GENERICITY OF SUBLINEARLY MORSE DIRECTIONS IN GENERAL METRIC SPACES

YULAN QING AND WENYUAN YANG

Abstract. In this paper, we show taht for any proper statistically convex-cocompact actions on proper metric spaces, the sublinearly Morse boundary has full Patterson-Sullivan measure in the horofundction boundary, .

Contents

1. Introduction

Suppose that a group G admits a proper and isometric action on a proper geodesic metric space (X, d). The group G is assumed to be *non-elementary*: there is no finite index subgroup isomorphic to the integer group $\mathbb{Z}$ or the trivial group.

The contracting property captures a key feature of quasi-geodesics in Gromov hyperbolic spaces, rank-1 geodesics in CAT(0) spaces, thick geodesics in Teichmüller

2000 *Mathematics Subject Classification.* Primary 20F65, 20F67.

(W.Y.) Partially supported by National Key R & D Program of China (SQ2020YFA070059) and National Natural Science Foundation of China (No. 12131009 and No.12326601).

spaces, and many others. In recent years, this notion and its variants have been proven fruitful in the study of general metric spaces.

Let A be a closed subset of X, and $\pi_A : \mathrm{X} \to A$ be the nearest-point projection map. We say that A is C*–contracting* for $C \geq 0$ if

$$\mathrm{diam}(\pi_A(B)) \leq C$$

for any metric ball B disjoint from A. An element of infinite order is called *contracting*, if it acts by translation on a contracting quasi-geodesic.

Contracting boundary for CAT(0) spaces was introduced by Charney-Sultan [2] as quasi-isometric invariant, and has attracted active research interests in recent years ([3, 13]). It is observed that the contracting boundary is measurably negligible in harmonic measures and conformal measures. The underlying set of the contracting boundary consists of the endpoints of contracting geodesic rays in the visual boundary.

On the other hand, Qing and Rafi [13] showed that a much larger class of sublinearly Morse geodesics equipped with an appropriate topology (called the sublinearly Morse boundary $\partial_\kappa \mathrm{X}$) is still quasi-isometrically invariant. See Definition 2.13. It is of interest to demonstrate that the sublinearly Morse boundary is indeed larger than its QI-invariant predecessors. In [7], the authors showed that a subset of the sublinearly Morse directions that is *regularly contracting* (see Definition 2.16) is generic in the Patterson-Sullivan measure on visual boundary of CAT(0) spaces under some conditions, and on the Thurston boundary of Teichmüller spaces.

The motivation of this paper is to extend the result of [7] to the conformal measure on horofunction boundary of any proper metric space. Recently, a general theory of a conformal density has been developed by Coulon and the second-named author independently in [4, 21] on horofunction boundary in presence of contracting elements. A key tool in this generalization is assuming that the given $G \curvearrowright X$ is a proper, *statistically convex-cocompact (SCC)* action, introduced by the second-named author in [18], with a contracting element. Except the negatively curved examples which lies out of the interests of this paper, we mention:

- CAT(0)-groups, including right-angled Coxeter/Artin groups, act geometrically on CAT(0) spaces with rank-1 elements.
- Mapping class groups act on Teichmüller space endowed with Teichmüller metric.

Using different and more geometrically flavored tool, we show that sublinearly Morse directions form a full measure set with respect to conformal measures on horofunction boundary.

Let $\partial_h \mathrm{X}$ be the horofunction boundary. Every sublinearly Morse equivalent class $[\gamma]$ in $\partial_\kappa \mathrm{X}$ contains a representative of geodesic ray γ, defining a horofunction γ^+ in $\partial \mathrm{X}$. By abuse of language, we also call the set of images $\{\gamma^+ \in \partial_h \mathrm{X} : [\gamma] \in \partial_\kappa \mathrm{X}\}$ *sublinearly Morse directions* in $\partial_h \mathrm{X}$ [1]. Our main result says that such directions form a generic set with respect to conformal measures.

Theorem A (Theorem 4.1)**.** *Let $G \curvearrowright \mathrm{X}$ be a proper, non-elementary SCC action on a proper geodesic metric space* X *with a contracting element. Let μ_o be the ω_G–dimensional conformal density on the horofunction boundary $\partial_h \mathrm{X}$. Then the set of sublinearly Morse directions is a μ_o–full measure subset.*

---

[1]We do not claim here the assignment $[\gamma] \to \gamma^+$ is well-defined, up to finite difference or sublinear difference.

If X is a proper CAT(0) space, then there is a unique geodesic ray representative, so gives an injective map from sublinearly Morse boundary $\partial_\kappa \mathrm{X}$ into the visual boundary (homeomorphic to $\partial_h \mathrm{X}$). We thus obtain the following result. This removes some technical assumption (e.g. geodesically complete) on X in the recent result of [7, Theorem 1.1], which is required to apply the results of Ricks [15].

Theorem B. *Let $G \curvearrowright \mathrm{X}$ be a proper, non-elementary SCC action on a proper CAT(0) space X with a rank-1 element. Let μ_o be the ω_G–dimensional conformal density on the the visual boundary $\partial_h \mathrm{X}$. Then the set of sublinearly Morse directions is μ_o–full measure.*

At last, let us emphasize that Theorem A holds for any convergence boundary including Thurston boundary of Teichmüller spaces. In particular, this gives a different proof of this result in [7]. We refer the reader to Theorem 4.1 for the precise statement.

Historic background and related works. In his celebrated proof of now called Mostow rigidity theorem, G. Mostow carried out a two-step strategy as follows:

(1) Quasi-isometry induces an homeomorphisms called boundary map between visual boundaries,
(2) Proving the boundary map that is quasi-conformal is actually conformal by using ergodicity of geodesic flows. Absence in dimension 2, the non-singularity of this map sending a positive Lebesgue measure to the target turns out to be a remarkable feature in this approach.

Sought for a quasi-isometric invariant boundary has been an active research topic, from a perspective of quasi-isometric classification in geometric group theory. The famous counter-example of Croke-Kleiner [5] on the visual boundary of CAT(0) spaces generates the interests to the subclass of hyperbolic-like directions as the quasi-isometric invariant boundary, as shown in the fore-mentioned works of Charney-Sultan [2] and Qing-Rafi-Tiozzo [14].

Stimulated by Step 2, it is natural to ask whether the Q.I. invariant boundary under construction is generic in a measure of interest. Lebesgue measure on the visual boundary of rank-1 symmetric spaces is an important instance of conformal measures considered in this paper. First construction of such measures appeared in the seminal work of Patterson [11] and has been further developed by Sullivan, setting the track for fruitful research in last several decades. Recently, the theory of conformal density on the horofunction boundary is generalized to groups with contracting element independently by Coulon [4] and by the second-named author in [21]. This is the main setup of the paper. We refer the reader to these papers and references therein for a detailed discussion.

Statistically convex-cocompact actions of groups has received wide attention in the last few years. The concept was introduced in [18] to generalize the idea of convex-cocompact subgroups in Kleinian groups to non-hyperbolic settings. Independently, SCC actions were also introduced by Schapira-Tapie [16] under the terminoledge of strongly positively recurrent actions (or manifolds) in dynamical system. The class of SCC actions encompasses many examples, but forms a strict subclass of actions with purely exponential growth. In many circumstances, the latter is equivalent to the so-called positively recurrent actions in [12].

Another measure of interest is harmonic measure arising from random walk on Poisson boundary. Recently Qing-Rafi-Tiozzo [14] showed that, when $\kappa = \log t$, the κ-boundary of the Cayley graph of the mapping class group can be identified with the Poisson boundary of the associated random walks. More general results concerning stationary measures were recently announced by Inhyeok Choi, who in place of our ergodic theoretic and boundary techniques uses a pivoting technique developed by Gouëzel. It will be interesting to understand that under suitable conditions and restricted to the correcting compacitifcation, whether the sublinearly Morse directions that supports the hitting measure and the ones that supports the Patterson-Sullivan measures are singular or equivalent to each other.

Let us close the introduction by mentioning the following conjecture of Sullivan:

Conjecture 1.1. *Suppose that a discrete action $G \curvearrowright \mathbb{H}^n$ is of divergence type. Let μ_o be the corresponding Patterson-Sullivan measure supported on $\partial\mathbb{H}^n$. Then the following set of boundary points $\xi \in \partial\mathbb{H}^n$ for which the geodesic ray $\gamma = [o,\xi]$ satisfies*

$$\lim_{t\to\infty} \frac{d(\gamma(t), Go)}{t} = 0$$

is generic.

This was confirmed by Sullivan if the Bowen-Margulis measure on geodesic flow is finite. It follows by definition that this limit tends to 0 along regularly contracting rays, so the genericty of regularly contracting rays implies the conjecture. Note that if the action is of divergence type, then μ_o is supported on conical limit points of G. The conjecture then predicts it is further supported on the "sublinear" limit points (without assuming SCC action or finiteness of BMS measures etc). Therefore, if a variant of Sullivan's conjecture holds for regularly contracting rays, then Theorem A would be valid under a proper action of divergence type.

Organization of the paper. We present relevant background in Section 2. Section 3 collects all tools from the recently developed conformal density theory on convergence boundary. Section 4 is devoted to the proof of Theorem Theorem A.

2. Preliminary

2.1. **Notations.** Let (X, d) be a proper geodesic metric space. Let A be a closed subset of X and x be a point in X. By $d(x, A)$ we mean the set-distance between x and A, i.e.

$$d(x, A) := \inf\{d(x,y) : y \in A\}.$$

Let

$$\pi_A(x) := \{y \in A : d(x,y) = d(x,A)\}$$

be the set of nearest-point projections from x to A. Since X is a proper metric space, $\pi_A(x)$ is non empty. We refer to $\pi_A(x)$ as the *projection set* of x to A. Define

$$\mathbf{d}_A(x,y) = \mathrm{diam}(\pi_A(x) \cup \pi_A(y)).$$

If a set A is countable then we use $\sharp A$ to denote the number of elements in A.

Let $\alpha : [a,b] \to \mathrm{X}$ be a path with arc-length parametrization from the initial point $\alpha^- = \alpha(a)$ to the terminal point $\alpha^+ = \alpha(b)$. If $x = \alpha(s)$ and $y = \alpha(t)$ are two points on α, $[x,y]_\alpha$ denotes the parametrized subpath of α going from x to y, that is, the restriction of α: $[s,t] \to \mathrm{X}$. We also write $\alpha[s,t]$ simply to denote the segment restricting on the interval $[s,t]$. Let $[x,y]$ denote a geodesic segment (not necessarily unique) between $x, y \in \mathrm{X}$.

If β is a geodesic ray emanating from the base-point, then $\beta|_{\mathsf{r}}$ denote the point on β that is distance r from o.

A continuous path α is called a (q, Q)*–quasi-geodesic* for constants $q \geq 1$, $Q > 0$ if for any rectifiable subpath β,

$$\ell(\beta) \leq q \cdot d(\beta^-, \beta^+) + Q$$

where $\ell(\beta)$ denotes the length of β. Equivalently, if $\alpha : [a, b] \subseteq \mathbb{R} \to \mathrm{X}$ is parametrized with arclength parametrization, for any $s, t \in [a, b]$, we have

$$\frac{|s - t|}{\mathsf{q}} - Q \leq d\big(\alpha(s), \alpha(t)\big) \leq q\,|s - t| + Q.$$

The additional assumption that quasi-geodesics are continuous is not necessary, but it is added for convenience and to make the exposition simpler. When X is a geodesic metric space, one can always adjust a quasi-geodesic ray slightly to make it continuous (see [1, Lemma III.1.11]).

Denote by $\alpha \cdot \beta$ (or simply $\alpha\beta$) the concatenation of two paths α, β provided that $\alpha^+ = \beta^-$.

We say that a subset A is r*–close* to B if A is contained in the r–neighborhood of B.

Let f, g be real-valued functions. Then $f \prec_{c_i} g$ means that there is a constant $C > 0$ depending on parameters c_i such that $f < Cg$. The symbol $\succ_{c_i}$ is defined similarly, and $\asymp_{c_i}$ means both $\prec_{c_i}$ and $\succ_{c_i}$ are true. The constant c_i will be omitted if it is a universal constant.

2.2. **Contracting subsets.**

Definition 2.1 (Contracting subset)**.** For given $C \geq 1$, a subset U in X is called C*–contracting* if for any geodesic γ with $d(\gamma, U) \geq C$, we have

$$\mathrm{diam}(\pi_U(\gamma)) \leq C.$$

A collection of C–contracting subsets is referred to as a C*–contracting system.*

A subset $U \subseteq \mathrm{X}$ is called *quasi-convex* if for a function $\sigma : \mathbb{R}^+ \to \mathbb{R}^+$: given $(q, Q) \geq (1, 0)$, any (q, Q)–quasi-geodesic with endpoints in U lies in $N_{\sigma((q,Q))}(U)$.

Lemma 2.2. *Let U be a C-contracting subset. Then*

1. *There exists $\sigma = \sigma(C)$ such that U is σ–quasi-convex.*
2. *For any $r > 0$, there exists $\hat{C} = \hat{C}(C, r)$ such that a subset $V \subseteq \mathrm{X}$ having Hausdorff distance at most r to U is $\hat{C}$–contracting.*
3. *For any $\lambda \geq 1, Q \geq 0$, there exists $\hat{C} = \hat{C}(\lambda, Q, C)$ so that any subpath of a (λ, Q)–quasi-geodesic U is $\hat{C}$–contracting.*
4. *There exists $\hat{C} = \hat{C}(C)$ such that $d_U(y, z) \leq d(y, z) + \hat{C}$ for any $y, z \in \mathrm{X}$.*

Proof. The assertion (3) is proved in [19, Prop 2.2.3]. □

In this paper, we are interested in a contracting system $\mathscr{F}$ with τ*–bounded intersection* property for a function $\tau : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ if the following holds

$$\forall U, V \in \mathscr{F} : \quad \mathrm{diam}(N_r(U) \cap N_r(V)) \leq \tau(r)$$

for any $r \geq 0$. This is, in fact, equivalent to a *bounded projection property* of $\mathscr{F}$: there exists a constant $B > 0$ such that the following holds

$$\mathrm{diam}(\pi_U(V)) \leq B$$

for any $U \neq V \in \mathscr{F}$. See [20, Lemma 2.3].

An infinite order element $h \in G$ is called *contracting* if the subgroup $\langle h \rangle$ acts by translation on a contracting quasi-geodesic. The set of contracting elements is preserved under conjugacy.

Lemma 2.3. [18, Lemma 2.11] *Let $h \in G$ be a contracting element. Then it is contained in the following unique maximal elementary subgroup*

$$E(h) = \{g \in G : \exists n \in \mathbb{N}_{>0}, (\ gh^n g^{-1} = h^n) \ \vee \ (gh^n g^{-1} = h^{-n})\}.$$

Keeping in mind the basepoint $o \in \mathrm{X}$, the *axis* of h is defined as the following quasi-geodesic

$$\mathrm{Ax}(h) = \{fo : f \in E(h)\}. \tag{1}$$

Notice that $\mathrm{Ax}(h) = \mathrm{Ax}(k)$ and $E(h) = E(k)$ for any contracting element $k \in E(h)$.

2.3. **Horofunction boundary.** We recall the construction of horofunction boundary, which are endowed with the so-called finite and sublinear difference partitions.

Fix a basepoint $o \in \mathrm{X}$. For each $y \in \mathrm{X}$, we define a Lipschitz map $b_y : \mathrm{X} \to \mathrm{X}$ by

$$\forall x \in \mathrm{X} : \quad b_y(x) = d(x, y) - d(o, y).$$

This family of 1–Lipschitz functions sits in the set of continuous functions on X vanishing at o. Endowed with the compact-open topology, the Arzela-Ascoli Lemma implies that the closure of $\{b_y : y \in \mathrm{X}\}$ gives a compactification of X. The complement, denote by $\partial_h \mathrm{X}$, of X is called the *horofunction boundary*.

A *Buseman cocycle* $B_\xi : \mathrm{X} \times \mathrm{X} \to \mathbb{R}$ (independent of o) is given by

$$\forall x_1, x_2 \in \mathrm{X} : \quad B_\xi(x_1, x_2) = b_\xi(x_1) - b_\xi(x_2).$$

The topological type of horofunction boundary is independent of the choice of basepoints. Every isometry ϕ of X induces a homeomorphism on $\overline{\mathrm{X}}$:

$$\forall y \in \mathrm{X} : \quad \phi(\xi)(y) := b_\xi(\phi^{-1}(y)) - b_\xi(\phi^{-1}(o)).$$

According to the context, both ξ and b_ξ are used to denote the boundary points.

Finite difference relation. Two horofunctions b_ξ, b_η have *K–finite difference* for $K \geq 0$ if the L^∞–norm of their difference is K–bounded:

$$\|b_\xi - b_\eta\|_\infty \leq K.$$

The *locus* of b_ξ consists of horofunctions b_η so that b_ξ, b_η have K–finite difference for some K depending on η. The loci $[b_\xi]$ of horofunctions b_ξ form a *finite difference equivalence relation* $[\cdot]$ on $\partial_h \mathrm{X}$. The *locus* $[\Lambda]$ of a subset $\Lambda \subseteq \partial_h \mathrm{X}$ is the union of loci of all points in Λ. We say that Λ is *saturated* if $[\Lambda] = \Lambda$.

Say that a sequence of subsets X_n is *escaping* if $d(o, X_n) \to \infty$. We summarize the following properties of horofunction boundary in [21, Sec. 5], which are called the convergence property of the horofunction boundary.

Sublinear difference relation. It will be also useful to consider another bigger relation than the finite difference relation. We say two horofunctions $b_\xi, b_\eta : [0,\infty) \to \mathrm{X}$ have *sublinear difference* if

$$\lim_{n\to\infty} \sup_{d(o,x)\ge n} \frac{|b_\xi(x) - b_\eta(x)|}{d(o,x)} = 0. \tag{2}$$

Since two horofunction representatives of a given point $\xi \in \partial \mathrm{X}$ differ by a constant for different basepoints, this relation is independent on the choice of basepoint. The sublinear difference relation is an equivalence relation. We denote by $[\xi]_s$ the equivalent class of $\xi \in \partial \mathrm{X}$, and $[\partial_h \mathrm{X}]_s$ the resulting quotient space of $\partial_h \mathrm{X}$. It is clear that $[\xi] \subseteq [\xi]_s$, so $[\partial_h \mathrm{X}]$ is also quotient of $[\partial_h \mathrm{X}]_s$. In particular, Theorem 2.15 holds for sublinear equivalence relation as well, where $[\cdot]$ could be replaced with $[\cdot]_s$.

Any geodesic ray α tends to a unique point, denoted by α^+, at the horofunction boundary $\partial_h \mathrm{X}$. Namely, the associated horofunction is as follows

$$\alpha^+(x) := \lim_{t\to\infty} [d(\alpha(t), x) - t]$$

Lemma 2.4. *Let $\alpha, \beta : [0,\infty) \to \mathrm{X}$ be two geodesic ways issuing from the same base-point $o \in \mathrm{X}$, ending at $\alpha^+, \beta^+ \in \partial_h \mathrm{X}$ respectively. If there are two sequences of real numbers $s_n, r_n \to \infty$ satisfying*

$$\limsup_{n\to\infty} r_n/s_n < 1 \tag{3}$$

$$\forall n \gg 0,\ \limsup_{t\to\infty} d(o, [\alpha(s_n), \beta(t)]) \le r_n \tag{4}$$

then $[\alpha^+]_s \ne [\beta^+]_s$.

Proof. Let $x_n = \alpha(s_n) \in \alpha$ so we have $\alpha^+(x_n) = -s_n$. We are going to prove that x_n violates (2). For small enough $\epsilon > 0$, according to (4), there are $t_m \to \infty$ so that

$$d(o, [\alpha(s_n), \beta(t_m)]) \le r_n + \epsilon$$

so by triangle inequality,

$$|d(\alpha(s_n), \beta(t_m)) - s_n - t_m| \le 2(r_n + \epsilon)$$

Thus, letting $s_n \to \infty$ and $\epsilon \to 0$,

$$\begin{aligned}\limsup_{s_n\to\infty} \frac{|\beta^+(x_n) - \alpha^+(x_n)|}{s_n} &= \limsup_{s_n\to\infty} \frac{|\beta^+(x_n) + s_n|}{s_n} \\ &= \frac{|\lim_{t_m\to\infty}[d(\beta(t_m), x_n) - t_m - s_n] + 2s_n|}{s_n} > 0\end{aligned}$$

where the last inequality uses (3). Hence, $[\alpha^+]_s \ne [\beta^+]_s$ follows by definition. □

Lemma 2.5. *Let α, β be two geodesic rays ending at $[\alpha^+] \ne [\beta^+]$ respectively. Assume that α contains infinitely many disjoint C–contracting segments of length L in its r–neighborhood. If $L \gg 0$, then $[\alpha^+]_s \ne [\beta^+]_s$.*

Proof. Pick up two unbounded sequences of $x_n \in \alpha$ and $y_n \in \beta$. By assumption, α contains infinitely many C–contracting segments p_m of length L. By the C–contracting property of p_m, if $[x_n, y_n] \cap N_C(p_m) = \varnothing$ for infinitely many p_m, then $\beta \cap N_C(p_m) \ne \varnothing$. As p_m is escaping, this implies the intersection of α and $N_{r+C}(\beta)$ is unbounded. By definition of horofunctions, $[\alpha^+] = [\beta^+]$ are in the same finite difference class, contradicting the assumption. Consequently, there must exist some

p_m so that $[x_n, y_n] \cap N_C(p_m) \neq \varnothing$. Again the C–contracting property of p_m shows $[x_n, y_n]$ intersects $N_C(p_m)$ as $n \to \infty$. Setting $D = d(o, p_m) + r + C$, we obtain $d(o, [x_n, y_n]) \leq D$, where D is independent of n. It thus follows from Lemma 2.4 that $[\xi]_s \neq [\eta]_s$. The proof is complete. □

For hyperbolic spaces, the sublinear and finite difference relation turns out to be the same on $\partial_h \mathrm{X}$.

Corollary 2.6. *If* X *is a Gromov hyperbolic geodesic space, then* $[\partial_h \mathrm{X}]_s$ *is homeomorphic to the Gromov boundary* $\partial \mathrm{X}$.

Proof. Any geodesic ray in a hyperbolic space is uniformly contracting. The conclusions follows immediately from Lemma 2.5. □

2.4. **Sublinearly Morse boundaries.** Now we introduce a large class of quasi-geodesic rays that are quasi-isometry invariant. Intuitively, these quasi-geodesics have a weak Morse-like property. To begin with, we fix a function that is sublinear in the following sense:

Sublinear functions. We fix a function

$$\kappa \colon [0, \infty) \to [1, \infty)$$

that is monotone increasing, concave and sublinear, that is

$$\lim_{t \to \infty} \frac{\kappa(t)}{t} = 0.$$

Note that using concavity, for any $a > 1$, we have

$$\kappa(at) \leq a \left(\frac{1}{a}\, \kappa(at) + \left(1 - \frac{1}{a}\right) \kappa(0) \right) \leq a\, \kappa(t). \tag{5}$$

We say a quantity D *is small compared to a radius* $\mathsf{r} > 0$ if

$$\mathsf{D} \leq \frac{\mathsf{r}}{2\kappa(\mathsf{r})}. \tag{6}$$

Remark. The assumption that κ is increasing and concave makes certain arguments cleaner, otherwise they are not really needed. One can always replace any sublinear function κ, with another sublinear function $\overline{\kappa}$ so that $\kappa(t) \leq \overline{\kappa}(t) \leq \mathsf{C}\, \kappa(t)$ for some constant C and $\overline{\kappa}$ is monotone increasing and concave. For example, define

$$\overline{\kappa}(t) = \sup \left\{ \lambda \kappa(u) + (1 - \lambda) \kappa(v) \;\middle|\; 0 \leq \lambda \leq 1,\ u, v > 0,\ \text{and } \lambda u + (1 - \lambda) v = t \right\}.$$

The requirement $\kappa(t) \geq 1$ is there to remove additive errors in the definition of κ–contracting geodesics.

Definition 2.7 (κ–neighborhood)**.** Recall that, for $x \in X$, we have $\|x\| = d(o, x)$. To simplify notation, we often drop $\|\cdot\|$. That is, for $x \in X$, we define

$$\kappa(x) := \kappa(\|x\|).$$

For a closed set Z and a constant n, define the (κ, n)–neighbourhood of Z to be

$$\mathcal{N}_\kappa(Z, \mathsf{n}) = \left\{ x \in \mathrm{X} \;\middle|\; d(x, Z) \leq \mathsf{n} \cdot \kappa(x) \right\}.$$

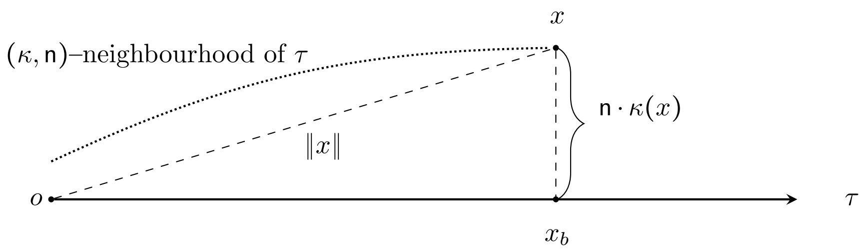

FIGURE 1. A κ–neighbourhood of a geodesic ray τ with multiplicative constant n.

In this paper, Z is either a geodesic or a quasi-geodesic. That is, we can write $\mathcal{N}_\kappa(\tau, \mathsf{n})$ to mean the (κ, n)–neighborhood of the image of the geodesic ray τ. Or, we can use phrases like "the quasi-geodesic β is κ–contracting" or "the geodesic τ is in a (κ, n)–neighbourhood of the geodesic c".

We recall the definition of κ–contracting and κ–Morse sets from [14].

Definition 2.8 (κ–Morse I)**.** We say a closed subset Z of X is κ*–Morse* if there is a function

$$\mathsf{m}_Z \colon \mathbb{R}_+^2 \to \mathbb{R}_+$$

so that if $\beta\colon [s,t] \to \mathrm{X}$ is a (q,Q)–quasi-geodesic with end points on Z then

$$\beta[s,t] \subseteq \mathcal{N}_\kappa\big(Z, \mathsf{m}_Z(q,Q)\big).$$

We refer to m_Z as a *Morse gauge* for Z. Without loss of generality one can assume

$$\mathsf{m}_Z(q,Q) \geq \max(q,Q). \tag{7}$$

Definition 2.9 (κ–Morse II)**.** We say a closed subset Z of X is κ*–Morse II* if there is a function $\mathsf{m}_Z \colon \mathbb{R}^2 \to \mathbb{R}$ such that, for every constants $\mathsf{r} > 0$, $\mathsf{n} > 0$ and every sublinear function κ', there is an $\mathsf{R} = \mathsf{R}(Z, \mathsf{r}, \mathsf{n}, \kappa') > 0$ where the following holds: Let $\eta\colon [0,\infty) \to \mathrm{X}$ be a (q,Q)–quasi-geodesic ray so that $\mathsf{m}_Z(q,Q)$ is small compared to r, let t_r be the first time $\|\eta(t_\mathsf{r})\| = \mathsf{r}$ and let t_R be the first time $\|\eta(t_\mathsf{R})\| = \mathsf{R}$. Then

$$d\big(\eta(t_\mathsf{R}), Z\big) \leq \mathsf{n} \cdot \kappa'(\mathsf{R}) \quad \Longrightarrow \quad \eta[0, t_\mathsf{r}] \subseteq \mathcal{N}_\kappa\big(Z, \mathsf{m}_Z(q,Q)\big).$$

It is also natural to generalize the notion of contracting to the sublinear setting:

Definition 2.10 (κ–contracting)**.** For a closed subspace Z of X, we say Z is κ*–contracting* if there is a constant c_Z so that, for every $x, y \in \mathrm{X}$

$$d(x,y) \leq d(x,Z) \quad \Longrightarrow \quad \mathbf{d}_Z\big(x,y\big) \leq \mathsf{c}_Z \cdot \kappa(x).$$

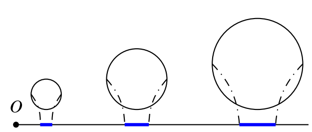

FIGURE 2. A sublinearly contracting geodesic ray

The following theorem summarizes the relation of the above three notions.

Theorem 2.11 ([14])**.** *Let* X *be a proper geodesic metric space and let τ be a quasi-geodesic ray in* X*. Then*

1. *τ is κ–Morse I if and only if τ is κ–Morse II. Thus we say a set is* kappa-*Morse if it is either κ–Morse I or κ–Morse II.*
2. *If τ is κ–contracting then τ is κ–Morse.*
3. *If τ is κ-Morse then it is κ'-contracting.*

Analogous to Lemma 2.3 (2), κ–Morse is a property than can also be established by approximity. That is to say, if a quasi-geodesic ray α sublinearly fellow travel a κ–Morse ray β, then α is also κ–Morse. We now make this precise. First assume without loss of generality that a quasi-geodesic ray is a continuous path. Define

$$\alpha_r := \{\alpha(t_0) \,|\, \alpha(t_0) \in (\alpha \cap X \backslash B(o,r)), \text{ and for any other } \alpha(t) \in \alpha \cap X \backslash B(o,r), \text{ we have } t_0 \le t\}$$

We say two quasi-geodesic rays *sublinearly fellow travel* if

$$\lim_{r\to\infty} \frac{d(\alpha_r, \beta_r)}{r} = 0.$$

Theorem 2.12. [14] *Let α be a (q_1, Q_2)–quasi-geodesic ray and let β be a (q_2, Q_2)–quasi-geodesic ray that is κ–Morse. If α and β sublinearly fellow travel, then there exists a κ–neighborhood depending only on (q_1, Q_1) and (q_2, Q_2) such that*

$$\alpha \in N_\kappa\big(\beta, m((q_1, Q_1), (q_2, Q_2))\big).$$

Furthermore, α is a κ–Morse ray.

Lastly, a quasi-geodesic is called *sublinearly Morse* if it is κ–Morse for some sublinearly growing function κ.

Definition 2.13 (Sublinearly Morse boundary)**.** Given a sublinear function κ, let $\partial_\kappa X$ denote the set of equivalence classes of κ–Morse quasi-geodesics. Equipped with a coarse version of cone topology, we call this set the *κ–Morse boundary* of X and denote it ∂_κX (for more details, see [14]).

It is shown in [14] that X $\cup\, \partial_\kappa$X with the coarse cone topology is a QI-invariant space and a metrizable topological space.

2.5. **Convergence boundary.** In this subsection, we discuss an axiomatic approach introduced in [21] to the boundary of proper geodesic metric spaces in presence of contracting subsets. The motivating examples are Gromov boundary of hyperbolic spaces and visual boundary of CAT(0) spaces. Before describing more examples in 2.5, we need to introduce a few terminologies.

Let (X, d) be a proper metric space admitting an isometric action of a non-elementary countable group Γ with a contracting element. Consider a metrizable compactification $\overline{\mathrm{X}} := \partial\mathrm{X} \cup \mathrm{X}$, so that X is open and dense in $\overline{\mathrm{X}}$. We also assume that the action of Isom(X) extends by homeomorphism to ∂X.

We equip ∂X with a Isom(X)–invariant partition $[\cdot]$: $[\xi] = [\eta]$ implies $[g\xi] = [g\eta]$ for any $g \in$ Isom(X). The *locus* $[Z]$ of a subset $Z \subseteq \partial$X is the union of all $[\cdot]$–classes of $\xi \in Z$. We say that ξ is *minimal* if $[\xi] = \{\xi\}$, and a subset U is *$[\cdot]$–saturated* if $U = [U]$.

We say that $[\cdot]$ restricts to be a *closed* partition on a $[\cdot]$–saturated subset $U \subseteq \partial$X if $x_n \in U \to \xi \in \partial$X and $y_n \in U \to \eta \in \partial$X are two sequences with $[x_n] = [y_n]$, then

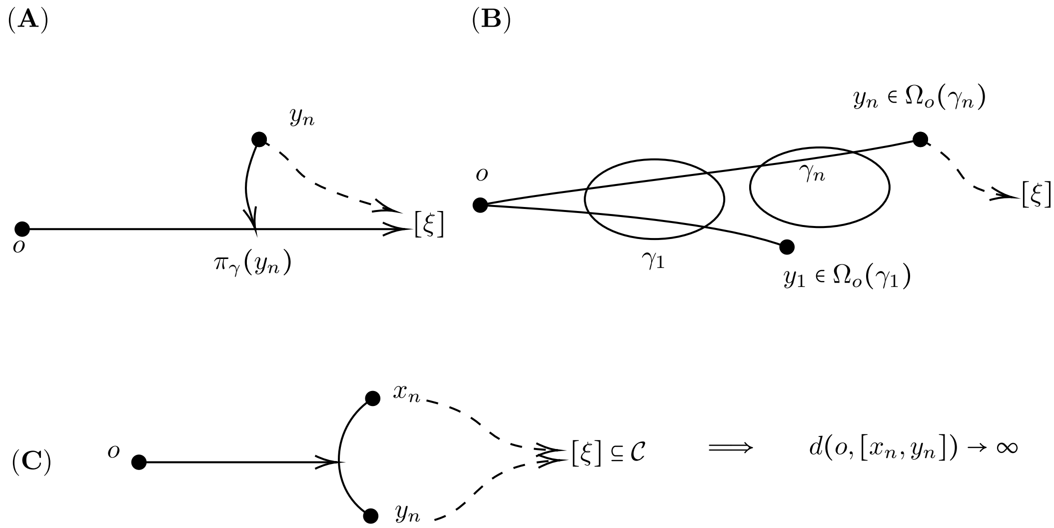

FIGURE 3. Illustrate Assumptions (A)(B)(C) in Definition 2.14

$[\xi] = [\eta]$. (Possibly ξ, η are not in U anymore.) If $U = \partial \mathrm{X}$, this is equivalent to say that the relation $\{(\xi, \eta) : [\xi] = [\eta]\}$ is a closed subset in $\partial \mathrm{X} \times \partial \mathrm{X}$, so the quotient space $[\partial \mathrm{X}]$ is Hausdorff. In general, $[\cdot]$ may be not closed over the whole $\partial \mathrm{X}$ (say the horofunction boundary with finite difference relation), but is closed on certain interesting subsets as requested in Assumption (C) below.

With respect to the given partition, we say that a sequence of points x_n *tend* (resp. *accumulates*) to $[\xi]$ if the limit point (resp. any accumulate point) is contained in $[\xi]$. This implies that $[x_n]$ tends or accumulates to $[\xi]$ in the quotient space $[\partial \mathrm{X}]$. So, an infinite ray γ *terminates* at a point in $[\xi] \in \partial \mathrm{X}$ if any sequence of points on γ accumulates in $[\xi]$.

Recall that $\Omega_o(A) = \{x \in \mathrm{X} : [o, x] \cap A \neq \varnothing\}$ is the cone of a subset $A \subseteq \mathrm{X}$ with light source at o. A sequence of subsets A_n is called *escaping* if $d(o, A_n) \to \infty$ for some (or any) $o \in \mathrm{X}$.

Definition 2.14. We say that $(\overline{\mathrm{X}}, [\cdot])$ is a *convergence compactification* if the following assumptions hold.

- **(A)** Any contracting geodesic ray γ accumulates into a closed subset $[\xi]$ for some $\xi \in \partial \mathrm{X}$ such that any sequence of $y_n \in \mathrm{X}$ with escaping projections $\pi_\gamma(y_n)$ tends to $[\xi]$.
- **(B)** Let $\{\gamma_n \subseteq \mathrm{X} : n \geq 1\}$ be an escaping sequence of C–contracting quasi-geodesics for some $C > 0$. Then for any given $o \in \mathrm{X}$, there exist a subsequence of $\{A_n := \Omega_o(\gamma_n) : n \geq 1\}$ (still denoted by A_n) and $\xi \in \partial \mathrm{X}$ such that A_n accumulates into $[\xi]$:

  Any convergent sequence of points $x_n \in A_n$ tends to a point in $[\xi]$.
- **(C)** The set $\mathcal{C}$ of *non-pinched* points $\xi \in \partial \mathrm{X}$ is non-empty. If $x_n, y_n \in \mathrm{X}$ are two sequence of points converging to $[\xi]$, then $[x_n, y_n]$ is an escaping sequence of geodesic segments.

Examples. The first three convergence boundaries are equipped with a *maximal* partition $[\cdot]$ (i.e. each $[\cdot]$-class is singleton). See [21] for more details.

(1) Hyperbolic space X with Gromov boundary ∂X, where all boundary points are non-pinched.
(2) CAT(0) space X with visual boundary ∂X (homeomorphic to horofunction boundary), where all boundary points are non-pinched.
(3) The Cayley graph X of a relatively hyperbolic group with Bowditch or Floyd boundary ∂X, where conical limit points are non-pinched.
    If X is infinitely ended, we could also take ∂X as the end boundary with the same statement.
(4) Teichmüller space X with Thurston boundary ∂X, where $[\cdot]$ is given by Kaimanovich-Masur partition [9] and uniquely ergodic points are non-pinched.
(5) Any proper metric space X with horofunction boundary ∂X, where $[\cdot]$ is given by finite difference partition and all boundary points are non-pinched. If X is the cubical CAT(0) space, the horofunction boundary is exactly the Roller boundary. If X is the Teichmüller space with Teichmüller metric, the horofunction boundary is the Gardiner-Masur boundary ([10, 17]).

From these examples, we see that the convergence boundary is not unique for a fixed proper metric space. In applications, the horofunction boundary provides a non-trivial convergence boundary for any proper action.

Theorem 2.15. [21, Theorem 1.1] *The horofunction boundary is a convergence boundary with finite difference relation* $[\cdot]$*, where all boundary points are non-pinched.*

As finite difference partition is finer than the sublinear difference partition, we see by definition that the above conclusion holds for sublinear difference partition as well.

2.6. **Regularly contracting geodesic rays.** The notion of the regularly contracting segment introduced in [7] shall be recalled in the subsection. This is a metric notion without involving any group action.

For any $\theta \in [0,1]$, if γ is a geodesic, a *θ–segment* means a connected and open subsegment of γ with length $\theta\ell(\gamma)$.

Definition 2.16. Fix $r, C > 0$.

(1) Given $L > 0, \theta \in (0,1]$, we say that a geodesic γ is *(r,C,L)–contracting at θ–frequency* if every θ–segment of γ contains a subsegment of length L that is r–close to a C–contracting geodesic. That is to say, for any $0 < t < (1-\theta)\ell(\gamma)$ there is an interval of time $[s-L, s+L] \subset [t, t+\theta\ell(\gamma)]$ and a C–contracting geodesic p such that,

$$u \in [s-L, s+L] \quad \Longrightarrow \quad d(\gamma(u), p) \le r. \tag{8}$$

(2) A geodesic ray $\gamma\colon [0,\infty) \to X$ is *(r,C,L)–contracting at θ–frequency* if there is an $R_0 > 0$ such that any initial segment of γ of length at least R_0 (i.e. the segment $\gamma[0,t]$ for any $t \ge R_0$) is (r,C,L)–contracting at θ–frequency.
(3) A geodesic ray $\gamma\colon [0,\infty) \to X$ is *(r,C)–regularly contracting* if τ is (r,C,L)–contracting at θ–frequency for each $L > 0$ and $\theta \in (0,1]$.
    A bi-infinite geodesic γ is *regularly contracting* if the rays $t \to \gamma(t)$ and $t \to \gamma(-t)$ are both regularly contracting.

Let us note the following connection with convergence boundary.

Lemma 2.17. *Let* $(\mathrm{X}, \partial \mathrm{X})$ *be a convergence compactification. Then a* (r, C, L)*–contracting geodesic ray* γ *at* θ*–frequency accumulates into a unique* $[\cdot]$*–class denoted by* $[\gamma^+]$ *in* $[\partial \mathrm{X}]$.

Proof. Let γ be an (r, C, L)–contracting geodesic ray at θ–frequency, where p_n is a sequence of the C–contracting geodesic segments satisfying (8). Up to taking subsegment of length L, we may assume that p_n is escaping.

Apply Definition 2.14(B) to the sequence of $(C+1)$-contracting quasi-geodesics $N_r(p_n)$. We thus obtain a unique $[\cdot]$–class in $\partial \mathrm{X}$, denoted by $[\gamma^+]$, and that $\Omega_o(N_r(p_n))$ accumulates to $[\gamma^+]$. In particular, any sequence of points on γ are eventually contained in $\Omega_o(N_r(p_n))$, so tend to $[\gamma^+]$. The proof is complete. □

As fore-mentioned, regularly contracting rays are the set of κ–contracting rays.

Theorem 2.18. [7] *If* τ *is regularly contracting, then it is* κ*–contracting for some sublinear function* κ*. In particular, it is also* κ*–Morse.*

Let us equip the space X of interest with a convergence boundary $\partial \mathrm{X}$. A universal choice is to consider the horofunction boundary endowed with finite or sublinear difference partition. However, the discussion in what follows equally works for any convergence boundary, e.g. in Examples 2.5.

By Lemma 2.17, let $\mathcal{FC}(\theta, r, C, L)$ be the set of all $[\cdot]_s$–classes $[\xi] \in [\partial \mathrm{X}]$ so that some geodesic ray γ in X starting at o and ending at $[\xi]$ is (r, C, L)–contracting at θ–frequency.

Accordingly, let $\mathcal{RC}(r, C)$ be the set of all $[\cdot]$–classes $[\xi] \in [\partial \mathrm{X}]$ so that some geodesic ray γ in X starting at o and ending at $[\xi]_s$ is (r, C)–regular contracting.

Corollary 2.19. *For any* $r, C > 0$*, the following holds*

$$\mathcal{RC}(r, C) = \bigcap_{L \in \mathbb{N}} \left[\bigcap_{\theta \in (0,1] \cap \mathbb{Q}} \mathcal{FC}(\theta, r, C, L) \right].$$

where $\mathbb{Q}$ *denotes the set of rational numbers.*

2.7. **Statistically convex-cocompact actions.** In this subsection, we first recall a class of statistically convex-cocompact actions introduced in [18].

Consider the ball of radius n centered at $o \in \mathrm{X}$:

$$N(o, n) = \{v \in Go : d(o, v) \le n\} \tag{9}$$

Fix $\Delta \ge 1$. Consider the annulus of radius n centered at $o \in \mathrm{X}$:

$$A(o, n, \Delta) = \{v \in Go : |d(o, v) - n| \le \Delta\} \tag{10}$$

Let Γ be a subset of G. The *critical exponent* of Γ is defined as

$$\omega_\Gamma = \limsup_{n \to \infty} \frac{\log \sharp (N(o, n) \cap \Gamma o)}{n}.$$

Given constants $0 \le M_1 \le M_2$, let $\mathcal{O}_{M_1, M_2}$ be the set of element $g \in G$ such that there exists some geodesic γ between $N_{M_2}(o)$ and $N_{M_2}(go)$ with the property that the interior of γ lies outside $N_{M_1}(\Gamma o)$.

Definition 2.20 (SCC action)**.** If there exist positive constants $M_1, M_2 > 0$ such that $\omega_{\mathcal{O}_{M_1, M_2}} < \omega_G < \infty$, then the proper action of G on X is called *statistically convex-cocompact* (SCC).

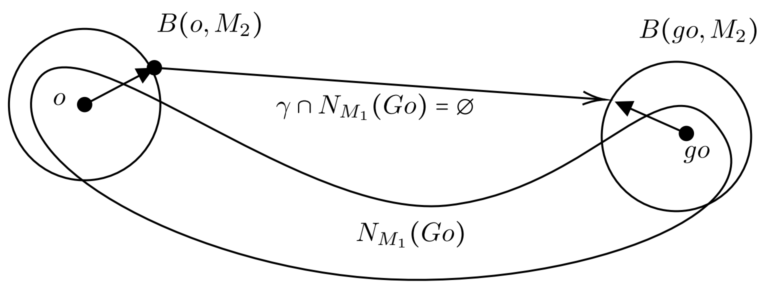

FIGURE 4. Illustration of $\mathcal{O}_{M_1,M_2}$.

Remark. The idea to define the set $\mathcal{O}_{M_1,M_2}$ is to look at the action of the fundamental group of a finite volume negatively curved Hadamard manifold on its universal cover. It is then easy to see that for appropriate constants $M_1, M_2 > 0$, the set $\mathcal{O}_{M_1,M_2}$ coincides with the union of the orbits of cusp subgroups up to a finite Hausdorff distance. The assumption in SCC actions was called the *parabolic gap condition* by Dal'bo, Otal and Peigné in [6].

When the SCC action contains a contracting element, the definition is independent of the basepoint (see [18, Lemma 6.2]).

A key notion in studying growth gaps is that of a barrier-free element introduced in [18].

Definition 2.21. Fix constants $r, M > 0$.

(1) Given $r > 0$ and $f \in G$, we say that a geodesic γ contains an (r, f)*–barrier* if there exists an element $t \in G$ so that

$$\max\{d(t \cdot o, \gamma), d(t \cdot fo, \gamma)\} \le r. \tag{11}$$

If no such $t \in G$ exists so that (11) holds, then γ is called (r, f)*–barrier-free.*

(2) An element $g \in G$ is (r, M, f)*–barrier-free* if there exists an (r, f)–barrier-free geodesic between $N_M(o)$ and $N_M(go)$.

We have chosen two parameters M_1, M_2 so that the definition of a statistically convex-cocompact action is flexible and easy to verify. It is enough to take $M_1 = M_2 = M$ in our use. Henceforth, we set $\mathcal{O}_M \coloneqq \mathcal{O}_{M,M}$ for ease of notation.

Given $r, M > 0$ and any $f \in G$, let $\mathcal{V}_{r,M,f}$ be the collection of all (r, M, f)–barrier-free elements in G. The following results will be crucially used in next sections.

Proposition 2.22. [18, Theorems B & C] *If G admits a SCC action on a proper geodesic space* (X, d) *with a contracting element, then*

(1) G has purely exponential growth.

(2) Let M_0 be the constant in the definition of SCC action, then for any $M > M_0$, there exists $r = r(M) > 0$ such that $\mathcal{V}_{r,M,f}$ is exponentially negligible for any $f \in G$.

3. CONFORMAL DENSITY ON CONVERGENCE BOUNDARY

Consider a proper action $G \curvearrowright \mathrm{X}$ with a contracting element, endowed with a convergence compactification $\overline{\mathrm{X}} = \mathrm{X} \cup \partial\mathrm{X}$. This section recalls the general setup of quasi-conformal density supported on $\partial\mathrm{X}$, and explains Patterson's famous construction of such density from the proper action $G \curvearrowright \mathrm{X}$.

3.1. **Patterson-Sullivan measures on convergence boundary.** Let $\mathcal{M}^+(\overline{\mathrm{X}})$ be the set of finite positive Borel measures on $\overline{\mathrm{X}} \coloneqq \partial\mathrm{X} \cup \mathrm{X}$, on which G acts by push-forward:

$$g_\star \mu(A) = \mu(g^{-1}A)$$

for any Borel set A. Let $\mathcal{C}$ be the (non-empty) set of non-pinched points in $\partial\mathrm{X}$ in Definition 2.14(C).

Definition 3.1. Let $\omega \in [0, \infty[$. The following map

$$\begin{aligned} \mu : \mathrm{X} &\longrightarrow \mathcal{M}^+(\overline{\mathrm{X}}) \\ x &\longmapsto \mu_x \end{aligned}$$

is called a *ω–dimensional G–quasi-equivariant quasi-conformal density* if for any $g, h \in G$ and any $x \in \mathrm{X}$, we have

$$\mu_y - \text{ a.e. } \xi \in \partial\mathrm{X}: \quad \frac{dg_\star\mu_x}{d\mu_x}(\xi) \in [\frac{1}{\lambda}, \lambda], \tag{12}$$

$$\mu_y - \text{ a.e. } \xi \in \mathcal{C}: \quad \frac{1}{\lambda} e^{-\omega B_\xi(x,y)} \le \frac{d\mu_x}{d\mu_y}(\xi) \le \lambda e^{-\omega B_\xi(x,y)} \tag{13}$$

for a universal constant $\lambda \ge 1$. We normalize μ_o to be a probability measure: its mass $\|\mu_o\| = \mu_o(\overline{\mathrm{X}}) = 1$.

We say that $\{\mu_x : x \in \mathrm{X}\}$ is *non-trivial* if μ_x is supported on $\mathcal{C}$.

If $\lambda = 1$ for (12), the map $\mu : \mathrm{X} \to \mathcal{M}^+(\overline{\mathrm{X}})$ is G–equivariant, that is, $\mu_{gx} = g_\star\mu_x$; equivalently, $\mu_{gx}(gA) = \mu_x(A)$. If both $\lambda = 1$, we call μ a conformal density.

Patterson-Sullivan measures. Choose a basepoint $o \in \mathrm{X}$. The Poincaré series for the action of $G \curvearrowright \mathrm{X}$

$$\mathcal{P}_G(s, x, y) = \sum_{g \in G} e^{-sd(x, gy)}, \ s \ge 0$$

diverges at $0 \le s < \omega_G$ and converges at $s > \omega_G$. The action of G on X is called *divergent* if $\mathcal{P}_G(s, x, y)$ diverges at the critical exponent ω_G. Otherwise, G is called *convergent*. Recall that $[\Lambda G o]$ is the limit set for the action $G \curvearrowright \mathrm{X}$, i.e. the set of accumulation points of a G-orbit, up to taking $[\cdot]$-closure.

We start to construct a family of measures $\{\mu_x^{s,y}\}_{x \in \mathrm{X}}$ supported on Gy for any given $s > \omega_G$ and $y \in \mathrm{X}$. Assume that $\mathcal{P}_G(s, x, y)$ is divergent at $s = \omega_G$. Set

$$\mu_x^{s,y} = \frac{1}{\mathcal{P}_G(s, o, y)} \sum_{g \in G} e^{-sd(x, gy)} \cdot \mathrm{Dirac}(gy), \tag{14}$$

where $s > \omega_G$ and $x \in \mathrm{X}$. Note that $\mu_o^{s,y}$ is a probability measure supported on Gy. If $\mathcal{P}_G(s, x, y)$ is convergent at $s = \omega_G$, the Poincaré series in (14) needs to be replaced by a modified series as in [11].

Fix $y \in \mathrm{X}$. Choose $s_i \to \omega_G$ such that $\mu_x^{s_i,y}$ are convergent in $\mathcal{M}^+(\Lambda G o)$. The *Patterson-Sullivan measures* $\mu_x^y = \lim \mu_v^{s_i,y}$ are the limit measures. Note that $\mu_o(\Lambda G o) = 1$. In what follows, we usually write $\mu_x = \mu_x^o$ for $x \in \mathrm{X}$.

Theorem 3.2. *Suppose that G acts properly on a proper geodesic space* X *compactified with horofunction boundary $\partial_h\mathrm{X}$. Then the family $\{\mu_x : x \in \mathrm{X}\}$ of Patterson-Sullivan measures is a ω_G-dimensional G-equivariant conformal density supported on $[\Lambda G o]$.*

In the sequel, we write PS-measures as shorthand for Patterson-Sullivan measures.

3.2. **Shadow Lemma.** In what follows, we make the standing assumption:

Convention 3.3. *Let F be a set of three (mutually) independent contracting elements f_i ($i = 1, 2, 3$), which form a contracting system*

$$\mathscr{F} = \{g \cdot \mathrm{Ax}(f_i) : g \in G\} \tag{15}$$

where the axis $\mathrm{Ax}(f_i)$ defined in (1) depending on the choice of a basepoint $o \in \mathrm{X}$ is C–contracting for some $C > 0$.

We may often assume $d(o, fo)$ is large as possible, by taking sufficiently high power of $f \in F$. The contracting constant C is not effected.

Let $r > 0$ and $x, y \in \mathrm{X}$. First of all, define the usual cone and shadow:

$$\Omega_x(y, r) \quad := \quad \{z \in \mathrm{X} : \exists [x, z] \cap B(y, r) \neq \varnothing\}$$

and $\Pi_x(y, r) \subseteq \partial\mathrm{X}$ be the topological closure in $\partial\mathrm{X}$ of $\Omega_x(y, r)$.

The partial shadows $\Pi_o^F(go, r)$ and cones $\Omega_o^F(go, r)$ given in Definition 3.4 depends on the choice of a contracting system $\mathscr{F}$ as in (15). Without index F, $\Pi_o(go, r)$ denotes the usual shadow.

Definition 3.4 (Partial cone and shadow)**.** For $x \in \mathrm{X}, y \in Go$, the (r, F)*–cone* $\Omega_x^F(y, r)$ is the set of elements $z \in \mathrm{X}$ such that y is an (r, F)–barrier for some geodesic $[x, z]$.

The (r, F)*–shadow* $\Pi_x^F(y, r) \subseteq \partial\mathrm{X}$ is the topological closure in $\partial\mathrm{X}$ of the cone $\Omega_x^F(y, r)$.

The key fact in the theory of conformal density is the Sullivan's shadow lemma.

Lemma 3.5. [21, Lemma 6.3] *Let $\{\mu_x\}_{x \in \mathrm{X}}$ be a* nontrivial *ω–dimensional G–equivariant conformal density for some $\omega > 0$ (i.e. supported on the set $\mathcal{C} \subseteq \partial\mathrm{X}$ of non-pinched points). Then there exist $r_0, L_0 > 0$ with the following property.*

Assume that $d(o, fo) > L_0$ for each $f \in F$. For given $r \geq r_0$, there exist $C_0 = C_0(F), C_1 = C_1(F, r)$ such that

$$C_0 \mathrm{e}^{-\omega \cdot d(o, go)} \leq \mu_o(\Pi_o^F(go, r)) \leq \mu_o(\Pi_o(go, r)) \leq C_1 \mathrm{e}^{-\omega \cdot d(o, go)}$$

for any $go \in Go$.

As a corollary, we obtain a lower bound on the dimension of a conformal density.

Lemma 3.6. [21, Prop. 6.6] *Under the assumption of Lemma 3.5, we have $\omega \geq \omega_G$. Moreover, ω_G can be realized as stated in Theorem 3.2.*

3.3. **Conical points.** We give the definition of a conical point relative to the above C–contracting system $\mathscr{F}$ in (3.3).

Definition 3.7. A point $\xi \in \partial\mathrm{X}$ is called (r, F)*–conical* if for some $x \in Go$, the point ξ lies in infinitely many (r, F)–shadows $\Pi_x^F(y_n, r)$ for $y_n \in Go$. We denote by $\Lambda_r^F(Go)$ the set of (r, F)–conical points.

The following useful property [21, Lemma 4.4] resembles the usual definition of conical points in Kleinian groups.

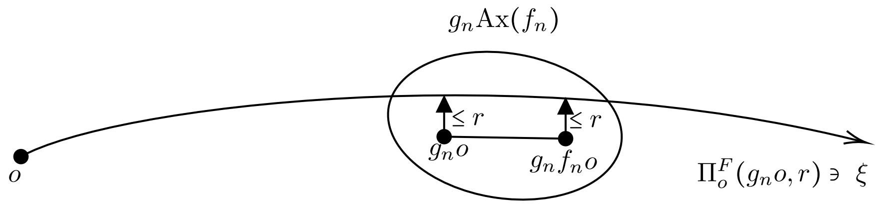

FIGURE 5. Conical points

Lemma 3.8. *Let $\xi \in \Lambda_r^F(Go)$. Then there exists $\hat{r} > 0$ with the following property. For any basepoint $o \in \mathrm{X}$ there exists a geodesic ray γ starting at o ending at $[\xi]$ with infinitely many $(\hat{r}, F)$-barriers. That is,*

- *there exist $g_n o \in G$ and $f_n \in F$ so that $d(g_n o, \gamma), d(g_n f_n o, \gamma) \le \hat{r}$.*

Combined with Lemma 2.4, we obtain the following corollary.

Lemma 3.9. *The sublinear difference relation restricted on the set of conical points coincides with finite difference relation.*

At last, we recall the following fact saying that conical points are generic for divergence type action. It is a part of the Hopf-Tsuji-Sullivan dichotomy [21, Theorem 1.10], where the converse (easier direction) is also true.

Lemma 3.10. *If $G \curvearrowright \mathrm{X}$ is of divergence type, then μ_o charges full measure on $[\Lambda_r^F(Go)]$ and every $[\xi]$–class is μ_o–null.*

4. Genericity of sublinearly Morse directions

This section is devoted to the proof of Theorem 4.1, recalled as follows.

Theorem 4.1. *Let $G \curvearrowright \mathrm{X}$ be a non-elementary SCC action with a contracting element, equipped with a convergence boundary $\partial\mathrm{X}$. Let $\{\mu_x\}_{x\in\mathrm{X}}$ be a ω–dimensional G–quasi-equivariant, quasi-conformal density for some $\omega > 0$, supported on the subset of non-pinched points $\mathcal{C} \subseteq \partial\mathrm{X}$. Then the set of regularly contracting rays has μ_o–full measure.*

4.1. Growth tightness of elements without frequent barriers.

Let γ be a geodesic segment. For any $\theta \in [0,1]$, a *θ–segment* of γ means a connected and open subsegment of γ with length $\theta\ell(\gamma)$. Recall that if γ has (r, f)–barriers, by the definition 2.21, there exists $t \in G$ so that

$$to, tfo \in N_r(\gamma).$$

Recall from Convention 3.3 that $F \subseteq G$ is a set of three independent contracting elements, so that $C > 0$ is the common contracting constant for each $f \in F$. That is, the axes $\mathrm{Ax}(f) = E(f)\cdot o$ is a C–contracting quasi-geodesic. For simplicity, we also assume that any geodesic segment with two endpoints in $\mathrm{Ax}(f)$ is C–contracting (by Lemma 2.2(2)(3)).

Definition 4.2. Fix $\theta \in (0,1]$, $r > 0$ and $f \in G$. We say that a geodesic γ contains *(r, f)–barriers at θ–frequency* if for every θ–segment of γ has (r, f)–barriers.

An element $g \in G$ has *(r, f)–barriers at θ–frequency* if there exists a geodesic γ between $B(o, M)$ and $B(go, M)$ such that γ has (r, f)–barriers at θ–frequency.

Fix any $\theta \in (0,1]$. Form now on,

Convention 4.3. *we fix a constant $r \geq C$ satisfy Proposition 2.22 and Lemma 3.5.*

We need the following two elementary lemmas.

Lemma 4.4. *Fix $M > 0$. There exist $\hat{r} = \hat{r}(C,r)$ and $L_1 = L_1(C,M) > 0$ with the following property. Choose $h \in E(f)$ so that $d(o,ho) > L_1$. Let α, β be two geodesics from $B(o,M)$ to $B(go,M)$. If α contains a θ–segment with (r,h)–barriers, then β contains some θ–segment with $(\hat{r},h)$–barriers.*

Proof. Let p be a θ–segment of α with an (r,h)–barrier. That is, $d(to,p), d(tho,p) \leq r$ for some $t \in G$. Let u,v be the entry and exit points of $\alpha \cap N_r(t\mathrm{Ax}(f))$, so $[u,v]$ contains this (r,f)–barrier to and

$$d(u,v) \geq d(o,fo) - 2r \geq L_1 - 2r.$$

As the shortest projection map to $t\mathrm{Ax}(f)$ is coarsely 1–Lipschitz by Lemma 2.2(4), $B(o,M)$ and $B(go,M)$ projects to a bounded set in $t\mathrm{Ax}(f)$ of diameter B depending on M and C. If β is disjoint with $N_r(t\mathrm{Ax}(f))$ where $r \geq C$, we project β to $t\mathrm{Ax}(f)$ with diameter at most C, yielding that $d(u,v) \leq 2B + C$. If we choose

$$L_1 \gg 2B + C + 2r$$

we would obtain a contradiction, whence β has to intersect $N_r(t\mathrm{Ax}(f))$.

Let x and y be the entry and exit points of β in $N_r(t\mathrm{Ax}(f))$. Arguing with a similar projection argument, we see that x,y are within D distance to u and v respectively, where $D = 2B + 2C$. By Lemma 2.2, $[x,y]$ is $\hat{C}$–contracting and the quasi-convexity implies $[u,v]$ is r'–close to $[x,y]$ for some $r' = r'(\hat{C})$. Thus,

$$d(to,[u,v]), d(tfo,[u,v]) \leq \hat{r} := r + r'.$$

Extending $[u,v]$ to be a θ–segment of β, we proved that β contains $(\hat{r},h)$–barriers in a θ–segment. □

The following lemma will be used in the proof of Lemma 4.12. Roughly speaking, it says that elements in $\mathcal{O}_{M_1,M_2}$ have no (r,f)–barrier.

Lemma 4.5. *Fix any $M \geq C$. Let γ be a geodesic with two endpoints in $N_M(Go)$ and α be a component in the complement $\gamma \setminus N_M(Go)$. Then there exists a constant $L_2 > 0$ depending only on r,C so that $\alpha \cap N_r(h\mathrm{Ax}(f))$ has diameter at most L_2 for any $h \in G$.*

Proof. Let x and y be the entry and exit points of α in $N_r(h\mathrm{Ax}(f))$ respectively. If x' and y' are the corresponding projection points to $h\mathrm{Ax}(f)$, we have $d(x,x'), d(y,y') \leq r$. By assumption, $\alpha \cap N_M(Go) = \varnothing$. Noting $C \leq M$ and $\mathrm{Ax}(f) \subseteq Go$, we have $\alpha \cap N_C(h\mathrm{Ax}(f)) = \varnothing$. The C–contracting property implies the projection of α to $h\mathrm{Ax}(f)$ has diameter at most C. Consequently, we obtain $d(x',y') \leq C$ and thus $d(x,y) \leq d(x',y') + 2r \leq C + 2r$. Setting $L_0 = C + 2r_0$ completes the proof. □

Convention 4.6. *We fix the constants $\hat{r}$ satisfying Lemma 4.4 and Lemma 4.5. We choose $M \geq \max\{\hat{r}, C\}$ as in the definition of SCC actions (2.20).*

Fix an element $h \in E(f)$ where $f \in F$ is given as in Convention (3.3).

Let $\mathcal{W}(\theta,r,h)$ denote the set of elements g in G having **no** (r,h)–barriers at θ–frequency: for any geodesic γ from $B(o,M)$ to $B(go,M)$, γ contains some θ–segment without (r,h)–barriers.

Lemma 4.7. *Let $G \curvearrowright \mathrm{X}$ be a non-elementary action with a contracting element. Assume that $G \curvearrowright \mathrm{X}$ has purely exponential growth. Then the set $\mathcal{W}(\theta, r, h)$ is growth negligible.*

In addition, if $G \curvearrowright \mathrm{X}$ is a SCC action, then the set $\mathcal{W}(\theta, r, h)$ is growth tight.

Proof. For simplicity, write $W := \mathcal{W}(\theta, r, h)$ in this proof. Given $g \in W$, let γ be a fixed geodesic from $B(o, M)$ to $B(go, M)$ so that some θ–segment of γ, denoted by α, is (r, h)–barrier-free. That is to say, $\ell(\alpha) \ge \theta\ell(\gamma)$ and α contains no (r, h)–barrier.

We may assume α is a maximal open segment with this property. We must have the two endpoints of α lies in $N_r(Go)$, otherwise we can extend α until reaching $N_r(Go)$. Thus, there exist $g_1, g_2 \in G$ such that

$$d(g_1 o, \alpha_-), d(g_2 o, \alpha_+) \le r,$$

in particular, $\hat{g} = g_1^{-1} g_2$ is (r, h)–barrier-free by definition. As a consequence, this implies that any $g \in W$ can be written as a product of three elements $g = g_1 \cdot \hat{g} \cdot (g_2^{-1} g)$ so that

$$|d(o, go) - d(o, g_1 o) - d(o, g_2 o) - d(o, \hat{g} o)| \le 4M.$$

By Proposition 2.22, the set of elements $\hat{g} \in \mathcal{V}_{r,M,h}$ is growth negligible: as $n \to \infty$,

$$\frac{\sharp N(o,n) \cap \mathcal{V}_{r,M,h}}{\sharp N(o,n)} \to 0.$$

If the action is SCC, then $\mathcal{V}_{r,M,h}$ is growth tight: for some $\epsilon > 0$,

$$\frac{\sharp N(o,n) \cap \mathcal{V}_{r,M,h}}{\sharp N(o,n)} \le \mathrm{e}^{-\epsilon n}.$$

Moreover, the element $\hat{g}$ in the above product takes a definite proportion:

$$d(o, \hat{g} o) \ge \ell(\alpha) - 2M \ge \theta d(o, go) - 2M.$$

Since $G \curvearrowright \mathrm{X}$ has purely exponential growth, we have $\sharp N(o,n) \asymp \mathrm{e}^{n\omega_G}$. By a similar argument as in [8, Lemma 3.9], a straightforward computation shows that W is growth tight if the action is SCC, and is growth negligible in general. The lemma is then proved. □

Corollary 4.8. *If the action is SCC, then the Poincaré series associated to $W := \mathcal{W}(\theta, r, h)$ is convergent at $s = \omega_G$. Namely,*

$$\sum_{g \in W} \mathrm{e}^{-\omega_G d(o, go)} < \infty.$$

Remark. We expect this same conclusion holds for proper action with purely exponential growth.

Let $\mathcal{NFB}(\theta, r, h)$ denote the set of limit points $\xi \in \partial \mathrm{X}$ that is contained in infinitely many shadows at elements in $\mathcal{W}(\theta, r, h)$. Precisely, denoting $W := \mathcal{W}(\theta, r, h)$, $\mathcal{NFB}(\theta, r, h)$ is the limit supper of the following sequence of sets

$$\bigcup_{v \in A(o,n,\Delta) \cap W} \Pi_o^F(v, r)$$

where $A(o, n, \Delta)$ is defined in (10), as $n \to \infty$. Hence, the sequence

$$\Lambda_n := \bigcup_{k \ge n} \left[\bigcup_{v \in A(o,k,\Delta) \cap W} \Pi_o^F(v, r) \right]$$

tends to $\mathcal{NFB}(\theta, r, h)$.

We clarify the set $\mathcal{NFB}(\theta, r, h)$ with the following result, which shall not be used in other places.

Lemma 4.9. *Let $g_n \in \mathcal{W}(\theta, r, h)$ be a sequence of elements without (r, h)–barriers at θ–frequency. Assume that ξ is contained in a sequence of shadows $\Pi_o^F(y_n, r)$ where $y_n := g_n o$. Then there exists a geodesic ray γ starting at o and ending at $[\xi]$ such that γ contains no (r, h)–barriers at θ–frequency.*

Proof. For $\xi \in \Pi_o^F(y_n, r)$, there exists $z_m \in \Omega_o^F(y_n, r) \to \xi$ such that $[o, z_m]$ contains a (r, F)–barrier at $y_n = g_n o$. That is to say, for some $f' \in F$,

$$d(g_n o, [o, z_m]), d(g_n f' o, [o, z_m]) \le r.$$

This inequality holds for any limit of $[o, z_m]$ by Ascoli-Arzela Lemma with respect to the local uniform convergence topology. A Cantor diagonal argument then provides a limiting geodesic ray γ, which has the above (r, F)–barrier at every y_n. As $[o, y_n]$ has no (r, f)–barriers at θ–frequency, any limit γ does so. It remains to note that γ ends at a point in $[\xi]$. To see this, first note that any limit point η of y_n lies in $[\xi]$, as the direct computation shows the difference b_ξ and b_η is uniformly bounded using the above inequality. As $d(y_n, \gamma) \le r$, the same reasoning shows that the Buseman function to γ has finite difference to b_η. This then concludes the proof. □

4.2. **Non-frequently contracting segments have non-frequent barriers.** We now relate the metric notion of frequently contracting segments (Def. 2.16) to the analogous one, but involving a group action, with frequent barriers (Def. 4.2). The title says the main result, Lemma 4.12, of this subsection. First of all, we state a preparatory lemma.

Lemma 4.10. *Let γ be a geodesic segment with (r, h)–barriers at θ–frequency for some $h \in E(f)$. Then γ is (r, C, L)–contracting at θ–frequency, where $L = d(o, ho)$.*

Proof. The proof follows by unravelling the definitions. By definition, if γ contains an (r, h)–barrier, then γ contains a segment $[o, ho]$ in the r–neighborhood of $\mathrm{Ax}(f)$, which is C–contracting. □

Lemma 4.11. *Let $h_n \in E(f)$ be a sequence of elements with $d(o, h_n o) \to \infty$. If γ is a geodesic ray with (r, h_n)–barriers at θ–frequency for any $\theta \in (0, 1]$, then γ is (r, C)–regular contracting.*

Proof. As $\theta \in (0, 1]$ is arbitrary and $d(o, ho) \to \infty$, it follows from Lemma 4.10 that γ is (r, C)–frequently contracting. □

We now look at the set of (r, C, L)–contracting rays at θ–frequency. Precisely, let $\mathcal{FC}(\theta, r, C, L)$ consist of all $[\cdot]_s$–classes $[\xi]_s \in [\partial \mathrm{X}]_s$ so that some geodesic ray γ in X starting at o and ending at $[\xi]_s$ is (r, C, L)–contracting at θ–frequency.

The relation with the previously defined set $\mathcal{NFB}(\theta, r, f)$ with no frequent (r, f)–barriers is explained in the following lemma. This is crucial in the proof of Theorem Theorem A later on.

Lemma 4.12. *Assume that*

- *L is a constant bigger than $\max\{L_0, L_1\}$, where L_0 are given by Lemma 4.5 and L_1 are given by Lemma 4.4.*
- *$h \in E(f)$ is an element with $d(o, ho) > 2L$,*

Then the complementary set $\Lambda_r^F(Go)\setminus\mathcal{FC}(\theta,\hat r,C,L)$ *is contained in the set* $\mathcal{NFB}(\theta,r,h)$.

The relevance of $f\in F$ to the constant C is that any segment in $N_r(\mathrm{Ax}(f))$ is assumed to be C–contracting. See Convention 3.3.

Proof. Let $\xi\in[\Lambda_r^F(Go)]$ be not contained in $[\mathcal{FC}(\theta,\hat r,C,L)]$. By definition, there exists a geodesic ray $\gamma=[o,\xi]$ that is not (r,C,L)–contracting at θ–frequency. By definition, there exist a sequence of positive numbers $R_n\to\infty$ with the following property :

$$(\circledast)\quad \begin{array}{l}\gamma[0,R_n]\text{ contains a }\theta\text{-interval }\alpha_n\text{ so that}\\ \text{no subsegment of }\alpha_n\text{ with length }L\text{ is }\hat r\text{–close to a }C\text{-contracting segment.}\end{array}$$

By Lemma 4.10, $\gamma[0,R_n]$ has no $(\hat r,h)$–barriers at θ–frequency.

If the point $\gamma[R_n]$ lies in the closed neighborhood $N_M(Go)$, there exists $g_n\in G$ such that $d(g_no,\gamma[R_n])\le M$. By Lemma 4.4, any geodesic from $B(o,M)$ to $B(g_no,M)$ has no (r,h)–barriers at θ–frequency. By definition, g_n has no (r,h)–barriers at θ–frequency, so $g_n\in\mathcal{W}(\theta,r,h)$ by definition.

Otherwise, we have $\gamma[R_n]$ lies outside $N_M(Go)$. The remainder of the proof is then to find another time R_n' so that $\gamma[0,R_n']$ has the above property $(\circledast)$(i.e. not (r,C,L)–contracting at θ–frequency), and the new point $\gamma[R_n']$ lies in $N_M(Go)$. To that end, our discussion is divided into the following two cases. Write explicitly $\alpha_n:=\gamma[s_n,t_n]$ for some $0\le s_n<t_n\le R_n$.

Case 1. Assume that $\gamma[0,R_n]$ contains a point $\gamma[R_n']$ in $N_M(Go)$ for some time $t_n\le R_n'<R_n$. As α_n is contained in $\gamma[0,R_n']$ and $R_n'\le R_n$ is an earlier time, the segment $\gamma[0,R_n']$ has the above property $(\circledast)$. Also, $d(g_no,\gamma[R_n'])\le M$ for some $g_n\in G$.

Case 2. After $\alpha_n=\gamma[s_n,t_n]$, $\gamma[0,R_n]$ contains no point in $N_M(Go)$. That is to say, for any $t\in[t_n,R_n]$, we have $d(\gamma[t],Go)>M$. In this case, we shall extend the segment $\gamma[0,R_n]$ to a future time $R_n'\ge R_n$, which still has the above property $(\circledast)$, so that $d(g_no,\gamma[R_n'])\le M$ for some $g_n\in G$. See Figure 6.

Indeed, let $R_n'>R_n$ be the least number so that $\gamma[R_n']$ is contained in $N_M(Go)$. The number $R_n'<\infty$ exists, since γ ends at $[\cdot]_s$–class of a conical point $\xi\in[\Lambda_r^F(Go)]$, which according to Lemma 3.8 intersects $N_{\hat r}(Go)$ in a unbounded diameter.

Let $g_n\in G$ satisfy $d(g_no,\gamma[R_n'])\le M$. We need to verify

Claim. g_n *has no* (r,h)*–barriers at* θ*–frequency.*

Proof of the claim. The segment $\alpha_n':=\gamma[s_n,R_n']$ is the union of α_n and $\gamma[t_n,R_n']$. As α_n is a θ–interval of $\gamma[0,R_n]$, it follows that α_n' is a θ–interval of $\gamma[0,R_n']$.

Now, assume by contradiction that g_n has (r,f)–barriers at θ–frequency. By definition, any geodesic (say $\gamma[0,R_n']$) between $B(o,M)$ and $B(g_no,M)$ has (r,h)–barriers at θ–frequency. In particular, the θ–interval α_n' has (r,h)–barriers. Hence, we have some $g\in G$ so that go,gho lie in the r–neighborhood of α_n'. That is, $\alpha_n'\cap N_r(g\mathrm{Ax}(f))$ has diameter at least $d(o,ho)>2L$.

As $\gamma[t_n,R_n']$ lies outside $N_M(Go)$, we obtain from Lemma 4.5 that $\gamma[t_n,R_n']\cap N_r(g\mathrm{Ax}(f))$ has diameter at most L_0. Thus, $\alpha_n\cap N_r(g\mathrm{Ax}(f))$ has diameter at least $2L-L_0>L$. By assumption, any segment with two endpoints in $g\mathrm{Ax}(h)$ is C–contracting, we see that α_n contains a segment of length L, that is r–close to a C–contracting segment. This is a contradiction with $(\circledast)$, so we proved that $g_n\in\mathcal{W}(\theta,r,h)$. $\square$

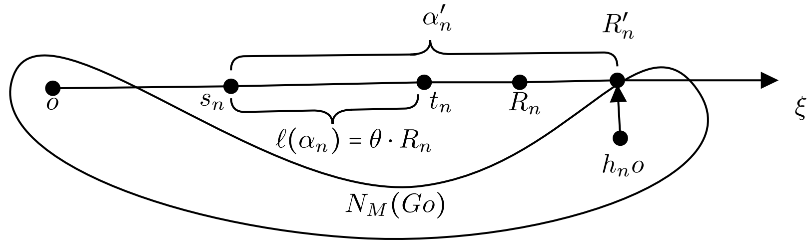

FIGURE 6. Case (2) in the proof of Lemma 4.12

In summary, we produced a sequence of elements $g_n \in \mathcal{W}(\theta, r, h)$ so that $d(g_n o, \gamma) \leq C$. Hence, ξ lies in $\mathcal{NFB}(\theta, r, h)$. The lemma is proved. □

In order to prove that $\mathcal{FC}(\theta, r, C, L)$ is μ_o–full, we only need to prove that $\mathcal{NFB}(\theta, r, h)$ is μ_o–null.

Lemma 4.13. *Assume that the action $G \curvearrowright \mathrm{X}$ is SCC. Fix $\theta \in (0,1)$ and a contracting element $h \in E(f)$. Then the set $\mathcal{NFB}(\theta, r, h)$ is μ_o–null.*

Proof. In this proof, we write $W := \mathcal{W}(\theta, r, h)$ for simplicity. The following argument is standard, but we write it for the completeness.

By way of contradiction, assume that $\mu_o(\mathcal{NFB}(\theta, r, h)) > 0$. By assumption, $\mu_o(\Lambda_n) \geq \frac{1}{2}\mu_o(\mathcal{NFB}) > 0$ for all $n \gg 0$. As $W \subseteq G$ is growth tight and ω is the critical exponent of G, the following Poincaré series associated with W

$$\mathcal{P}_W(\omega, o, o) = \sum_{g \in W} e^{-\omega d(o, go)}$$

is convergent. By Lemma 3.5, we obtain a uniform lower bound on the partial sum of $\mathcal{P}_W(\omega, o, o)$ for any $n \gg 0$:

$$\bigcup_{k \geq n} \left[\bigcup_{v \in A(o,k,\Delta) \cap W} e^{-\omega n} \right] \succ \mu_o(\Lambda_n)$$

thus contradicting to the convergence of $\mathcal{P}_W(\omega, o, o)$. This shows that $\mathcal{NFB}$ is μ_o–null. □

4.3. **Completion of the proof of Theorem A.** As shown in (2.19), the set of regularly contracting rays $\mathcal{RC}(r, C)$ for given $r, C > 0$ is the countable intersection of $\mathcal{FC}(\theta, r, C, L)$ over rationals $\theta \in \mathbb{Q}$ and integers $L \in \mathbb{N}$.

According to Lemma 4.12, $\mathcal{FC}(\theta, r, C, L)$ is contained in $\mathcal{NFB}(\theta, r, h)$ for some $h \in E(f)$, so is μ_o–full for every L, θ by Lemma 4.13. Hence, the countable intersection $\mathcal{RC}(r, C)$ is μ_o–full.

By Theorem 2.18, $\mathcal{RC}(r, C)$ is a subset of the sublinearly Morse directions, thus the set of sublinearly Morse directions is μ_o–full.

References

1. M. Bridson and A. Haefliger, *Metric spaces of non-positive curvature*, vol. 319, Grundlehren der Mathematischen Wissenschaften, Springer-Verlag, Berlin, 1999.
2. R. Charney and H. Sultan, *Contracting boundaries of CAT(0) spaces*, J Topology **8** (2015), no. 1, 93–117.

3. M. Cordes, *Morse boundaries of proper geodesic metric spaces*, Groups Geom. Dyn. **11** (2017), no. 4, 1281–1306. MR 3737283
4. R. Coulon, *Patterson-Sullivan theory for groups with a strongly contracting element*, preprint, arXiv:2206.07361.
5. C. Croke and B. Kleiner, *Spaces with nonpositive curvature and their ideal boundaries*, Topology **39** (2000), no. 3, 549–556. MR 1746908
6. F. Dal'bo, P. Otal, and M. Peigné, *Séries de Poincaré des groupes géométriquement finis*, Israel Journal of math. **118** (2000), no. 3, 109–124.
7. I. Gekhtman, Y.L. Qing, and K. Rafi, *Genericity of sublinearly morse directions in CAT(0) spaces and the Teichmüler space*, preprint, arXiv:2208.04778.
8. I. Gekhtman and W.Y. Yang, *Counting conjugacy classes in groups with contracting elements*, J. Topol. **15** (2022), no. 2, 620–665. MR 4441600
9. V. Kaimanovich and H. Masur, *The poisson boundary of the mapping class group*, Invent. math. **125** (1996), 221–264.
10. L.X. Liu and W.X. Su, *The horofunction compactification of the Teichmüller metric*, Handbook of Teichmüller theory. Vol. IV, IRMA Lect. Math. Theor. Phys., vol. 19, Eur. Math. Soc., Zürich, 2014, pp. 355–374. MR 3289706
11. S. Patterson, *The limit set of a Fuchsian group*, Acta Mathematica (1976), no. 1, 241–273.
12. Vincent Pit and Barbara Schapira, *Finiteness of Gibbs measures on noncompact manifolds with pinched negative curvature*, Ann. Inst. Fourier (Grenoble) **68** (2018), no. 2, 457–510. MR 3803108
13. Y. L. Qing and K. Rafi, *Sublinearly Morse boundary I: CAT(0) spaces*, Adv. Math. **404** (2022), Paper No. 108442, 51. MR 4423805
14. Y. L. Qing, K. Rafi, and G. Tiozzo, *Sublinearly morse boundary ii: Proper geodesic spaces*, to appear in Geometry & Topology.
15. R. Ricks, *Flat strips, Bowen-Margulis measures, and mixing of the geodesic flow for rank one* CAT(0) *spaces*, Ergodic Theory Dynam. Systems **37** (2017), no. 3, 939–970. MR 3628926
16. B. Schapira and S. Tapie, *Regularity of entropy, geodesic currents and entropy at infinity*, Ann. Sci. Éc. Norm. Supér. (4) **54** (2021), no. 1, 1–68. MR 4245867
17. C. Walsh, *The asymptotic geometry of the Teichmüller metric*, Geom. Dedicata **200** (2019), 115–152. MR 3956189
18. W. Y. Yang, *Statistically convex-cocompact actions of groups with contracting elements*, Int. Math. Res. Not. IMRN (2019), no. 23, 7259–7323. MR 4039013
19. ______, *Genericity of contracting elements in groups*, Math. Ann. **376** (2020), no. 3-4, 823–861. MR 4081104
20. Wenyuan Yang, *Growth tightness for groups with contracting elements*, Math. Proc. Cambridge Philos. Soc **157** (2014), 297 – 319.
21. ______, *Conformal dynamics at infinity for groups with contracting elements*, arXiv: 2208.04861, 2023.

Department of Mathematics, University of Tennessee at Knoxville, 1403 Circle Drive, Knoxville, TN 37916, USA
Email address: yqing@utk.edu

Beijing International Center for Mathematical Research, Peking University, Beijing 100871, China P.R.
Email address: wyang@math.pku.edu.cn