma) \leq d\cdot\kappa(\|\mathsf{p}(t)\|)\}$. By the Lemma 3.5, $Q^c$ is a collection of disjoint unions of open intervals in $\mathbb{R}$. If $(t,t') \in Q^c$ then $d(\mathsf{p}(t),\mathsf{p}(t')) < 12d\cdot A(\mathsf{p}(t),\mathsf{p}(t'))$. If $\eta \in Q^c$ then $\eta \in (t,t') \subset Q^c$ for some $t,t' \in \mathbb{R}$. Hence,

$$d(\mathsf{p}(\eta),\gamma) < d(\mathsf{p}(\eta),\mathsf{p}(t)) + d\cdot\kappa(\|\mathsf{p}(t))\| < 12d\cdot A(\mathsf{p}(\eta),\mathsf{p}(t)) + d\cdot\kappa(\|\mathsf{p}(t)\|).$$

Since $d(\mathsf{p}(\eta),\mathsf{p}(t)) \leq 12d\cdot\max\{\kappa(\|\mathsf{p}(\eta)\|,\kappa(\|\mathsf{p}(t))\|\}$ then by Lemma 2.3,

$$\kappa(\|\mathsf{p}(\eta))\| \leq D_2\cdot\kappa(\|\mathsf{p}(t))\|$$

for some $D_2 > 0$. Therefore,

$$d(\mathsf{p}(\eta),\gamma) \leq 12d\cdot(\kappa(\|\mathsf{p}(\eta)\|) + D_2\cdot\kappa(\|\mathsf{p}\|(\eta)) + d\cdot D_2\cdot\kappa(\|\mathsf{p}(\eta)\|).$$

This implies,

$$d(\mathsf{p}(\eta),\gamma) \leq d\cdot(13D_2+12)\cdot\kappa(\|\mathsf{p}(\eta)\|).$$

Thus, there exists a sublinear function $\kappa'$ depends on $\kappa$ such that $\mathsf{p} \subset \mathcal{N}_{\kappa'}(d,\gamma)$. Hence, the Proposition follows. ■

**Theorem 3.6.** *Let $X$ be a proper geodesic metric space. Let $\beta$ be a $\mathsf{q}$-quasi-geodesic ray. Then $\beta$ is sublinearly Morse if and only if $\beta$ is $\kappa$-middle recurrent for some $\kappa$.;*

*Proof.* Proposition 3.3, Proposition 3.4 establish proof in each direction. □

**Corollary 3.7.** *Let $\gamma$ be a sublinearly Morse geodesic ray. Then there exists a sublinear function $\kappa^{\text{path}}$ and a function $p : \mathbb{R}^2 \to \mathbb{R}$ such that for any $c, L > 0$. Suppose $\mathsf{p}$ be a path with local slope bound $L$ (i.e. for any subpath $\mathsf{p}'$ of $\mathsf{p}$, $sl(\mathsf{p}') \leq L$) and starting distance $\leq cL$ is contained in a $p(c,L)\kappa^{\text{path}}(t)$-neighborhood of $\gamma$.*

**Lemma 3.8.** *Let $X$ be a graph of bounded degree with a fixed base-point. Let $\gamma$ be any geodesic ray emanating from the base-point. Then the set of $\omega$ that maps any sublinear neighborhood of any $\gamma$ in $X$ to a linear neighborhood in $X_\omega$ of $\gamma_\omega$ is a set of measure zero.*

*Proof.* Suppose $\mathcal{N}_\kappa(\gamma, n)$ be a sublinear neighborhood of $\gamma$ in $X$ for some sublinear function $\kappa$ and gauge $n$. Let $x \in \mathcal{N}_\kappa(\gamma, n)$. Then by Proposition 2.12,

$$d_\omega(x, \gamma) \leq b \cdot n \cdot \kappa(||x||) + r_o(\omega).$$

Consider $[\mathfrak{o}, x]_\omega$ be the $\omega$-geodesic in $X_\omega$. Then the preimage of $[\mathfrak{o}, x]_\omega$ in $X$ satisfies the assumption of Proposition 2.14. Therefore, we get

$$||x||_\omega = \ell_\omega([\mathfrak{o}, x]_\omega) \geq c \cdot ||x|| - r_1(\omega).$$

Hence for a.e. $\omega$ we have,

$$\begin{aligned} d_\omega(x, \gamma) \leq & \frac{b \cdot n}{c} \cdot \kappa(||x||_\omega + r_1(\omega)) + r_0(\omega) \\ & \leq \left( \frac{b \cdot n}{c} + r_0(\omega) \right) \cdot \kappa'(||x||), \text{ where, } \kappa'(t) = \kappa(t + r_1(\omega)). \end{aligned}$$

Therefore, almost surely, FPP sends a sublinear neighborhood of a geodesic ray in $X$ to a sublinear neighborhood of the image ray in $X_\omega$.

□

**Theorem 3.9.** *The image of a $\kappa$-Morse geodesic is almost surely sublinearly Morse. Specifically, let $\gamma$ be a $\kappa$-Morse geodesic ray emanating from a fixed base point in $X$. Then for a.e. $\omega$ there exists $\kappa'$, depending on $\kappa$ and $\omega$ such that $\gamma_\omega$ is $\kappa'$-Morse.*

*Proof.* Let $\gamma \in X$ be a $\kappa$-Morse geodesic ray. Consider $\gamma_\omega \subset X_\omega$. Suppose that $\gamma_\omega$ is not sublinearly Morse, then there exists $(q, Q)$, $c_1 > 0$ and a family of $(q, Q)$-quasi-geodesic segments $\{\alpha_i\}$ with endpoints on $\gamma_\omega$ such that for each $i \in \mathbb{N}$, there exists $\alpha_i$ whose starting point is of norm greater than $i$ and $\alpha_i$ is not contained in the $c_1$-linear neighborhood of $\gamma_\omega$.

Because $\alpha_i$ is not contained in the neighborhood, they have to be linearly long compared to their distance to the origin, thus there exists a constant $c_2(c_1, q, Q)$ such that for all $\alpha_i$

$$i < d(\alpha_i, \mathfrak{o}) \leq c_2 |\alpha_i|.$$

Now consider the pre images of $\alpha_i$ (say, $\alpha_i^X$) in $X$. For all $i$, suppose the endpoints of $\alpha_i^X$ on $\gamma$ are $s_i, e_i$ respectively. We make the following claim:

**Claim 3.10.** *Up to an infinite subsequence $\{i_k\}$, the base of $\alpha_{i_k}^X$ are bounded below by $c_3 \cdot d(\mathfrak{o}, s_{i_k})$ for some $c_3$, i.e. there exists $c_3$ such that $d(s_{i_k}, e_{i_k}) \geq c_3 \cdot d(s_{i_k}, \mathfrak{o})$.*

*Proof.* Since for all $i$, $\alpha_i$ is outside of the $c_1$-linear neighborhood in $X_\omega$ then there exists $x_i \in \alpha_i$ such that,

$$d_\omega(x_i, \gamma_\omega) \geq c_1 \cdot d_\omega(\mathfrak{o}, x_i).$$

On the other hand, $d_\omega(\mathfrak{o}, s_i) \leq d_\omega(\mathfrak{o}, x_i) + d_\omega(x_i, s_i) \leq \left(\frac{1+c_1}{c_1}\right) d_\omega(x_i, s_i)$. Therefore,

$$\ell_\omega(\alpha_i) \geq \frac{c_1}{1+c_1} \cdot d_\omega(\mathfrak{o}, s_i). \tag{6}$$

By Proposition 2.14, for a.e. $\omega$ we get,

$$d_\omega(\mathfrak{o}, s_i) \geq c \cdot d(\mathfrak{o}, s_i) - r_1(\omega). \tag{7}$$

Since, $\alpha_i$ is a $(q, Q)$ quasi-geodesic, we have,

$$q \cdot d_\omega(s_i, e_i) + Q \geq \ell_\omega(\alpha_i). \tag{8}$$

Now, by combing Equations 6, 7, 8 and using Proposition 2.12, there exists two constants $L_1, L_2$ (which depends on $\omega$) such that,

$$d(s_i, e_i) \geq L_1 \cdot d(\mathfrak{o}, s_i) - L_2.$$

By assumption, $d(s_i, \mathfrak{o}) \to \infty$ as $i \to \infty$. Therefore, for large $i$, we have

$$d(s_i, e_i) \geq c_3 \cdot d(\mathfrak{o}, s_i).$$

□

**Claim 3.11.** *The paths $\alpha_i^X$ have bounded slope.*

*Proof.* Suppose otherwise, that as $i \to \infty$, the slope of $\alpha_i^X$ tends to infinity i.e. $\frac{\ell(\alpha_i^X)}{d(s_i, e_i)} \to \infty$. Therefore, there exists $R > 0$ such that $\ell(\alpha_i^X) \geq \frac{1}{c_3} \cdot d(s_i, e_i)$ Since the base grows linearly with respect to $d(\mathfrak{o}, s_i^X)$, the lengths of the curves grows superlinearly with respect to the base and distance to the origin. Thus all but finite of $\{\alpha_i^X\}$ satisfies the condition:

$$d_X(\mathfrak{o}, s_i^X) \leq \ell(\alpha_i^X).$$

By Proposition 2.14, for a.e. $\omega$ we have,

$$\ell_\omega(\alpha_i) \geq c \cdot \ell(\alpha_i^X) - r_1 \tag{9}$$

Hence by Equation 9 and Proposition 2.12,

$$sl(\alpha_i) = \frac{\ell_\omega(\alpha_i)}{d_\omega(s_i, e_i)} \geq \frac{c \cdot \ell(\alpha_i^X) - r_1}{b \cdot d(s_i, e_i) + r_0}$$

Thus, $sl(\alpha_i)$ grows unboundedly as $i$ goes to infinity because the slope of the curves $\alpha_i^X$ goes to infinity in $X$. This is impossible as $\alpha_i$'s are quasi-geodesics in $X_\omega$. Therefore, it follows that the paths $\{\alpha_i^X\}$ also have bounded slopes. □

Thus the preimages all have bounded slope and by the path characterization, they all intersect the middle third of a sublinearly neighborhood. Which contradicts lemma 3.8. Thus $\gamma_\omega$ is a sublinearly Morse set. □

3.1. **Construction of the bijective map between sublinearly Morse boundaries.** First we observe that a sublinear Morse geodesic ray can be mapped to a sublinearly Morse geodesic ray. We first observe that one can always find a geodesic ray in a sublinear bound of the set $\gamma_\omega$. This follows from Theorem 3.8 in [JQ25]. Furthermore, the construction of the geodesic line is the following: consider a sequence of points on the set $\gamma_\omega$ whose norm goes to infinity. The sequence $\{x_i | d(x_i, \mathfrak{o}_\omega) > i\}$ exists as $\gamma_\omega$ is an unbounded set. Consider the geodesic images $[\mathfrak{o}, x_i]$ and take the limit. We conclude that since $\gamma_\omega$ is sublinearly Morse, by definition there exists a sublinear neighborhood such that every $[\mathfrak{o}, x_i]$ ( a (q,0)-quasigeodesic segment with endpoints on $\gamma_\omega$). Hence the limit set $\gamma_0$ is in a sublinear neighborhood of $\gamma_\omega$ and is a geodesic ray we will refer to as a *geodesic representative* of the class $[\gamma_\omega]$.

**Definition 3.12.** Define:

$$f^\star_\omega(\gamma) := \gamma_0$$

It follows from [QRT23, Lemma 4.2] that $\gamma_0$ is also sublinearly Morse.

First it needs to be verified that $f^\star_\omega$ is well-defined.

**Lemma 3.13.** *Suppose $\alpha$ and $\beta$ are two geodesics in the same class $[\gamma]$, and from the above construction, we get two geodesic representatives $\alpha^\omega$ and $\beta^\omega$ in $X_\omega$. $\alpha_\omega$ and $\beta_\omega$ are in same class in $X_\omega$ i.e. $\frac{d_\omega\left(\alpha_\omega(t), \beta_\omega(t)\right)}{t} \to 0$, as $t \to \infty$.*

*Proof.* Let $\alpha^\omega$ and $\beta^\omega$ be the image of $\alpha$ and $\beta$ under a first passage percolation with the stated assumptions. Suppose $\alpha$ and $\beta$ are in the same class. Let $\gamma_t$ be a geodesic joining $\alpha(t)$ and $\beta(t)$. Therefore by Proposition 2.12 we have,

$$d_\omega(\alpha(t), \beta(t)) \leq |\gamma_t|_\omega \leq b \cdot d(\alpha(t), \beta(t)) + r_0$$

implies,

$$\frac{d_\omega(\alpha(t), \beta(t))}{t} \leq b \cdot \frac{d(\alpha(t), \beta(t))}{t} + \frac{r_0}{t}$$

Since $[\alpha] = [\beta]$, then $\frac{d_\omega(\alpha(t), \beta(t))}{t} \to 0$, as $t \to \infty$. Hence, $\alpha^\omega \sim \beta^\omega$ in $X_\omega$. From the construction in line 3.1 we get $\alpha_\omega$ and $\beta_\omega$ refers to the geodesic representative of the class $[\alpha^\omega]$and $[\beta^\omega]$ in $X_\omega$. Since, by transitivity we have $\alpha_\omega$ and $\beta_\omega$ are in same class in $X_\omega$. □

Now are ready produced the map constructed so far that is well-defined.

**Definition 3.14** (First passage percolation induceds a map from $\partial_s X$ to $\partial_s X_\omega$.)**.** From the Theorem 3.9, Construction 3.1, for almost every $\omega \in (\Omega, \mathbb{P})$ there is a map $f^\star_\omega$ that takes sublinear geodesic rays to sublinear geodesic rays. Lemma **??** shows that the maps preserves sublinear tracking of geodesic rays. Therefore we use the same notation to denote the associated map on a sublinear equivalence class for every $\kappa$-Morse geodesic $\alpha$ in $X$ starting at $\mathfrak{o}$.. Define:

$$f^\star_\omega([\alpha]) := [\alpha^\omega].$$

**Lemma 3.15.** *The map $f^\star_\omega$ is one-to-one.*

*Proof.* Suppose $f^\star_\omega([\alpha]) = [\alpha_\omega]$ and $f^\star_\omega([\beta])$=$[\beta_\omega]$ represents the same point in $\partial_s X_\omega$, where $\alpha, \beta$ are geodesics in $X$, starting at $\mathfrak{o}$. We aim to show that $[\alpha] = [\beta]$ in $\partial_s X$. Since $[\alpha_\omega] = [\beta_\omega]$ then $\frac{d_\omega\left(\alpha^\omega_r, \beta^\omega_r\right)}{r} \to 0$ as $r \to \infty$. Suppose by way of contradiction, $\alpha$ and $\beta$ do not fellow travel each other sublinearly in $X$. Now, consider $\{r_n\}_n$, a sequence of natural number such that for each $n \in \mathbb{N}$, $\alpha^\omega_{r_n} := \alpha^\omega(t_{r_n})$, and $\beta^\omega_{r_n} := \beta^\omega(t_{r_n})$ be two points on $\alpha_\omega$ and $\beta_\omega$, where, $t_{r_n}$ be the first time where $||\alpha^\omega(t_{r_n})||_\omega = ||\beta^\omega(t_{r_n})||_\omega = r_n$. Let $s_{r_n}, t_{r_n}$ be the nearest point projection of $\alpha^\omega_{r_n}$ and $\beta^\omega_{r_n}$ on $f_\omega(\alpha)$ and $f_\omega(\beta)$ respectively. Hence we have, $||s_{r_n}||_\omega = 2||\beta^\omega_{r_n}||_\omega \leq 2r_n$ and similiarly, $||t_{r_n}||_\omega \leq 2r_n$. By definition 3.14,

$$d_\omega(\alpha^\omega_{r_n}, s_{r_n}) \leq m_1 \cdot \kappa(r_n) \text{ and } d_\omega(\beta^\omega_{r_n}, t_{r_n}) \leq m_2 \cdot \kappa'(r_n).$$

Therefore,

$$r_n - m_1 \cdot \kappa(r_n) \leq ||s_{r_n}||_\omega \leq 2r_n \text{ and } r_n - m_2 \cdot \kappa'(r_n) \leq ||t_{r_n}||_\omega \leq 2r_n.$$

If we see $s_{r_n}, t_{r_n}$ in $X$, then by Proposition 2.12 and 2.14,

$$\frac{r_n - m_1 \cdot \kappa(r_n) - r_0}{2b} \leq ||s_{r_n}|| \leq \frac{2r_n + r_1}{c} \text{ and } \frac{r_n - m_2 \cdot \kappa'(r_n) - r_0}{2b} \leq ||s_{r_n}|| \leq \frac{2r_n + r_1}{c}.$$

Therefore, we have $\frac{c}{4b} \leq \lim_{n\to\infty} \frac{||s_{r_n}||}{||t_{r_n}||} \leq \frac{4b}{c}$. Moreover, $||s_{r_n}||, ||t_{r_n}|| \to \infty$.

Let $P_{r_n}$ be a geodesic in $X_\omega$ joining $s_{r_n}$ and $t_{r_n}$ in $X_\omega$. Now, $P^\omega_{r_n}$ be the preimage path of $P_{r_n}$ joining $s_{r_n}$ and $t_{r_n}$ in $X$. Since $s_{r_n}$ and $t_{r_n}$ have normals of bounded ratio, there exists $C' > 0$ such that

$$d_X(\mathfrak{o}, s_{r_n}) \leq C' \frac{s_{r_n}}{t_{r_n}},$$

By Proposition 2.14, the lengths of $P^\omega_{r_n}$ are bounded above sublinearly by $\|P\|^\omega_{r_n}$. However, since $\alpha, \beta$ do not sublinearly fellow travel, there exists $C''$ and an infinite collection of radii $\{\tau_i\}$ such that

$$d(\beta(\tau_i), \alpha) \geq C''\|\beta\|(\tau_i).$$

Since $\|P\|^\omega_{r_n}$ occurs infinitely many times and so do $\{\tau_i\}$, then $\beta$ travels back and forth from linearly far away from $\alpha$ to sublinearly far away from $\alpha$. This is only possible if the $P^\omega_{r_n}$'s (thus accordingly the $s_{r_n}$) are spaced linearly apart. However, since $\alpha^\omega$ and $\beta^\omega$ sublinearly fellow travel each other, any time sampling will render the same result for the associated $P^\omega_{r_n}$'s. Thus, pick the initial sequence $\{r_n\}$ such that there are $n_2$ points in any interval $[0, n]$, and we achieve the desired contradiction. □

**Lemma 3.16.** *The map $f^\star_\omega$ is onto.*

*Proof.* Suppose the map is not onto. Therefore, there doesn't exist any $[a] \in \partial_\kappa X$ such that $f_\omega([a]) = [a^\omega]$ for some $\kappa$- Morse geodesic ray $a^\omega$ in $X_w$ starting at $\mathfrak{o}$. That is, there does not exists any $\kappa$-Morse geodesic ray starting at $\mathfrak{o}$ whose image is in $[a^\omega]$ under $f_\omega$. Hence, for $\gamma^\omega \in [a^\omega]$ the pre image of $\gamma^\omega$ (say $\gamma$) under FPP map is not $\kappa$-Morse. Then there exists $(q, Q)$, $c_1 > 0$ and a family of $(q, Q)$-segments $\{\alpha_i\}$

with endpoints on $\gamma$ such that for each $i \in \mathbb{N}$, there exists $\alpha_i$ whose starting point is of norm greater than $i$ and $\alpha_i$ is not contained in the $c_1$-linear neighborhood of $\gamma_\omega$. There is an infinite subsequence of $\alpha_{i_k}$ with $d(s_{i_k}, e_{i_k}) \geq C \cdot d(s_{i_k}, \mathfrak{o})$ for some constant $C$ where $s_{i_k}, e_{i_k}$ are the endpoints of $\alpha_{i_k}$ on $\gamma$. By proposition 2.14 for a.e. $\omega$ we have $d_\omega(s_{i_k}, e_{i_k}) \geq c \cdot d(s_{i_k}, e_{i_k}) - r_1$. Therefore,

$$\frac{|\alpha^\omega_{i_k}|_\omega}{d_\omega(s_{i_k}, e_{i_k})} \leq \frac{b \cdot |\alpha_{i_k}| + r_o}{c \cdot d(s_{i_k}, e_{i_k}) - r_0}$$

Now, $\alpha^\omega_{i_k}$ are quasi-geodesics so they have bounded slope. Hence, $\alpha^\omega_{i_k}$ are paths with endpoints on $\gamma^\omega$ with bounded slope. Since $\gamma^\omega$ is $\kappa$-Morse so by proposition 3.3 $\alpha^\omega_{i_k}$ intersects a sublinear neighborhood of middle third of $\gamma^\omega[s_{i_k}, e_{i_k}]$. Suppose $p_{i_k} \in \alpha_{i_k}$ be the point on intersection.

**Claim 3.17.** *For a.e $\omega$ there exist $A = A(\omega), B = B(\omega)$ such that,*

$$\kappa(d_\omega(\mathfrak{o}, p_{i_k}) \geq d_\omega(p_{i_k}, \gamma^\omega_{\frac{1}{3}[s_{i_k}, e_{i_k}]}) \geq A \cdot d(\mathfrak{o}, p) - B$$

*Proof.* Since $\alpha_{ik} \cap \mathcal{N}_{\kappa'}(\gamma^\omega_{\frac{1}{3}[s_{i_k}, e_{i_k}]}, m) \neq \emptyset$ then there exist $x$ on $\gamma^\omega$ such that,

$$d_\omega(p_{i_k}, \gamma^\omega) = d_\omega(p_{i_k}, x) \leq m \cdot \kappa(d_\omega(\mathfrak{o}, p_{i_k}).$$

Suppose $\sigma^\omega$ be the geodesic joining $p_{i_k}$ and $x$ in $X_\omega$ and under FPP map the preimage $\sigma$ be the path joining $p_{i_k}$,$x$ in $X$. Since $p_{i_k} \in \alpha_{i_k}$ then $p_{i_k}$ is outside the $c_1$ linear neighborhood. Therefore we get,

$$|\sigma| \geq d(p_{i_k}, x) \geq d(p_{i_k}, \gamma) \geq c_1 \cdot d(\mathfrak{o}, p_{i_k}) \geq c_1 \cdot d(\mathfrak{o}, \sigma)$$

Thus, by proposition 2.14 for a.e $\omega$ there exist $r_1 = r_1(\omega)$ such that, $d_\omega(p_{i_k}, x) \geq c \cdot d(p_{i_k}, x) - r_1$. Hence, the claim follows. □

By proposition 2.12, 3.17 and monotonicity of $\kappa$ we have,

$$\kappa\big(b \cdot d(\mathfrak{o}, p_{i_k}) + r_o\big) \geq A \cdot d(\mathfrak{o}, p_{i_k}) - B.$$

Which is a contradiction. □

**Lemma 3.18.** *Let $\gamma \in X$ be a sublinearly Morse geodesic ray. Then in $X_\omega$, $\gamma_\omega$ and the geodesic ray $\gamma_0$ are in sublinear neighborhoods of each other.*

*Proof.* Consider the limit geodesic of $[\mathfrak{o}, c_i]$ and call that $\gamma_1$, By Definition 3.12, $\gamma_1$ is a sublienarly Morse geodesic ray. By the construction of $\gamma_1$, $\gamma_1$ is in a sublinear neighborhood of $\gamma_\omega$. Given $\epsilon > 0$, there exists a subsequence of $c_{ij}$ that are within $\epsilon$-neighborhood of $\gamma_1$, therefore also in a sublinear neighbohrood of $\gamma_0$, contradicting the choice of the $c_i$.

Suppose $\gamma$ be a $\kappa$-Morse geodesic ray with morse gauge $M$ and $\omega(\alpha)$ is a sublinear Morse set for some sublinear function $\rho$. Now, by the construction of the map 3.1, $\gamma_0$ is the geodesic representative. Suppose $\{c_i\}_{i\geq 1}$ be the points on $\gamma$(also on $\omega(\gamma)$) for which equation **??** satifies. Now if we take $[\mathfrak{o}, c_i]_\omega$ then the limit set namely $\gamma_1$ is in a $\rho$-neighbourhood of $\omega(\gamma)$. Suppose by the construction the geodesic representative of

preimage of $\gamma_1$ under $f^\star_\omega$ is $\alpha$. Then by construction, $\{c_i\}$'s are on $\alpha$. For each $i$, the segment $\alpha|_{[c_i,c_{i+1}]}$ is a geodesic therefore, $\alpha|_{[c_i,c_{i+1}]} \subset \mathcal{N}_\kappa(\gamma, m)$, where $m = M(1,0)$. Therefore, $\alpha \subset \mathcal{N}_\kappa(\gamma, m)$. Which implies $\alpha \sim \gamma$. By the injectivity of $f^\star_\omega$, $\gamma_0 \sim \gamma_1$ in $X_\omega$. By lemma ?? $\gamma_0 \subset \mathcal{N}_\rho(\gamma_1, m_{\gamma_1}(1,0))$ and $\gamma_1 \subset \mathcal{N}_\rho(\gamma_0, 2m_{\gamma_1}(1,0))$.
Suppose $x_{ij}$ are the points on $\gamma_1$ such that $d_\omega(x_{ij}, c_{ij}) < \epsilon$. Also observe that, $d_\omega(\mathfrak{o}, c_{ij}) \leq d_\omega(\mathfrak{o}, c_{ij}) + \epsilon$. Hence,

$$c \cdot d_\omega(\mathfrak{o}, c_{ij}) \leq d(c_{ij}, \gamma_0) \leq d_\omega(c_{ij}, x_{ij}) + d(x_{ij}, \gamma_0)$$

Which implies, $c \cdot d_\omega(\mathfrak{o}, c_{ij}) \leq \epsilon + 2m \cdot \rho(d_\omega(\mathfrak{o}, x_{ij})) \leq \epsilon + 2m \cdot \rho(d_\omega(\mathfrak{o}, c_{ij}) + \epsilon)$. Which is a contradiction to the fact that $d_\omega(c_{ij}, \mathfrak{o}) \to \infty$.

□

**Theorem 3.19.** *Let $f_\omega$ be a first passage percolation. Then for a.e $\omega$ $f_\omega$ induces a homeomorphism $f^\star_\omega \colon \partial_s X \to \partial_s Y$.*

*Proof.* By Lemma 3.15 and Lemma 3.16, $f^\star_\omega$ defined in Definition 3.12 gives a bijection between $\partial_s X$ and $\partial_s Y$. To prove the homeomorphism, we first show that $f^{\star^{-1}}_\omega$ is continuous, and secondly, we will show $f^\star_\omega$ is continuous.
To prove the first claim, we will look into the classical topological definitions. Which says the following, $f^\star_\omega$ is open, then the following are equivalent,

(1) For every $\mathbf{b}_X \in \partial_s X$ and an open set $\mathbf{b}_X \in \mathcal{U}$ there exists an open set $\mathcal{V}$ containing $f^\star_\omega(\mathbf{b}_X) = \mathbf{b}_Y$ such that $\mathcal{V} \subset f^\star_\omega(\mathcal{U})$.
(2) For $\mathbf{b}_X \in \mathcal{U}_\kappa(b_x, \mathsf{r})$ there exists $\mathsf{r}'$ such that $f^\star_\omega(\mathbf{b}_X) = \mathbf{b}_Y \in \mathcal{U}_\kappa(\mathbf{b}_Y, \mathsf{r}')$ and $\mathcal{U}_\kappa(\mathbf{b}_Y, \mathsf{r}') \subset f^\star_\omega(\mathcal{U}_\kappa(b_x, \mathsf{r}))$
(3) If $\mathsf{a}_Y \in \mathcal{U}_\kappa(\mathbf{b}_Y, \mathsf{r}')$ the there exist $\mathsf{a}_X \in \mathcal{U}_\kappa(\mathbf{b}_X, \mathsf{r})$ such that $(f^\star_\omega)^{-1}(\mathsf{a}_Y) = \mathsf{a}_X$.

We will use (3) above to prove our claim.

Let $\mathcal{V}$ be an open set in $\partial_s X$ where $\mathbf{b}_X \in \mathcal{V}$. Assume without loss of generality that $\mathbf{b}_X$ is $\kappa$-Morse for some $\kappa$. Then there exists a neighborhood $\mathcal{U}_s(\mathbf{b}_X, \mathsf{r}) \subset \mathcal{V}$. Let $\mathbf{b}_Y = f^\star_\omega(\mathbf{b}_X)$. If $b_X$ is the geodesic representative $\mathbf{b}_Y$ then $f_\omega(b_X)$ is a $\kappa'$-Morse set and $b_Y$ is the geodesic representative of $\mathbf{b}_Y \in \partial_s X$ from the definition. Therefore by construction $b_Y \subset \mathcal{N}_{\kappa'}(f_\omega(b_X), \mathsf{m}_{f_\omega(b_X)})$ similarly if $\mathsf{a}_X \in \partial_s X$ then $a_Y \subset \mathcal{N}_{\kappa'}(f_\omega(a_X), \mathsf{m}_{f_\omega(a_X)})$ where $a_X$ is a $\rho$-Morse geodesic representative in $\mathsf{a}_X$ whereas the image $f_\omega(a_X)$ is $\rho'$-Morse. By Lemma 3.18, there is a constant $\mathsf{n}_1$ depending on $\kappa, \kappa'$ and $\mathsf{m}_{b_Y}$ such that

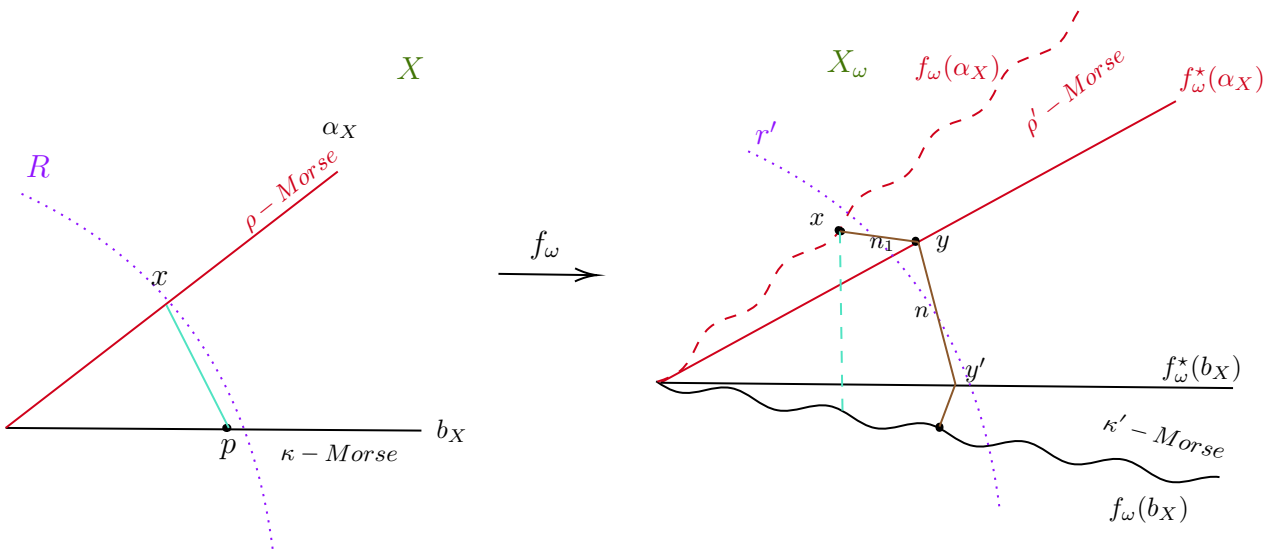
$$f_\omega(a_X) \subset \mathcal{N}_{\rho'}(a_Y, \mathsf{n}_1).$$

Next, we let

$$\mathsf{n} = C(r)^{-1}[\mathsf{n}_1 + 2n + 4\mathsf{m}_{f_\omega(b_X)}(1,0) + r_0(\omega)] \cdot (b' + r'_0(\omega))$$

and let $\mathsf{R} = \mathsf{R}(b_X, \mathsf{r}, \mathsf{n}, \kappa')$ as in Definition of sublinear Morse and choose $\mathsf{r}'$ such that:

(1) $\mathsf{r}' \geq 2bR + r_o \cdot \kappa(r)$, and
(2) $n$ is small compare to $\mathsf{r}'$.




Now we are ready to consider any given $\alpha \in \mathsf{a}_X$ be a $(\mathsf{q}, \mathsf{Q})$–quasi-geodesic in $X$, where

(1) $\mathsf{q}, \mathsf{Q}$ is such that $\mathsf{m}_{b_X}(\mathsf{q}, \mathsf{Q})$ is small compared to $\mathsf{r}$ with respect to $\kappa$; and
(2) $f_\omega(\alpha)$ is a ray that is associated with an element in $\mathcal{U}_{\kappa'}(\mathbf{b}_Y, \mathsf{r}')$.

By the choice of $\mathsf{r}'$, $n$ is small compared to $\mathsf{r}'$. Hence,

$$(f^\star_\omega)(\alpha_X) = \alpha_Y|_{\mathsf{r}'} \subset \mathcal{N}_{\kappa'}\big(b_Y, n\big)$$

Pick a point $x \in \alpha_X|_{\mathsf{R}}$ such that $||x|| = R$ in $X$. Lets consider $d(x, b_X)$ then $d(x, b_X) = d(x, p)$ where $p \in \Pi_{b_X}(x)$. Then if $[x, p]_\omega$ is the $\omega$-geodesic joining $x$ and $p$. Then the pre image path namely, $\beta$ satisfy, $d(\mathfrak{o}, \beta) \leq ||x|| = R \leq R \cdot \ell(\beta)$. Therefore by Proposition we have

$$d(x, b_X) \leq C(R)^{-1}(d_\omega(x, b_X) + r_o)$$

$$\begin{aligned}
d_X(x, b_X) &\leq C(R)^{-1}(d_\omega(x, f_\omega(b_X)) + r_o) \\
&\leq C(R)^{-1}[d_\omega(x, \mathsf{y}) + d_\omega(\mathsf{y}, \mathsf{y}') + d_\omega(\mathsf{y}', f_\omega(b_X)] \quad (\text{where, } \mathsf{y} \in \Pi_{\alpha_Y}(x), \mathsf{y}' \in \Pi_{b_Y}(x)) \\
&\leq C(R)^{-1}[\mathsf{n}_1 \cdot \rho'(||x||_\omega) + n \cdot \kappa'(||\mathsf{y}||_\omega) + \mathsf{m}_{f_\omega(b_X)}(1, 0) \cdot \kappa'(||\mathsf{y}'||_\omega) + r_0(\omega)] \\
&\leq C(R)^{-1}[\mathsf{n}_1 + 2n + 4\mathsf{m}_{f_\omega(b_X)}(1, 0) + r_0(\omega)] \cdot \mathcal{A}(||x||_\omega)
\end{aligned}$$

where, $\mathcal{A} = \max\{\rho', \kappa'\}$. We also have

$$\mathcal{A}(||x||_\omega) \leq b' \cdot \mathcal{A}(||x||) + r'_o \leq (b' + r'_0(\omega))\mathcal{A}(||x||)$$

Note that by Proposition 2.14, $C(.)$ is a decreasing function, hence, combining the preceding inequalities, we have,

$$\begin{aligned} d_X(x, b_X) &\leq C(R)^{-1}[\mathsf{n}_1 + 2n + 4\mathsf{m}_{f_\omega(b_X)}(1,0) + r_0(\omega)] \cdot \mathcal{A}(||x||_\omega) \\ &\leq C(R)^{-1}[\mathsf{n}_1 + 2n + 4\mathsf{m}_{f_\omega(b_X)}(1,0) + r_0(\omega)] \cdot (b' + r_0'(\omega)\mathcal{A}(||x||) \\ &\leq C(r)^{-1}[\mathsf{n}_1 + 2n + 4\mathsf{m}_{f_\omega(b_X)}(1,0) + r_0(\omega)] \cdot (b' + r_0'(\omega)\mathcal{A}(||x||) \\ &\leq \mathsf{n} \cdot \mathcal{A}(||x||) \end{aligned}$$

Now, the Definition implies that

$$\alpha_X|_{\mathsf{r}} \subset \mathcal{N}_\kappa(b_X, \mathsf{m}_{b_X}(1,0)).$$

Therefore, $\mathsf{a}_X \in \mathcal{U}_\kappa(\mathbf{b}_X, \mathsf{r})$ and

$$(f_\omega^\star)^{-1}\mathcal{U}_s(\mathbf{b}_Y, \mathsf{r}') \subset \mathcal{U}_s(\mathbf{b}_X, \mathsf{r}).$$

For the other direction, it is equivalent to saying, the $(f_\omega^\star)^{-1}$ is open. Fix a $r > 0$, then we aim to show that, for any $\mathcal{U}_\kappa(\mathsf{a}_Y, r)$ the preimage $(f_\omega^\star)^{-1}(\mathcal{U}_\kappa(\mathsf{a}_Y, r))$ is open in $\partial_s X$. That is, for every $\mathsf{a}_X \in (f_\omega^\star)^{-1}(\mathcal{U}_\kappa(\mathsf{a}_Y, r))$ there exists a $r'$, such that $\mathcal{U}_s(\mathsf{a}_X, r') \subset (f_\omega^\star)^{-1}(\mathcal{U}_\kappa(\mathsf{a}_Y, r))$.Without loss of generality, suppose $f_\omega^\star(\mathsf{a}_X) = \mathsf{a}_Y$. Let $R(a_Y, r, \mathsf{n}, \rho')$ then choose $r'$ such that:

(1) $r' \geq \frac{R+r_1}{c}$, where $r_1, c$ are the constant from Proposition 2.14.
(2) $\mathsf{n}'$ is small compared to $r'$.

Suppose, $\mathbf{b}_X \in \mathcal{U}_s(\mathsf{a}_X, r')$ and $b_X$, be a geodesic representative of $\mathbf{b}_X \in \partial_s X$ and $f_\omega^\star(\mathbf{b}_X) = \mathbf{b}_Y$. Suppose $a_Y$ and $b_Y$ are geodesic representatives of $\mathsf{a}_Y$ and $\mathbf{b}_Y$ in $\partial_s X_\omega$ respectively.

Suppose, $y$ be a point on $b_Y$ with $||y||_\omega = R$, and let $y' \in \Pi_y(\omega(b_X))$. Now, $y''$ be the nearest point projection of $y'$ on $a_X$ in $X$. Then,

$$\begin{aligned} d_\omega(y, a_Y) &\leq d_\omega(y, y') + d_\omega(y', y'') + d_\omega(y'', a_Y) \\ &\leq m_{b_Y}(1,0) \cdot \kappa'(R) + 2b \cdot m_{a_X}(1,0) \cdot \rho(||y'||) + r_0(\omega) + n_1 \cdot \rho'(||y''||_\omega) \end{aligned}$$

Similarly like the previous proof there exists $\mathsf{n}'$ and a sublinear function $\eta$, we have

$$d_\omega(y, a_Y) \leq \mathsf{n}' \cdot \eta(R)$$

.

$$b_Y|_{\mathsf{r}} \subset \mathcal{N}_\kappa(a_Y, \mathsf{m}_{a_Y}(1,0)).$$

Hence, $f_\omega^*$ is continuous.

□

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